```
> ## Consider the Yukawa potential and try to find the ground state.
> ## As we saw, the minimal energy is
> Emin:=(beta0,beta)->(1/2/beta^2-4*beta0/beta/(beta+2)^2);
```

$$Emin := (\beta 0,\, \beta) \to \frac{1}{2}\,\frac{1}{\beta^2} - \frac{4\,\beta 0}{\beta\,(\beta+2)^2}$$

```
> # where beta is determined from minimizing Emin which gives
> betaeq:=beta->(beta+2)^3-8*beta*beta0*(1+3/2*beta);
```

$$betaeq := \beta \to (\beta+2)^3 - 8\,\beta\,\beta 0\,(1+\frac{3}{2}\,\beta)$$

```
> # for example consider  beta0=10
```

$$S1 := 8 + \beta^3 + (6 - 12\,\beta 0)\,\beta^2 + (-8\,\beta 0 + 12)\,\beta$$

```
> res:=[evalf(solve(subs(beta0=10,betaeq(beta)),beta))];
```

$res := [114.5927963 - 0.2\,10^{-9}\,I,\ -0.6934677818 - 0.3464101616\,10^{-7}\,I,$
$0.1006714618 + 0.3464101616\,10^{-7}\,I]$

```
> [Emin(10,res[1]),Emin(10,res[2]),Emin(10,res[3])];
```

$[0.00001239847096 + 0.1\,10^{-24}\,I,\ 34.83013199 + 0.66\,10^{-14}\,I,\ -40.70514436 - 0.56\,10^{-12}\,I$
$]$

```
> # the third one is negative indicating a bound state with
> E=-40.70514436 (at least from variationl method)
> # Let's compare with numerical solution of the Schroedinger
> equation
> Eq1:=(beta0,E)->-1/2*(diff(psi(u),u$2)+2/u*diff(psi(u),u))-beta0*exp(
> -u)/u*psi(u)-E*psi(u);
```

$$Eq1 := (\beta 0,\, E) \to -\frac{1}{2}\,(\tfrac{d^2}{du^2}\,\psi(u)) - \frac{\frac{d}{du}\,\psi(u)}{u} - \frac{\beta 0\,e^{(-u)}\,\psi(u)}{u} - E\,\psi(u)$$

```
> dsolve({Eq1(10,-0.1),psi(0)=1,D(psi)(0)=0
> },numeric,output=listprocedure);
```

```
Error, (in dsolve/numeric/checksing) ode system has a removable
singularity at u=0. Initial data is restricted to {psi(u) =
-.10000000000000*diff(psi(u),u)}
```

```
> # OK, we need to redefine the differential equation
> exp(beta0*u)*subs(psi(u)=exp(-beta0*u)*chi(u),Eq1(beta0,E));
```

$$e^{(\beta 0\,u)}\,\left(-\frac{1}{2}\,(\tfrac{\partial^2}{\partial u^2}\,(e^{(-\beta 0\,u)}\,\chi(u))) - \frac{\frac{\partial}{\partial u}\,(e^{(-\beta 0\,u)}\,\chi(u))}{u} - \frac{\beta 0\,e^{(-u)}\,e^{(-\beta 0\,u)}\,\chi(u)}{u} - E\,e^{(-\beta 0\,u)}\,\chi(u)\right)$$

```
> expand(%);
```

$$-\frac{1}{2}\,\beta 0^2\,\chi(u) + \beta 0\,(\tfrac{d}{du}\,\chi(u)) - \frac{1}{2}\,(\tfrac{d^2}{du^2}\,\chi(u)) + \frac{\beta 0\,\chi(u)}{u} - \frac{\frac{d}{du}\,\chi(u)}{u} - \frac{\beta 0\,\chi(u)}{e^u\,u} - E\,\chi(u)$$

```
> Eq2:=(beta0,E)->-1/2*beta0^2*chi(u)+beta0*diff(chi(u),u)-1/2*diff(chi
> (u),`$`(u,2))+1/u*beta0*chi(u)-1/u*diff(chi(u),u)-beta0/exp(u)/u*chi(u
> )-E*chi(u);
```

$Eq2 := (\beta 0, E) \rightarrow$

$$-\frac{1}{2}\beta 0^2\chi(u) + \beta 0\left(\frac{d}{du}\chi(u)\right) - \frac{1}{2}\left(\frac{d^2}{du^2}\chi(u)\right) + \frac{\beta 0\,\chi(u)}{u} - \frac{\frac{d}{du}\chi(u)}{u} - \frac{\beta 0\,\chi(u)}{e^u\,u} - E\,\chi(u)$$

```
> # OK now it's working
> sol1:=dsolve({Eq2(10,-40.705803),chi(0)=1,D(chi)(0)=0
> },numeric,output=listprocedure):
> chi1s:=subs(sol1,chi(u));
```

$$chi1s := \mathbf{proc}(u)\ \ldots\ \mathbf{end\ proc}$$

```
> # Here we need to explore the values of E and see where the sign
of
> the aymptotic function changes.
> # We look for a function with no zeros (nodes) since we want
the
> ground state.
> plot([exp(-10*u)*chi1s(u)],u=0..3);
```

Plot: Yukawa-pot01.eps

```
> #If the value of E is change in the last decimal place the function
> goes to plus infinity.
> #We find  E=-40.705803  showing that the variationl procedure
gave a
> quite accurate result.
> #To see why lt's plot the wave functions
> plot([exp(-u/.1006714618),exp(-10*u)*chi1s(u)],u=0..1
```

Plot: Yukawa-pot02.eps

```
>  # They cannot be distinguish within the precision of the plot.
It was
>  a good guess.
```