Phys 661 HW 4 Solutions

Problem 1 – Yicheng Feng

PHYS 661 HW4 Yicheng Feng

1. Spin $\frac{1}{2}$ means that those particles are fermions, so no 2 particles can coexist in the same state.

The eigenvalues (energy) and eigenstates of the Hamiltonian are:

$E_n = (n+\frac{1}{2})\hbar\omega \qquad |E_n\rangle = |n\rangle$

The total state of each particle could be $|\psi_{n,s}\rangle = |n\rangle \otimes |s\rangle$

$\Rightarrow$ there are at most 2 particles in each energy eigenstate.

For the ground state, we let the N identical particles to fill up the states with energy as low as possible.

$$E_g = \begin{cases} 2 \sum_{j=1}^{N/2} \left[(j-1)+\frac{1}{2}\right]\hbar\omega = 2\hbar\omega\left[\frac{1}{2}\frac{N}{2}\left(\frac{N}{2}-1\right)+\frac{N}{4}\right] = \frac{1}{4}N^2\hbar\omega & \text{(even } N\text{)} \\ 2\sum_{j=1}^{(N+1)/2}\left[(j-1)+\frac{1}{2}\right]\hbar\omega - \left[\left(\frac{N+1}{2}-1\right)+\frac{1}{2}\right]\hbar\omega = \frac{1}{4}(N^2+1)\hbar\omega & \text{(odd } N\text{)} \end{cases}$$

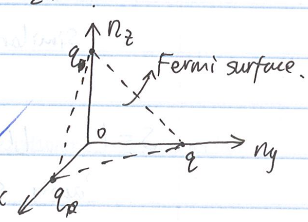
When N is large enough, we can just keep the N^2 term:

$E_g \approx \frac{1}{4}N^2\hbar\omega$.

For 3-D isotropic harmonic oscillator

$E_{(n_x,n_y,n_z)} = (n_x+n_y+n_z)\hbar\omega + \frac{3}{2}\hbar\omega = \varepsilon(n_x,n_y,n_z) + \frac{3}{2}\hbar\omega$

$|\psi(n_x,n_y,n_z,s)\rangle = |n_x,n_y,n_z\rangle \otimes |s\rangle$



In the following discussion, we just drop the term $\frac{3}{2}\hbar\omega$, because its contribution to the total energy E is easy to get: $\frac{3}{2}N\hbar\omega$.

All the states with ~~$\varepsilon < \varepsilon_F$~~ energy ~~less~~ no more than $(\varepsilon + \frac{3}{2}\hbar\omega)$ are in the pyramid in the sketch. $q = \frac{\varepsilon}{\hbar\omega}$. They are "integral points".

When N is very big, we can use the volume of this pyramid to approximate the number of states $(E < \varepsilon + \frac{3}{2}\hbar\omega)$:

$\frac{1}{2}N = \frac{1}{6}q^3$ and on the Fermi surface $q_F = \sqrt[3]{3N} \qquad \varepsilon_F = \hbar\omega\sqrt[3]{3N}$

$\therefore\ dN = q^2 dq = (\hbar\omega)^{-3}\varepsilon^2 d\varepsilon$

Total energy:

$$E-\frac{3N\hbar\omega}{2} = \int_0^N \varepsilon\, dN' = \int_0^{q_F} \varepsilon(q)\, q^2 dq = \hbar\omega \int_0^{q_F} q^3 dq = \frac{1}{4}\hbar\omega\, q_F^4 = \frac{1}{4}\hbar\omega (3N)^{\frac{4}{3}}$$

$$\therefore E = \frac{1}{4}\hbar\omega(3N)^{\frac{4}{3}} + \frac{3}{2}N\hbar\omega. = E_g$$

2. For 2 identical spin 1 particles, we now use $|S_{z1}, S_{z2}\rangle$ to express $|S, S_z\rangle$. To indicate the difference, we underline $|S, S_z\rangle$ to be $|\underline{S, S_z}\rangle$

$S=2$: $\quad |\underline{2,2}\rangle = |1,1\rangle$

$S^- = S_1^- + S_2^- \qquad S^-|\underline{2,2}\rangle = \sqrt{2\times(2+1) - 2\times(2-1)}\,|\underline{2,1}\rangle = 2|\underline{2,1}\rangle$

$S^-|1,1\rangle = (S_1^- + S_2^-)|1,1\rangle = \sqrt{1\times(1+1) - 1\times(1-1)}\,(|0,1\rangle + |1,0\rangle) = \sqrt{2}(|0,1\rangle + |1,0\rangle)$

$\therefore |\underline{2,1}\rangle = \frac{\sqrt{2}}{2}(|0,1\rangle + |1,0\rangle)$

$S^-|\underline{2,1}\rangle = \sqrt{2\times(2+1) - 1\times(1-1)}\,|\underline{2,0}\rangle = \sqrt{6}\,|\underline{2,0}\rangle$

$S^-\left(\frac{\sqrt{2}}{2}|0,1\rangle + \frac{\sqrt{2}}{2}|1,0\rangle\right) = (S_1^- + S_2^-)\frac{\sqrt{2}}{2}(|0,1\rangle + |1,0\rangle)$

$= \frac{\sqrt{2}}{2}(\sqrt{2}|-1,1\rangle + \sqrt{2}|0,0\rangle + \sqrt{2}|0,0\rangle + \sqrt{2}|1,-1\rangle) = |-1,1\rangle + 2|0,0\rangle + |1,-1\rangle$

$\therefore |\underline{2,0}\rangle = \frac{1}{\sqrt{6}}(|-1,1\rangle + 2|0,0\rangle + |1,-1\rangle)$

Similarly, $\quad |\underline{2,-1}\rangle = \frac{1}{\sqrt{2}}(|-1,0\rangle + |0,-1\rangle) \qquad |\underline{2,-2}\rangle = |-1,-1\rangle$

$S=1$: $\quad |\underline{1,1}\rangle = \alpha|0,1\rangle + \beta|1,0\rangle$

apply S^+ to both sides: ~~$\sqrt{2}|1,0\rangle = \alpha\, 0 =$~~

$0 = \alpha\sqrt{2}|1,1\rangle + \beta\sqrt{2}|1,1\rangle \quad \therefore \alpha + \beta = 0 \quad$ let $\alpha = \frac{1}{\sqrt{2}},\ \beta = -\frac{1}{\sqrt{2}}$

$\therefore |\underline{1,1}\rangle = \frac{1}{\sqrt{2}}|0,1\rangle - \frac{1}{\sqrt{2}}|1,0\rangle$

apply S^- to both sides:

$\sqrt{2}|\underline{1,0}\rangle = |-1,1\rangle - |0,0\rangle + |0,0\rangle - |1,-1\rangle$

$\therefore |\underline{1,0}\rangle = \frac{1}{\sqrt{2}}(|-1,1\rangle - |1,-1\rangle)$

apply S^- to both sides:

$\sqrt{2}|\underline{1,-1}\rangle = |-1,0\rangle - |0,-1\rangle$

$\therefore |\underline{1,-1}\rangle = \frac{1}{\sqrt{2}}(|-1,0\rangle - |0,-1\rangle)$

Problem 2 – K Shiva Teja

2) Sol:) Spin 1 particles ⇒ bosons

⇒ Symmetric total wavefunctions.

If the Hamiltonian is independent of spin, the total wavefunction, $\bar{S}^2$ commutes trivially with H

$\Rightarrow \quad \Psi(\bar{x}, s) = \phi(x)\,\chi(s)$, where $\chi(s)$ has a unique value of total spin.

Spin = 2 $\quad |j\ m_j\rangle \longrightarrow |m_1\ m_2\rangle$

$|2\ 2\rangle = |1\ 1\rangle$

$|2\ 1\rangle = \frac{1}{\sqrt{2}}|0\ 1\rangle + \frac{1}{\sqrt{2}}|1\ 0\rangle$

$|2\ 0\rangle = \frac{1}{\sqrt{6}}\left(|1\ {-1}\rangle + |{-1}, 1\rangle\right) + \sqrt{\frac{2}{3}}|0\ 0\rangle$

$|2\ {-1}\rangle = \frac{1}{\sqrt{2}}|0\ {-1}\rangle + \frac{1}{\sqrt{2}}|{-1}, 0\rangle$

$|2\ {-2}\rangle = |{-1}\ {-1}\rangle$

We see that all the $\chi(s)$ are symmetric under the exc of particles

Spin = 1

$$\left.\begin{aligned} |1\ 1\rangle &= \frac{1}{\sqrt{2}}|0\ 1\rangle - \frac{1}{\sqrt{2}}|1\ 0\rangle \\ |1\ 0\rangle &= \frac{1}{\sqrt{2}}|1\ {-1}\rangle - \frac{1}{\sqrt{2}}|{-1}\ 1\rangle \\ |1\ {-1}\rangle &= \frac{1}{\sqrt{2}}|0\ {-1}\rangle - \frac{1}{\sqrt{2}}|{-1}, 0\rangle \end{aligned}\right\} \begin{gathered}\chi(s)\\ \text{Antisym-}\\ \text{-etric}\end{gathered}$$

K. Shiva Teja,

Spin = 0

$$|0\ 0\rangle = \frac{1}{\sqrt{3}}|1\ -1\rangle + \frac{1}{\sqrt{3}}|-1\ 1\rangle - \frac{1}{\sqrt{3}}|0\ 0\rangle$$

Obviously this state is also symmetric.

$\therefore$ If the spatial wavefunction is symmetric, we can only have total spin = 2 or 0.

If the spatial wavefunction is antisymmetric, we can only have total spin = 1.

Problem 3 – Michael Higgins

3. A one-dimensional harmonic oscillator in the ground state $|0\rangle$ @ $t=0$. An external potential $V(t) = V_0 x e^{-\frac{t}{\tau}}$ is then turned on:

a) Use time-dependent perturbation theory to find the probability of the harmonic oscillator to be in any of its possible states at $t>0$.

b) Study behavior @ $t \to \infty$.

Solution: $\hat{H} = \hat{H}_0 + \lambda V(t)$, $\hat{H}_0 = \hbar\omega\left(a^+a + \frac{1}{2}\right)$ where

- $\hat{H}_0|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle$, $[a^+, a] = -1$
- $\begin{cases} a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle \\ a\,|n\rangle = \sqrt{n}\,|n-1\rangle \end{cases}$, $x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^+)$

- From 1st order perturbation theory: $c_n^{(1)}(t) = \frac{-i}{\hbar}\int_0^t e^{i\omega_{ni}t'} V_{ni}(t')\,dt'$

where $\hbar\omega_{ni} = E_n - E_i$, $V_{ni} = \langle E_n | V(t) | E_i \rangle$.

- At $t=0$, $|i\rangle = |0\rangle$ $\therefore$ $c_n(t) = \frac{-i}{\hbar} V_0 \langle n|x|0\rangle \int_0^t e^{i\omega_{n0}t' - \frac{t'}{\tau}}\,dt'$

$$c_n^{(1)}(t) = \frac{-iV_0\langle n|x|0\rangle}{\hbar}\left(\frac{1}{(i\omega_{n0} - \frac{1}{\tau})}\right)\left(e^{i\omega_{n0}t}e^{-\frac{t}{\tau}} - 1\right)$$

- $\langle n|x|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle n|a + a^+|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle n|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\delta_{n1}$

$$\therefore \boxed{c_n^{(1)}(t) = \begin{cases} -i\sqrt{\frac{1}{2m\hbar\omega}}\,V_0\left(\frac{1}{i\omega_{n0} - \frac{1}{\tau}}\right)\left(e^{i\omega_{n0}t}e^{-\frac{t}{\tau}} - 1\right), & n=1 \\ 0 & n \neq 1 \end{cases}}$$

- The probability is $|c_n^{(1)}(t)|^2$:

$$P_{0\to n}(t) = \frac{V_0^2}{2m\hbar\omega}\left(\frac{1}{\omega_{n0}^2 + (\frac{1}{\tau})^2}\right)\left|\left(e^{-\frac{t}{\tau}}\cos(\omega_{n0}t) - 1\right) + i e^{-\frac{t}{\tau}}\sin(\omega_{n0}t)\right|^2$$

$$\boxed{P_{0\to n}(t) = \begin{cases} \frac{V_0^2}{2m\hbar\omega}\left(\frac{1}{\omega_{n0}^2 + (\frac{1}{\tau})^2}\right)\left[1 + e^{-\frac{2t}{\tau}} - 2e^{-\frac{t}{\tau}}\cos(\omega_{n0}t)\right], & n=1, \ \omega_{10}=\omega \\ 0 & n \neq 1 \end{cases}}$$

b) $t \to \infty$ $P_{0\to n}(t) \to \begin{cases} \frac{V_0^2}{2m\hbar\omega}\left[\frac{1}{\omega_{n0}^2 + (\frac{1}{\tau})^2}\right], & n=1, \ \omega_{10}=\omega \\ 0, & n \neq 1 \end{cases}$

3. b) (cont.)

- The probability for transitioning from $n=0$ to $n=1$ increases with time to a constant probability of $P_{0\to 1}(t\to\infty) = \frac{V_0^2}{2m\hbar\omega}\left(\frac{1}{\omega^2+\left(\frac{1}{\tau}\right)^2}\right)$ in the first order approximation. The probability of transitioning to a state $n\neq 1$ is zero in the first order limit. Once the potential is turned on, there will be a possibility of measuring the state to be in the $|1\rangle$ State for all $t>0$.

Problem 4 – Q Mirza

#4

$dV = -E\,dr = -E_0 e^{-t/\tau} dz \Rightarrow V = -E_0 z e^{-t/\tau}$

$U = qV = -eE_0 z e^{-t/\tau}$

$C_n' = \frac{-i}{\hbar}\int_{t_0}^{t} dt' e^{-i\omega_{ni}t} U_{ni}(t)$

$V_{ni} = -eE_0 e^{-t/\tau} \langle 2\, l'\, l_z' | z | 1\, 0\, 0\rangle$

$\Pi \quad + \quad - \quad + \quad - \quad 0$

$\langle 2\, 1\, l_z \,|\, T_0{}' \,|\, 1\, 0\, 0\rangle$

$\Pi : - \quad - \quad + \quad + \quad ?$

$l_z' = \pm 1$ gives zero matrix element thus $\underline{P_{1s \to 2p\, l_z = \pm 1} = 0}$

$C_{2p\, l_z = 0}'(t) = \frac{i}{\hbar} eE_0 \int_0^t dt' e^{-i\omega_{2p\, l_z=0\, 1s}t'} e^{-t/\tau} \langle 2\, 1\, 0 | z | 1\, 0\, 0\rangle$

$C'(t) = \frac{i}{\hbar} eE_0 \underbrace{\int_0^t dt' e^{-(i\omega + \frac{1}{\tau})t'}} \langle 2\, 1\, 0 | r\cos\theta | 1\, 0\, 0\rangle$

$C'(t) = -\frac{i}{\hbar} eE_0 \frac{\left(e^{-(i\omega + \frac{1}{\tau})t} - 1\right)}{i\omega + 1/\tau} \langle 2\, 1\, 0 | r\cos\theta | 1\, 0\, 0\rangle$

$|2\, 1\, 0\rangle = Y_1^0 R_{21} = \frac{1}{4\sqrt{2\pi}\, a_0^{3/2}} \frac{r}{a_0} \cos\theta\, e^{-r/2a_0}$

$$\Rightarrow \langle 210 | r\cos\theta | 100 \rangle = \int \frac{1}{4\sqrt{2\pi}\, a_0^{3/2}} \frac{r}{a_0} \cos\theta\, e^{-r/2a} (r\cos\theta) \frac{1}{\sqrt{\pi}} \frac{e^{-r/a_0}}{a_0^{3/2}} r^2 dr\, s\theta\, d\theta\, d\phi$$

let $u = \frac{3}{2}\frac{r}{a_0}$ $\quad dr = \frac{2}{3} a_0\, du$

$$= \frac{1}{2\sqrt{2}\, \cancel{a_0^3}} \int_0^\infty \frac{2}{3} u e^{-u} \left(\frac{2}{3}\right)^3 u^3 a_0^3 \left(\frac{2}{3}\right) a_0\, du \int_{-1}^{1} du\, u^2$$

$$= \left(\frac{\cancel{2}}{3}\right)\left(\frac{2}{3}\right)^5 \frac{a_0}{\cancel{2}\sqrt{2}} \underbrace{\int_0^\infty u^4 e^{-u} du}_{=4!} =$$

$$= \frac{2^5}{3^5} \frac{a_0}{\sqrt{2}} 2^3 = \frac{2^{8-1/2}}{3^5} a_0 = \frac{2^{15/2}}{3^5} a_0$$

$$C_g^1(t) = -\frac{i}{\hbar} eE_0 \frac{2^8 a_0}{3^5\sqrt{2}} \frac{e^{-(i\omega + 1/\tau)t} - 1}{i\omega + 1/\tau}$$

$$P_{1s \to 2p\, l_z = 0} = |c^1(t)|^2 = \frac{2^{15} e^2 E_0^2 a_0^2}{\hbar\, 3^{10}} \frac{\left(e^{-(i\omega + 1/\tau)t} - 1\right)^2}{(i\omega + 1/\tau)^2}$$

Problem 5 – Sean Myers

$$H = \frac{4\Delta}{\hbar^2}\, \vec{S}_1 \cdot \vec{S}_2$$

$$H = \frac{4\Delta}{\hbar^2}\left(S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z} \right)$$

$$V_{ni} = \frac{4\Delta}{\hbar^2}\langle n |\, S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z} \,| \uparrow\downarrow \rangle$$

$|n\rangle$ can be $|\uparrow\uparrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle$

each $S_{1i}S_{2i} = \frac{\hbar^2}{4}\sigma_{1i}\sigma_{2i}$

so I will factor out the $\hbar^2/4$

$$V_{ni} = \Delta \langle n | \underbrace{\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z} | \uparrow\downarrow \rangle}_{\left(|\downarrow\uparrow\rangle + |\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle\right)}$$

$$\boxed{V_{ni} = \Delta \langle n | \left(2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle \right)}$$

the state with non zero probability $|\downarrow\uparrow\rangle$ or $|\uparrow\downarrow\rangle$

$V_{\downarrow\uparrow,i} = 2\Delta$

$V_{\uparrow\downarrow,i} = -\Delta$

$$c_n^{(1)} t = \frac{-i}{\hbar}\int_0^t \cancel{e^{i\omega t'}}\; V_{ni}(t)\, dt' = \frac{-i}{\hbar} V_{ni} t$$

$$\boxed{\left| c_{\downarrow\uparrow}^{(1)}(t) \right|^2 = \frac{4\Delta^2}{\hbar^2} t^2 \qquad \left| c_{\uparrow\downarrow}^{(1)}(t) \right|^2 = 1 - \frac{4\Delta^2}{\hbar^2} t^2}$$

required to match initial conditions and $|c_{\uparrow\downarrow}|^2 + |c_{\downarrow\uparrow}|^2 = 1$

(b) $H = \frac{4\Delta}{\hbar^2} \vec{S}_1 \cdot \vec{S}_2$

rewrite as

$$H = \frac{4\Delta}{\hbar^2}\left(\frac{S^2 - S_1^2 - S_2^2}{2}\right)$$ note that $[\vec{S}_1, \vec{S}_2] = 0$

where $\vec{S} = \vec{S}_1 + \vec{S}_2$

eigenstates of H are $|s m_s\rangle$ where

S is total spin ~~and~~ $m_s = m_{1s} + m_{2s}$

writing the states in terms of the eigenstates

$$|\uparrow\uparrow\rangle = |1\,+1\rangle$$

$$|\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle + |00\rangle\right)$$

$$|\downarrow\uparrow\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle - |00\rangle\right)$$

$$|\downarrow\downarrow\rangle = |1\,-1\rangle$$

Starts in state $|\uparrow\downarrow\rangle$ $t=0$

$$|\uparrow\downarrow\rangle_{t=0} = \frac{1}{\sqrt{2}}\left(|10\rangle + |00\rangle\right)$$

by time evolution

$$|\uparrow\downarrow\rangle_t = e^{-i\frac{\hat{H}t}{\hbar}} |\uparrow\downarrow\rangle_{t=0} = \frac{1}{\sqrt{2}}\left(e^{-i\frac{\hat{H}t}{\hbar}}|10\rangle + e^{-i\hat{H}t/\hbar}|00\rangle\right)$$

For $$\hat{H}|10\rangle = \frac{2\Delta}{\hbar^2}\left(S^2 - S_1^2 - S_2^2\right)|10\rangle$$

$$= \frac{2\Delta}{\cancel{\hbar^2}}\left(\cancel{\hbar^2}2 - \frac{1}{2}\left(\frac{3}{2}\right)\cancel{\hbar^2} - \frac{1}{2}\left(\frac{3}{2}\right)\cancel{\hbar^2}\right)|10\rangle$$

$$= 2\Delta\left(2 - \frac{6}{4}\right)|10\rangle = \Delta$$

$$\hat{H}|00\rangle = \frac{2\Delta}{\hbar^2}\Big(0 \underbrace{- \hbar^2\frac{3}{4} - \hbar^2\frac{3}{4}}_{-\frac{6}{4} = -\frac{3}{2}}\Big)$$

$$= -3\Delta$$

$$\therefore \; |\uparrow\downarrow\rangle_t = \frac{1}{\sqrt{2}}\left(e^{-\frac{i\Delta t}{\hbar}}|10\rangle + e^{+\frac{i3\Delta t}{\hbar}}|00\rangle\right)$$

note that since $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ contain no combinations of $|00\rangle$ or $|10\rangle$ the probability to transition to either one is zero.

Probability to remain in the same state $|\uparrow\downarrow\rangle$

to make life easier I will multiply $|\uparrow\downarrow\rangle_t$ by a phase factor $e^{-i\Delta t/\hbar}$

$$|\uparrow\downarrow\rangle_t = \frac{1}{\sqrt{2}}\left(e^{-\frac{2i\Delta t}{\hbar}}|10\rangle + e^{+i2\Delta t/\hbar}|00\rangle\right)$$

$$|\langle\uparrow\downarrow|\uparrow\downarrow\rangle_t|^2 = \frac{1}{4}\left|\frac{2}{2}e^{-2i\Delta t} + e^{i2\Delta t}\right|^2$$

$$= \frac{1}{4}\left|2\cos\left(\frac{2\Delta t}{\hbar}\right)\right|^2 = \boxed{\cos^2\left(\frac{2\Delta t}{\hbar}\right)}$$

Note @ $t=0$

$Prob_{|\uparrow\downarrow\rangle} = 1$ ✓

Probability to transition to $|\downarrow\uparrow\rangle$

$$\left|\langle\downarrow\uparrow|\uparrow\downarrow\rangle_t\right|^2 = \frac{1}{4}\left| e^{-i2\Delta t/\hbar} - e^{+i2\Delta t/\hbar}\right|^2$$

$$\frac{2i}{2i}$$

$$= \frac{1}{4}\left| -2i\sin\left(\frac{2\Delta t}{\hbar}\right)\right|^2$$

$$\boxed{\mathrm{Prob}_{|\downarrow\uparrow\rangle}(t) = \sin^2\left(\frac{2\Delta t}{\hbar}\right)}$$

Looking $\mathrm{Prob}_{|\uparrow\downarrow\rangle}(t)$ and $\mathrm{Prob}_{|\downarrow\uparrow\rangle}(t)$ in the limit $t \ll \frac{\hbar}{2\Delta}$

$$\cos\left(\frac{2\Delta t}{\hbar}\right) \simeq 1 - \frac{1}{2}\left(\frac{4\Delta^2}{\hbar^2}t^2\right)$$

$$\cos^2\left(\frac{2\Delta t}{\hbar}\right) \simeq 1 - \frac{4\Delta^2}{\hbar^2}t^2 + O\,t^4$$

$$\sin^2\left(\frac{2\Delta t}{\hbar}\right) \simeq \left(\frac{2\Delta t}{\hbar}\right) \simeq \frac{4\Delta^2}{\hbar^2}t^2$$

$$\mathrm{Prob}_{|\downarrow\uparrow\rangle}(t) \simeq \frac{4\Delta^2}{\hbar^2}t^2$$

$$\mathrm{Prob}_{|\uparrow\downarrow\rangle}(t) \simeq 1 - \frac{4\Delta^2}{\hbar^2}t^2$$

In agreement for small $t \ll \frac{\hbar}{2\Delta}$