Homework VI

P1

$V(r>R)=0.$

$$\sigma_T = \frac{m^2}{4\pi^2\hbar^4}\int d\Omega\, |\tilde V(\vec k-\vec k')|^2$$

For $|\vec k|\to 0$ also $|\vec k'|=|\vec k|\to 0$ then $\tilde V(\vec k-\vec k') = \tilde V(0)$
indep. of θ & φ

$$\sigma_T = \frac{m^2}{\pi\hbar^4}|\tilde V(0)|^2 = 4\pi a_s^2$$

$$a_s^2 = \frac{m^2}{4\pi^2\hbar^4}|\tilde V(0)|^2 \Rightarrow \boxed{a_s = \frac{m}{2\pi\hbar^2}|\tilde V(0)|}$$

$$\boxed{\tilde V(0) = \int d^3x\; V(\vec x)}$$

If $V=V(r)$ $\quad\boxed{\tilde V(0) = 4\pi\int_0^R r^2dr\, V(r)}$

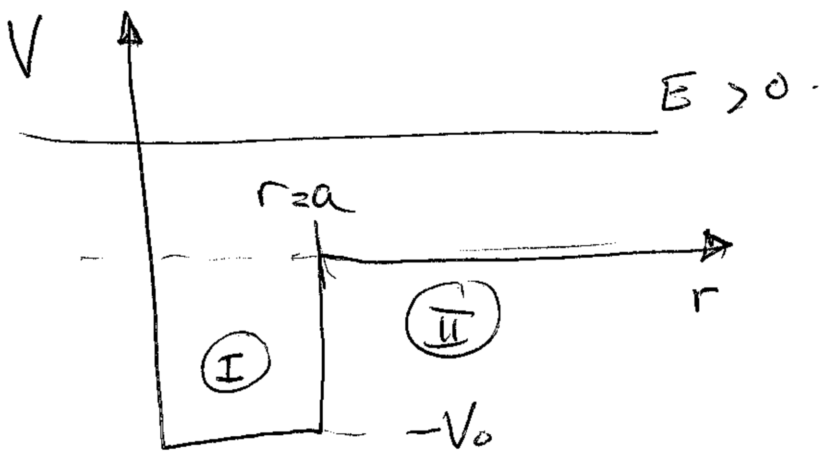
..) $$\operatorname{Im} f(k,k) = \frac{k}{4\pi}\sigma_T \simeq \frac{m^2k}{4\pi^2\hbar^4}|\tilde V(0)|^2 \quad \text{as } k\to 0$$

P2

$$V(r) = -V_0 \qquad r<a.$$

For s-waves. there is no centrifugal barrier.

$$-\frac{\hbar^2}{2m}\partial_r^2 \chi + V(r)\chi = E\chi$$



In region I $\qquad \chi = Ae^{ik_1 r} + Be^{-ik_1 r}$

$$k_1 = \sqrt{\frac{2m(E+V_0)}{\hbar^2}} \qquad \chi(0)=0$$

$$\boxed{\chi = A\sin k_1 r}$$

In region II $\qquad \chi = Ce^{ikr} + De^{-ikr} \quad ; \quad k=\sqrt{\frac{2mE}{\hbar}}$

$$A\sin k_1 a = Ce^{ika} + De^{-ika}$$

$$k_1 A\cos k_1 a = ikCe^{ika} - ikDe^{-ika}$$

$$\boxed{\begin{aligned} C e^{ika} + D e^{-ika} &= A \sin k_1 a \\ C e^{ika} - D e^{-ika} &= -\frac{ik_1}{k} A \cos k_1 a \end{aligned}}$$

$$2 C e^{ika} = A \left(\sin k_1 a - \frac{ik_1}{k} \cos k_1 a \right)$$

$$2 D e^{-ika} = A \left(\sin k_1 a + \frac{ik_1}{k} \cos k_1 a \right)$$

$$\frac{C}{D} e^{2ika} = \frac{\sin k_1 a - \frac{ik_1}{k} \cos k_1 a}{\sin k_1 a + \frac{ik_1}{k} \cos k_1 a}$$

$$\boxed{\frac{C}{D} = e^{-2ika} \frac{1 - \frac{ik_1}{k} \cot k_1 a}{1 + \frac{ik_1}{k} \cot k_1 a}}$$

$$\boxed{\frac{C}{D} = - e^{2i\delta_0}}$$ definition of δ_0.

$$e^{2i\delta_0} = - e^{-2ika} e^{-2i\varphi} \qquad \tan\varphi = \frac{k_1}{k} \cot k_1 a$$

$$\boxed{\delta_0 = \frac{\pi}{2} - ka - \varphi}$$

$$\tan\delta_0 = \tan\left(\frac{\pi}{2} - ka - \varphi\right) = \frac{\tan\left(\frac{\pi}{2} - ka\right) - \tan\varphi}{1 + \tan\left(\frac{\pi}{2} - ka\right)\tan\varphi}$$

$$= \frac{\cot ka - k_1/k \,\cot k_1 a}{1 + \cot ka \,\frac{k_1}{k} \cot k_1 a}$$

$$\boxed{\tan\delta_0 = \frac{k \cos ka \sin k_1 a - k_1 \cos k_1 a \sin ka}{k \sin k_1 a \sin ka + k_1 \cos ka \cos k_1 a}}$$

or $\delta_0 = \frac{\pi}{2} - ka - \operatorname{atan}\left(\frac{k_1}{k} \cot k_1 a\right)$

$\boxed{\text{low energy } E \to 0}$

$$k \to 0 \qquad k_1 \to \sqrt{\frac{2mV_0}{\hbar}}$$

$$\tan\delta_0 = \frac{k \sin k_1 a - k k_1 a \cos k_1 a}{k_1 \cos k_1 a} = \frac{k}{k_1}\left(\tan k_1 a - k_1 a\right)$$

$k \to 0 \qquad \tan\delta_0 \to 0 \qquad \tan\delta_0 \simeq \delta_0$

$$\boxed{\delta_0 \simeq \frac{k}{k_1}\left(\tan k_1 a - k_1 a\right)}$$

$$\boxed{\underset{k\to 0}{\sigma_T} \simeq \frac{4\pi}{k^2}\,\delta_0^2 = \frac{4\pi}{k_1^2}\left(\tan k_1 a - k_1 a\right)^2}$$

$E \to \infty$

$k_1 \simeq k + \delta k$

$$\tan\delta_0 \simeq \frac{k\cos ka \cdot a \cdot \cos(ka)\,\delta k + k\sin ka \cdot a \cdot \delta k \sin ka - \delta k \sin ka \cos ka}{k}$$

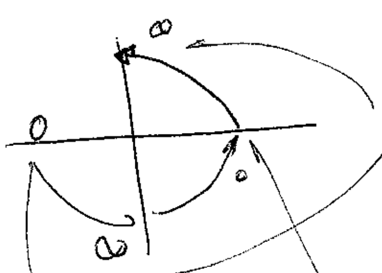
$$\simeq ka\frac{\delta k}{k} - \frac{\delta k}{k}\sin ka \cos ka$$

$$k_1 = \sqrt{\frac{2m(E+V_0)}{\hbar^2}} = \sqrt{k^2 + \frac{2mV_0}{\hbar^2}} = k\sqrt{1+\frac{2mV_0}{\hbar^2 k^2}} \simeq k\left(1+\frac{mV_0}{\hbar^2 k^2}\right)$$

$$\delta k = \frac{mV_0}{\hbar^2 k}$$

$\delta_0 \to 0$ also at ∞

$\Rightarrow$ $\delta_0(E=\infty)$ and $\delta_0(E=0)$ differ by integer $\times\ \pi$.



# of zeros of denominator $k\cos ka \sin k_1 a = k_1 \cos k_1 a \sin ka$

$$\boxed{k\tan k_1 a = k_1 \tan ka}$$

zeros of numerator $k\sin k_1 a \sin ka \pm k_1 \cos ka \cos k_1 a$

Bound states.

$$\chi_I = A \sin(\tilde{k}_1 r) \qquad \tilde{k}_1 = \sqrt{\frac{2m(E+V_0)}{\hbar^2}} \qquad \underline{\underline{E<0}}$$

$$\chi_{II} = C e^{-kr}$$

$$A \sin \tilde{k}_1 a = C e^{-ka} \qquad k = \sqrt{\frac{2m|E|}{\hbar^2}}$$

$$\tilde{k}_1 A \cos \tilde{k}_1 a = -kC e^{-ka}$$

$$\frac{1}{\tilde{k}_1} \frac{\sin \tilde{k}_1 a}{\cos \tilde{k}_1 a} = -\frac{1}{k}$$

$$\tan \tilde{k}_1 a = -\tilde{k}_1/k$$

$$\tan\left(\sqrt{\frac{2m(V_0-E)a^2}{\hbar^2}}\right) = -\frac{1}{k}\sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

$$\tilde{k}_1^2 = -\frac{2mE}{\hbar^2} + \frac{2mV_0}{\hbar^2} = -k^2 + k_0^2$$

$$k_1^2 = k^2 + k_0^2$$

$$1 + \tan^2 \tilde{k}_1 a = 1 + \tilde{k}_1^2/k^2 = \frac{1}{\cancel{k}}$$

$$\frac{1}{\cos^2 \tilde{k}_1 a} = \frac{\tilde{k}_1^2 + k^2}{k^2}$$

$$\cos^2 \tilde{k}_1 a = \frac{k^2}{\tilde{k}_1^2 + k^2}$$

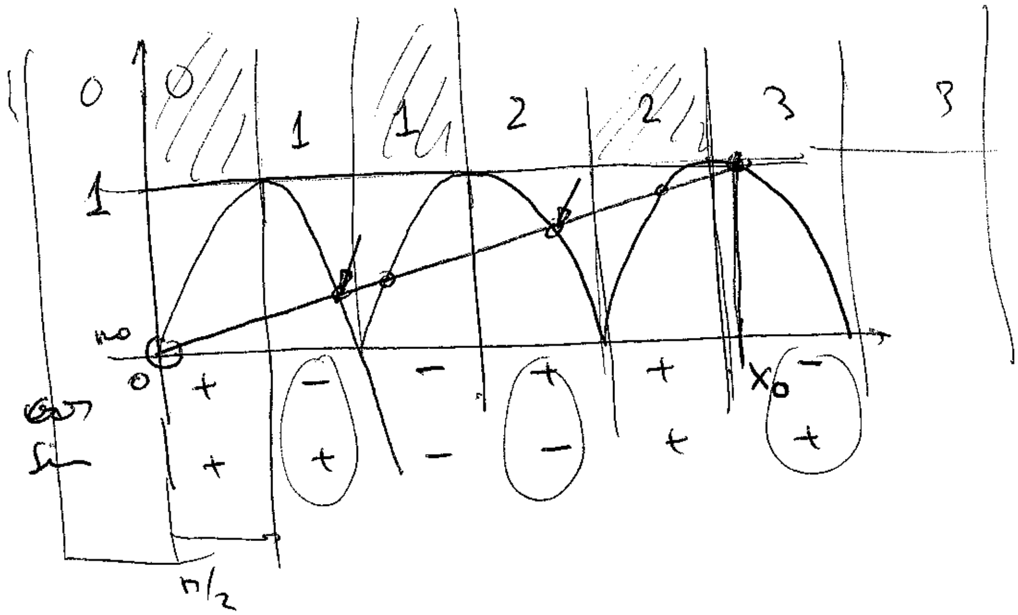
$$1 - \cos^2 \tilde{k}_1 a = \sin^2 \tilde{k}_1 a = \frac{\tilde{k}_1^2}{\tilde{k}_1^2 + k^2}$$

$$\sin^2 \tilde{k}_1 a = \frac{\tilde{k}_1^2}{\tilde{k}_1^2 + k_0^2}$$

$\sin \tilde{k}_1 a = \tilde{k}_1 / k_0$ $\qquad \tan \tilde{k}_1 a < 0$

$x = \tilde{k}_1 a \qquad x_0 = k_0 a \qquad ; \qquad x : 0 \to x_0$

$$\sin x = \pm \frac{x}{x_0}$$



$$\frac{(x_0 + \pi/2)}{\pi} \; .$$

$$N_{\text{bound states}} = \text{integer part } \frac{x_0 + \pi/2}{\pi}$$

One can check numerically that this agrees with

$$N_0 \pi = \delta(0) - \delta(\infty)$$

B

$f(\theta)$ indep. of θ

$\mathrm{Im}\, f(\theta=0) = \dfrac{k\,\sigma_T}{4\pi}$

$\mathrm{Im}\, f(\theta) = \dfrac{k}{\cancel{4\pi}} \cancel{4\pi} |f|^2$

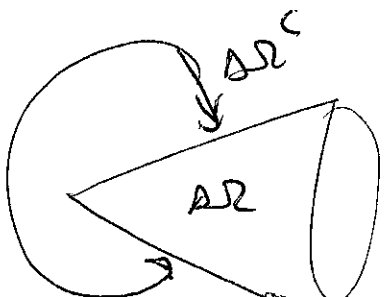
$f = f_1 + i f_2$

$f_2 = k\,(f_1^2 + f_2^2) \Rightarrow f_1^2 + f_2^2 = f_2/k \Rightarrow \left.\begin{array}{l} f_2 > 0 \\ \text{and } f_2^2 < f_2/k \end{array}\right\} \Rightarrow$

$\Rightarrow f_2 < 1/k \Rightarrow f_2/k < 1/k^2 \Rightarrow \boxed{\sigma_T \leqslant \dfrac{4\pi}{k^2}}$

$\boxed{k \to \infty \quad \sigma_T \to 0}$

ee)



$f \sim$ cst. $= f_0$

$\sigma_T = \Delta\Omega |f_0|^2 + \int_{\Omega - \Delta\Omega^c} |f|^2 \Rightarrow \sigma_T > \Delta\Omega |f_0|^2$

$\Rightarrow \boxed{\Delta\Omega < \sigma_T / |f_0|^2}$

$$\sigma_T = \frac{4\pi}{k} \, \mathrm{Im} f_0$$

$$\frac{4\pi}{k} \mathrm{Im} f_0 = \Delta\Omega \, |f_0|^2 + \int_{\Delta\Omega^c} |f|^2$$

$$\Delta\Omega (\mathrm{Im} f_0)^2 \leq \Delta\Omega \, |f_0|^2 \leq \frac{4\pi}{k} \mathrm{Im} f_0$$

$$\Delta\Omega \, (\mathrm{Im} f_0)^{\cancel{2}} \leq \frac{4\pi}{k} \, \cancel{\mathrm{Im} f_0}$$

$$\Delta\Omega \leq \frac{4\pi}{k \, \mathrm{Im} f_0}$$

...)

$\sigma_T \to$ cst. as $k \to \infty$

$\Rightarrow \mathrm{Im} f_0 \sim k \quad \Rightarrow \mathrm{Im} f_0$ grows $\Rightarrow |f_0|^2 \sim k^2$

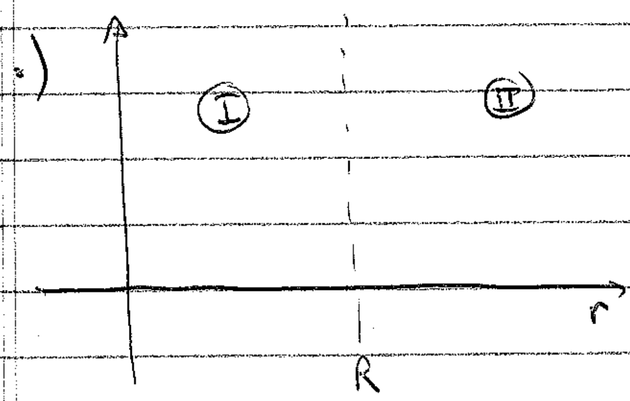
Large contribution from forward direction.

$$\Delta\Omega \leq \frac{4\pi}{k^2}$$

$\boxed{P4}$

$$V(r) = \frac{\hbar^2}{2m}\, \gamma\, \delta(r-R) \quad ; \quad \gamma > 0$$

a)



$\boxed{l=0}$

$$\begin{cases} \chi_{I}(r) = A \sin(kr) \\ \chi_{II}(r) = B\left(e^{-ikr} - e^{2i\delta_0} e^{ikr}\right) \end{cases} \quad ; \quad \frac{\hbar^2 k^2}{2m} = E$$

$(-)^{l+1}$

$$-\frac{\hbar^2}{2m} \partial_r^2 \chi + \frac{\hbar^2}{2m} \gamma\, \delta(r-R)\, \chi = \frac{\hbar^2 k^2}{2m} E$$

$$\int_{R-\varepsilon}^{R+\varepsilon} \rightarrow \quad -\partial_r \chi^{II} + \partial_r \chi^{I} + \gamma\, \chi(R) = 0$$

$$\boxed{\left.\partial_r \chi^{II} - \partial_r \chi^{I}\right|_{r=R} = \gamma\, \chi(R)}$$

$$\begin{cases} -ikB\left(e^{-ikR} + e^{2i\delta_0} e^{ikR}\right) - kA \cos kR = \gamma A \sin(kR) \\ B\left(e^{-ikR} - e^{2i\delta_0} e^{ikR}\right) = A \sin(kR) \end{cases}$$

②

$$B e^{-ikR}\left(1 - e^{2i\delta_0 + 2ikR}\right) = A\sin(kR)$$

$$B e^{-ikR}\left(1 + e^{2i\delta_0 + 2ikR}\right) = +\frac{iA}{k}\left(\gamma \sin kR + k\cos kR\right)$$

$$\frac{1 + e^{2i\delta_0 + 2ikR}}{1 - e^{2i\delta_0 + 2ikR}} = \frac{i}{k}\left(\gamma + k\frac{\cos kR}{\sin kR}\right)$$

$$= i\left(\frac{\gamma}{k} + \frac{\cos kR}{\sin kR}\right) = iC$$

$$\left(1 + e^{2i\delta_0 + 2ikR}\right) = iC - iC\, e^{2i\delta_0 + 2ikR}$$

$$(1 + iC)\, e^{2i\delta_0 + 2ikR} = -1 + iC$$

$$e^{2i\delta_0 + 2ikR} = -\frac{1 - iC}{1 + iC} = \frac{\cancel{i}\,(C + i)}{\cancel{i}\,(C - i)}$$

$$e^{2i\delta_0 + 2ikR} = \frac{\frac{\gamma}{k} + \frac{\cos kR}{\sin kR} + i}{\frac{\gamma}{k} + \frac{\cos kR}{\sin kR} - i}$$

$$\delta_0 + kR = \arg\left(\frac{\gamma}{k} + \frac{\cos kR}{\sin kR} + i\right)$$

$$= \operatorname{atan}\left(\frac{1}{\frac{\gamma}{k} + \frac{\cos kR}{\sin kR}}\right)$$

$$\boxed{\delta_0 = -kR + \operatorname{acotan}\left(\frac{\gamma}{k} + \operatorname{cotan}(kR)\right)}$$

or

$$e^{2i\delta_0 + 2ikR} = \frac{\frac{\gamma}{k}\sin kR + e^{ikR}}{\frac{\gamma}{k}\sin kR + e^{-ikR}}$$

$$e^{2i\delta_0} = \frac{1 + \frac{\gamma}{k}\sin kR\, e^{-ikR}}{1 + \frac{\gamma}{k}\sin kR\, e^{ikR}}$$

$$\boxed{\delta_0 = \arg\left(1 + \frac{\gamma}{k}\sin kR\, e^{-ikR}\right)}$$

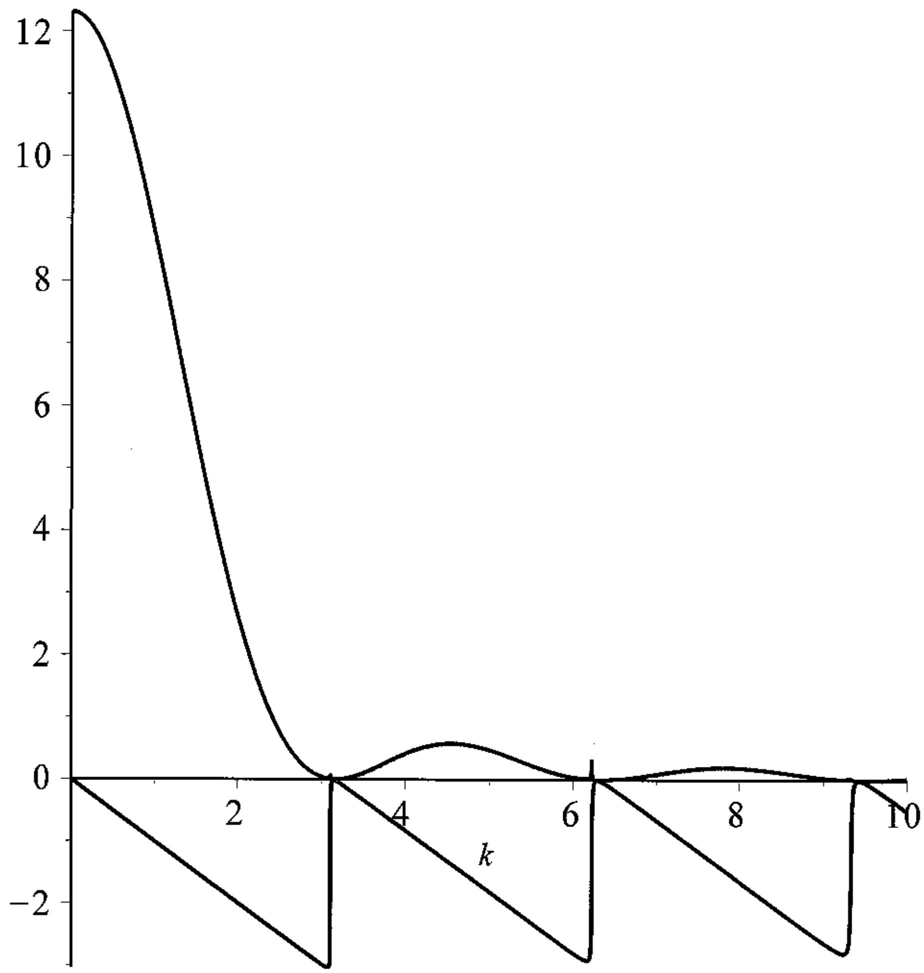
$$\boxed{\delta_0 = \arg\left(1 + \frac{\gamma R}{kR}\sin kR\, e^{-ikR}\right)}$$

$k \to 0 \quad \delta_0 \to 0 \quad ; \quad k \to \infty \quad \delta_0 \to 0$

```
> f1:=(g,k)->1+g/k*sin(k)*exp(-I*k);
```

$$f1 := (g,k) \rightarrow 1 + \frac{g \sin(k)\, e^{-I k}}{k} \quad (1)$$

```
> plot([argument(f1(100,k)),4*Pi*sin(argument(f1(100,k)))^2/k^2],k=0.
  .10);
```

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> plot([argument(f1(100,k)),4*Pi*sin(argument(f1(100,k)))^2/k^2],k=3.
  .3.5);
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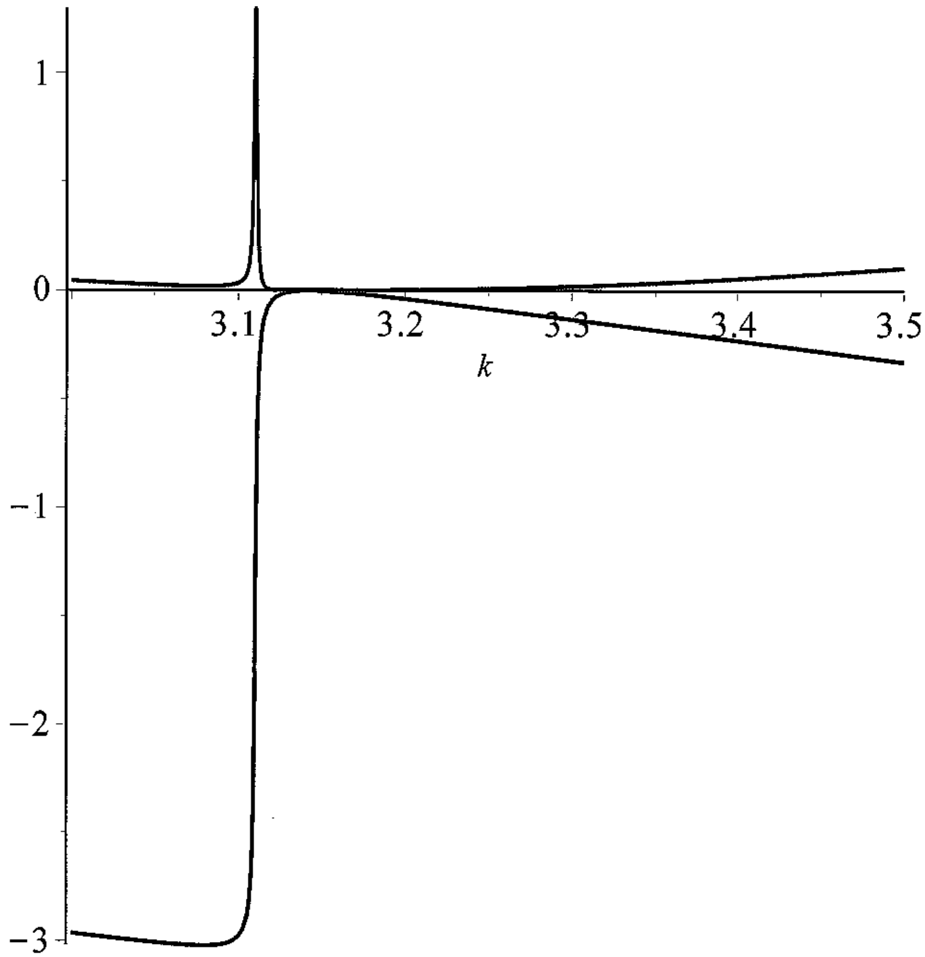
1
0
−1
−2
−3
3.1
3.2
3.3
3.4
3.5
k