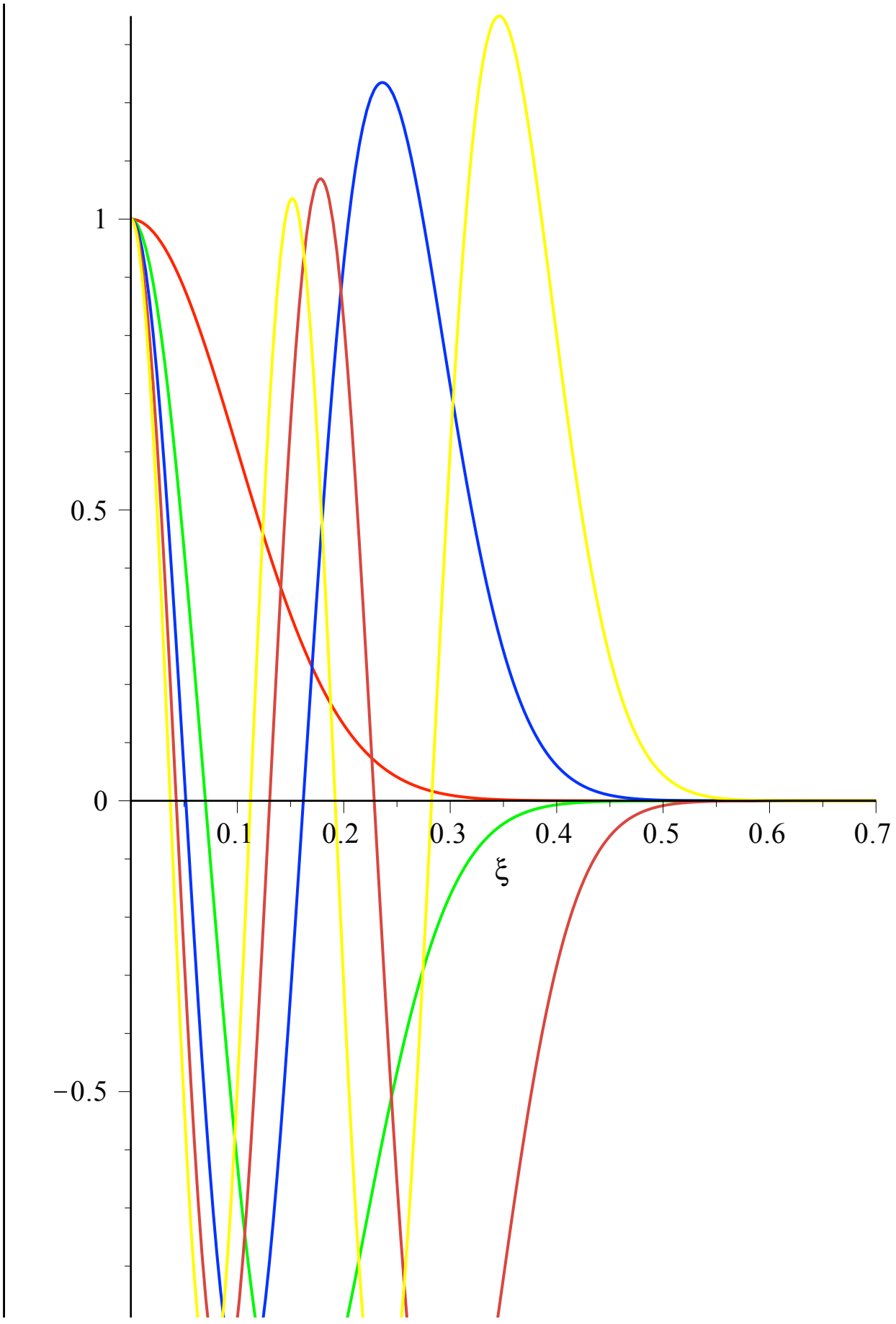
```
> with(plots): with(CurveFitting):
> # x^2 + x^4 potential

> eq1:=(Lambda,E)->-Lambda^2*diff(psi(xi),xi$2)+(xi^2+xi^4-E)*psi(xi)
  =0;
```

$$eq1 := (\Lambda, E) \rightarrow -\Lambda^2 \left(\frac{d^2}{d\xi^2} \psi(\xi) \right) + \left(\xi^2 + \xi^4 - E \right) \psi(\xi) = 0 \qquad (1)$$

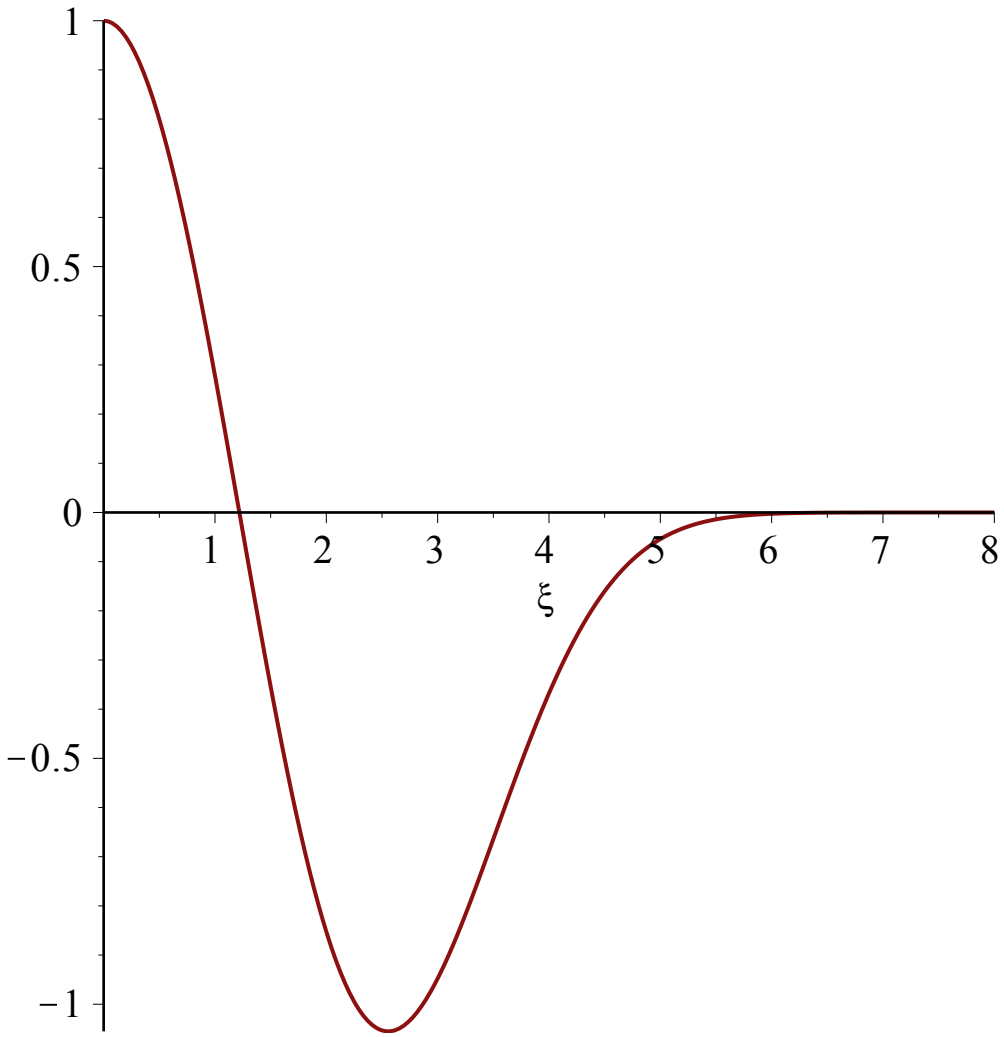
```
> # for different values of E.
> # we can write a procedure to plot the resulting solution

> plot_sol:=proc(Lambda,E,ran,colr) local sol1,psi1;
> sol1:=dsolve({eq1(Lambda,E),psi(0)=1,D(psi)(0)=0},numeric,output=
  listprocedure);
> psi1:=eval(psi(xi),sol1);
> plot(psi1(xi),ran,colr);
> end proc:
> find_sol:=proc(Lambda,E,ran) local sol1,psi1;
> sol1:=dsolve({eq1(Lambda,E),psi(0)=1,D(psi)(0)=0},numeric,output=
  listprocedure);
> psi1:=eval(psi(xi),sol1);
> return(psi1(ran)):
> end proc:

> colorF:=[red,green,blue,orange,yellow];
```

$$colorF := [red, green, blue, orange, yellow] \qquad (2)$$

```
> Lambda1:=0.01:
> for j from 1 to 5 do
>  gr[j]:=plot_sol(Lambda1,EvB[j],xi=0..0.7,color=colorF[j]);
> od:
> display(seq(gr[n],n=1..5));
```


1
0.5
0
−0.5
0.1
0.2
0.3
0.4
0.5
0.6
0.7
ξ

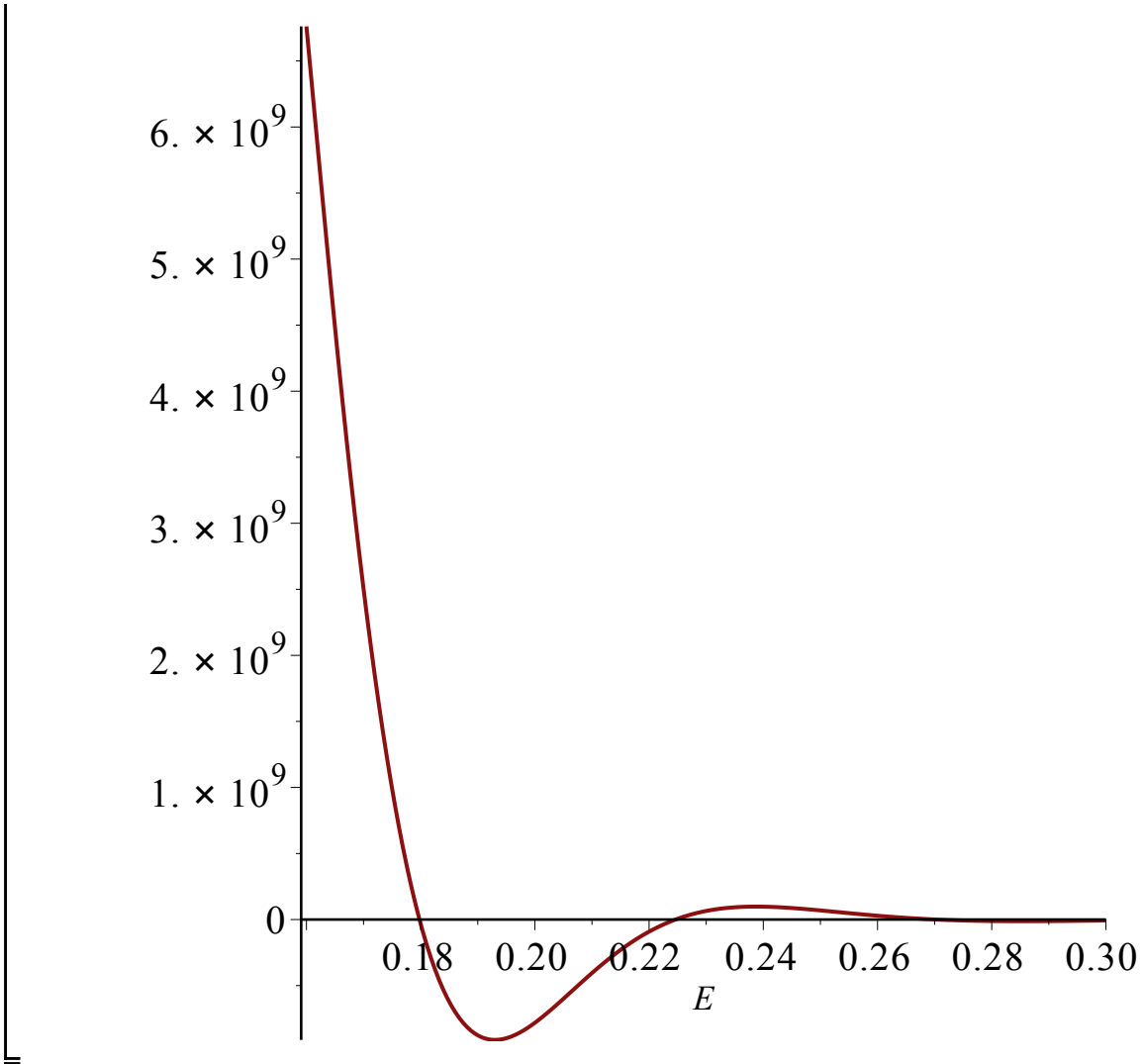
```
> plot_sol(10,166.3592213602,xi=0..8);
```




```
> Zf:=E->find_sol(0.01,E,0.8);
```

$$Zf := E \rightarrow find\_sol(0.0100000000, E, 0.8000000000) \quad \textbf{(3)}$$

```
> Ev:=[seq([0.001*i,Zf(0.001*i)],i=0..1000)]:
> Evf:=E->Spline(Ev,E):
> plot(Evf(E),E=0.16..0.3);
```




```
> EvA:=[fsolve(Evf(E),E=0.01),fsolve(Evf(E),E=0.05),fsolve(Evf(E),E=
  0.09),fsolve(Evf(E),E=0.14),fsolve(Evf(E),E=0.18)];
```

$$EvA := [0.0100737361, 0.0509393928, 0.0928948006, 0.1358671614, 0.1797951102] \quad \textbf{(4)}$$

```
> Fsolve:=proc(f,v,x1,x2)
> local xn,fn,fa,fb,i,xa,xb:
>  xa:=x1; xb:=x2;
> for i from 1 to 200 do
>  fa:=evalf(f(xa)-v): fb:=evalf(f(xb)-v):
>  xn:=(xa*fb-xb*fa)/(fb-fa): fn:=evalf(f(xn)-v):
>  if (abs(fn)<1e-20) then return xn: end if:
>  if (fa*fn)>0 then xa:=xn: else xb:=xn end if:
>  od:
> return xn:
> end proc:

> dE:=0.01:
> EvB:=[Fsolve(Zf,0,EvA[1]-dE,EvA[1]+dE),Fsolve(Zf,0,EvA[2]-dE,EvA[2]
  +dE),Fsolve(Zf,0,EvA[3]-dE,EvA[3]+dE),Fsolve(Zf,0,EvA[4]-dE,EvA[4]+
  dE),Fsolve(Zf,0,EvA[5]-dE,EvA[5]+dE)];
```

$$EvB := [0.0100737367, 0.0509393929, 0.0928948006, 0.1358671612, 0.1797951098] \quad \textbf{(5)}$$

```
> # ground state
> [0.1,0.1065285505], [0.2,0.2236585280], [0.01,0.0100737367], [10,
  24.4917398994];


> # second state

> [0.1,0.5747959459], [0.01,0.0509393929]; [10,166.3592213602];
```

$$[0.1000000000, 0.1065285505], [0.2000000000, 0.2236585280], [0.0100000000, 0.0100737367], [10, 24.4917398994]$$

$$[0.1000000000, 0.5747959459], [0.0100000000, 0.0509393929]$$

$$[10, 166.3592213602] \quad (6)$$

```
> #Lambda=10, spectrum
> [24.4917398994, 166.3592213602, 358.8517236745, 582.4130041754,
  830.0386925438];
```

$$[24.4917398994, 166.3592213602, 358.8517236745, 582.4130041754, 830.0386925438] \quad (7)$$

```
> # perturbation theory
> E0:=lambda->lambda+3/4*lambda^2;
```

$$E0 := \lambda \to \lambda + \frac{3}{4}\lambda^2 \quad (8)$$

```
> E2:=lambda->5*lambda+39/4*lambda^2;
```

$$E2 := \lambda \to 5\lambda + \frac{39}{4}\lambda^2 \quad (9)$$

```
> [0.1,E0(0.1)],[0.2,E0(0.2)],[0.01,E0(0.01)], [10,E0(10)];
```

$$[0.1000000000, 0.1075000000], [0.2000000000, 0.2300000000], [0.0100000000, 0.0100750000], [10, 85] \quad (10)$$

```
> [0.1,E2(0.1)],[0.2,E2(0.2)],[0.01,E2(0.01)], [10,E2(10)];
```

$$[0.1000000000, 0.5975000000], [0.2000000000, 1.3900000000], [0.0100000000, 0.0509750000], [10, 1025] \quad (11)$$

```
> #variational approach

> eq2:=Lambda->x^3*Lambda^2-x-3;
```

$$eq2 := \Lambda \to x^3 \Lambda^2 - x - 3 \quad (12)$$

```
> EV:=(Lambda,a)->Lambda^2/2*a+1/2/a+3/4/a^2;
```

$$EV := (\Lambda, a) \to \frac{1}{2}\Lambda^2 a + \frac{1}{2a} + \frac{3}{4a^2} \quad (13)$$

```
> fsolve(eq2(0.1));
```

$$-7.8648254120, -3.3893624160, 11.2541878300 \quad (14)$$

```
> EV(0.1,11.2541878300);
```

$$0.1066203617 \quad (15)$$

```
> fsolve(eq2(0.01));
```

$$-98.4648296800,\ -3.0027073160,\ 101.4675370000 \tag{16}$$

```
> EV(0.01,101.4675370000);
```

$$0.0100739075 \tag{17}$$

```
> fsolve(eq2(0.2));
```

$$6.1059834310 \tag{18}$$

```
> EV(0.2,6.1059834310);
```

$$0.2241229484 \tag{19}$$

```
> fsolve(eq2(10));
```

$$0.3214467950 \tag{20}$$

```
> EV(10,0.3214467950);
```

$$24.8862433100 \tag{21}$$

```
> #WKB
> xi12:=E->(-1+sqrt(1+4*E))/2;
```

$$\xi 12 := E \rightarrow -\frac{1}{2} + \frac{1}{2}\sqrt{1+4E} \tag{22}$$

```
> eta12:=E->(1+sqrt(1+4*E))/2;
```

$$\eta 12 := E \rightarrow \frac{1}{2} + \frac{1}{2}\sqrt{1+4E} \tag{23}$$

```
> BS:=E->2/3*sqrt(eta12(E)+xi12(E))*(eta12(E)*EllipticK(sqrt(xi12(E)/
  (xi12(E)+eta12(E))))+(xi12(E)-eta12(E))*EllipticE(sqrt(xi12(E)/
  (xi12(E)+eta12(E)))));
```

$$BS := E \rightarrow \frac{2}{3}\sqrt{\eta 12(E)+\xi 12(E)}\left(\eta 12(E)\,\mathrm{EllipticK}\left(\sqrt{\frac{\xi 12(E)}{\eta 12(E)+\xi 12(E)}}\right)+\left(\xi 12(E)-\eta 12(E)\right)\mathrm{EllipticE}\left(\sqrt{\frac{\xi 12(E)}{\eta 12(E)+\xi 12(E)}}\right)\right) \tag{24}$$

```
> fsolve(BS(E)=0.01*Pi/2);
> fsolve(BS(E)=0.1*Pi/2);
> fsolve(BS(E)=0.2*Pi/2);
> fsolve(BS(E)=10*Pi/2);
```

$$0.0100372380$$
$$0.1035155662$$
$$0.2133116175$$
$$20.6113956300 \tag{25}$$

```
> fsolve(BS(E)=10*(0+1/2)*Pi/2);
> fsolve(BS(E)=0.1*5*Pi/2);
> fsolve(BS(E)=0.2*5*Pi/2);
> fsolve(BS(E)=10*5*Pi/2);
```

$$0.0509064006$$
$$0.5730954235$$
$$1.2507687600$$

(26)

165.4582360000 **(26)**

```
> #Case Lambda=10, even states
> #Numerical
> [24.4917398994, 166.3592213602, 358.8517236745, 582.4130041754,
  830.0386925438];
> #WKB
> Lambda1:=10:[fsolve(BS(E)=Lambda1*(0+1/2)*Pi),fsolve(BS(E)=Lambda1*
  (2+1/2)*Pi),fsolve(BS(E)=Lambda1*(4+1/2)*Pi),fsolve(BS(E)=Lambda1*
  (6+1/2)*Pi),fsolve(BS(E)=Lambda1*(8+1/2)*Pi)];
> #Variational
> 24.8862433100;
> #perturbation
> [seq(2*(2*n+1/2)*Lambda+3/4*(2*(2*n)^2+2*(2*n)+1)*Lambda^2,n=0..4)]
  :
> subs(Lambda=10,%);
```

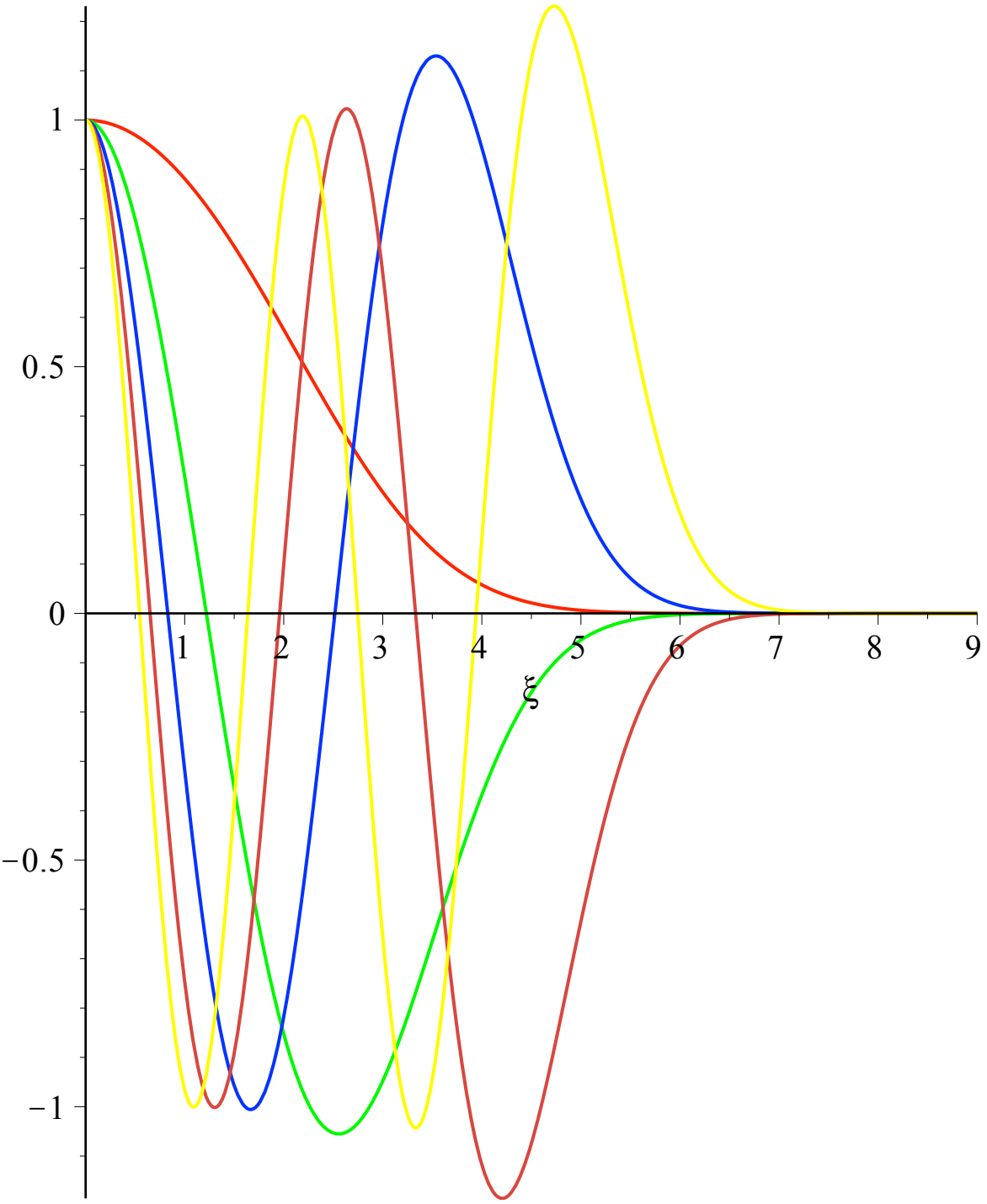
[24.4917398994, 166.3592213602, 358.8517236745, 582.4130041754, 830.0386925438]

[20.6113956300, 165.4582360000, 358.2416859000, 581.9347609000, 829.6385507000]

24.8862433100

[85, 1025, 3165, 6505, 11045] **(27)**

```
> #display(seq(gr[n],n=1..5));
```




```
> #Case Lambda=0.01, even states
> #Numerical
> [0.100737367e-1, 0.509393929e-1, 0.928948006e-1, .1358671612,
  .1797951098];
> #WKB
> Lambda1:=0.01:[fsolve(BS(E)=Lambda1*(0+1/2)*Pi),fsolve(BS(E)
  =Lambda1*(2+1/2)*Pi),fsolve(BS(E)=Lambda1*(4+1/2)*Pi),fsolve(BS(E)
  =Lambda1*(6+1/2)*Pi),fsolve(BS(E)=Lambda1*(8+1/2)*Pi)];
> #Variational
> 0.0100739075;
> #perturbation
```

```
> [seq(2*(2*n+1/2)*Lambda+3/4*(2*(2*n)^2+2*(2*n)+1)*Lambda^2,n=0..4)]
  :
> subs(Lambda=0.01,%);
```

$$[0.0100737367,\ 0.0509393929,\ 0.0928948006,\ 0.1358671612,\ 0.1797951098]$$

$$[0.0100372380,\ 0.0509064006,\ 0.0928646262,\ 0.1358393043,\ 0.1797691952]$$

$$0.0100739075$$

$$[0.0100750000,\ 0.0509750000,\ 0.0930750000,\ 0.1363750000,\ 0.1808750000] \qquad \textbf{(28)}$$

```
> display(seq(gr[n],n=1..5));
```