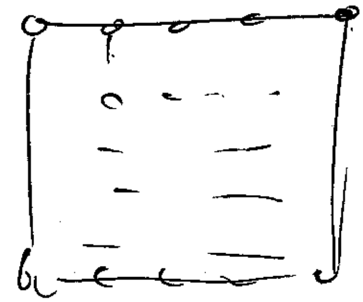
2d Ising model.



$\leftarrow \mu_\alpha = \{S_{j\alpha}\}$

$$Z = \sum_{\{\mu_\alpha\}} e^{-\beta \sum_\alpha E(\mu_\alpha, \mu_{\alpha+1}) - \sum_\alpha \beta E(\mu_\alpha)}$$

$$E_{\mu_\alpha} = -\sum_j S_j^\alpha S_{j+1}^\alpha$$

$$E_{\mu_\alpha \mu_{\alpha+1}} = -\sum_j S_j^\alpha S_j^{\alpha+1}$$

$$\langle \mu | P | \mu' \rangle = e^{-\beta E(\mu,\mu') - \beta E(\mu)}$$

$$Z = \mathrm{Tr}\, P^N = \lambda_1^N + \dots + \lambda_{2^N}^N$$

Assume $\lambda_j \geq 0$ $\qquad \lambda_{max}^N \leq Z \leq 2^N \lambda_{max}^N$

$$N \ln \lambda_{max} \leq \ln Z \leq N \ln 2 + N \ln \lambda_{max}$$

$$\ln \lambda_{max} \leq \frac{1}{N} \ln Z \leq \ln 2 + \ln \lambda_{max}$$

but we have N^2 sites $\qquad -\beta \frac{A}{N^2} = -\beta a$

$$\frac{1}{N}\ln\lambda_{max} \leq -\beta a \leq \frac{1}{N}\ln 2 + \frac{1}{N}\ln\lambda_{max}$$

$$N\to\infty \qquad -\beta a = \lim_{N\to\infty}\frac{1}{N}\ln\lambda_{max}.$$

$$\langle s_1 \ldots s_N | P | s_1' \ldots s_N'\rangle = \prod_{j=1}^{N} e^{\beta H s_j} e^{\beta s_j s_{j+1}} e^{\beta s_j s_j'}$$

$$\langle s_1 \ldots s_N | V_1' | s_1' \ldots s_N'\rangle = \prod_{j=1}^{N} e^{\beta s_j s_j'}$$

$$\langle s_1 \ldots s_N | V_2 | s_1' \ldots s_N'\rangle = \delta_{s_1 s_1'} \ldots \delta_{s_N s_N'} \prod_{j=1}^{N} e^{\beta s_j s_{j+1}}$$

$$\langle s_1 \ldots s_N | V_3 | s_1' \ldots s_N'\rangle = \delta_{s_1 s_1'} \ldots \delta_{s_N s_N'} \prod_{j=1}^{N} e^{\beta H s_j}$$

$$P = V_3 V_2 V_1'$$

$$V_1' = A\otimes \ldots \otimes A \qquad A_{s_j s_j'} = e^{\beta s_j s_j'} = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta}\end{pmatrix}$$

$$= e^{\beta}\mathbb{1} + e^{-\beta}\sigma_1$$

$$e^{\theta\sigma_1} = \operatorname{ch}\theta + \sigma_1\operatorname{sh}\theta = \operatorname{ch}\theta\,(1+\operatorname{th}\theta\,\sigma_1) = \cancel{\alpha}\, e^{\beta}(\mathbb{1} + e^{-2\beta}\sigma_1)$$

$$\operatorname{th}\theta = e^{-2\beta}$$

$$1-\mathrm{th}^2\theta = 1-\frac{\mathrm{sh}^2\theta}{\mathrm{ch}^2\theta} = \frac{1}{\mathrm{ch}^2\theta} = 1-e^{-4\beta}$$

$$\mathrm{ch}^2\theta = \frac{1}{1-e^{-4\beta}} = \frac{e^{2\beta}}{e^{2\beta}-e^{-2\beta}} = \frac{e^{2\beta}}{2\,\mathrm{sh}2\beta}$$

$$e^{\beta\sigma_1} = \frac{e^{\beta}}{\sqrt{2\,\mathrm{sh}2\beta}}\left(1+e^{-2\beta}\sigma_1\right) = \frac{A}{\sqrt{2\,\mathrm{sh}2\beta}}$$

$$A = \sqrt{2\,\mathrm{sh}2\beta}\; e^{\theta\sigma_1}$$

$$V_1' = (2\,\mathrm{sh}2\beta)^{N/2}\; e^{\theta\sigma_1}\otimes \text{——} \otimes e^{\theta\sigma_1}$$

$$\sigma_1^{\alpha} = \mathbb{1}\otimes - - \overset{\alpha}{\sigma_1} \otimes \text{—} \otimes \mathbb{1}$$

$$V_1' = (2\,\mathrm{sh}2\beta)^{N/2} \prod_{\alpha=1}^{N} e^{\theta\sigma_1^{\alpha}} = (2\,\mathrm{sh}2\beta)^{N/2}\, e^{\theta\sum_{\alpha=1}^{N}\sigma_1^{\alpha}}$$

Commute

$$V_1 = e^{\theta\sum_{\alpha=1}^{N}\sigma_1^{\alpha}} \qquad \mathrm{th}\,\theta = e^{-2\beta}$$

$$P = (2\,\mathrm{sh}2\beta)^{N/2}\, V_3 V_2 V_1$$

$$V_3 = A_3 \otimes \cdots \otimes A_3$$

$$A_3 = \begin{pmatrix} e^{\beta H} & 0 \\ 0 & e^{-\beta H} \end{pmatrix} = e^{\beta H \sigma_3}$$

$$\langle s_1 \cdots s_n | V_3 | s'_1 \cdots s'_n \rangle = A_{3\, s_1 s'_1} \cdots A_{3\, s_n s'_n} = e^{\beta H \Sigma \sigma_3^{(j)}} \checkmark$$

$$\langle s_1 \cdots s_n | V_2 | s'_1 \cdots s'_n \rangle = \delta_{s_1 s'_1} \cdots \delta_{s_n s'_n} \prod_{j=1}^{N} e^{\beta s_j s_{j+1}}$$

$$\prod_{\alpha=1}^{N} e^{\beta \sigma_3^{\alpha} \sigma_3^{\alpha+1}}$$

$$e^{\beta \sigma_3^{\alpha} \sigma_3^{\alpha+1}}$$

$$\sigma_3^{\alpha} \sigma_3^{\alpha+1} = 1 \otimes \cdots \otimes \overset{\alpha}{\sigma_3} \otimes \overset{\alpha+1}{\sigma_3} \otimes \cdots$$ ← diagonal.

$$\langle s_1 \cdots s_n | e^{\beta \sigma_3^{\alpha} \sigma_3^{\alpha+1}} | s'_1 \cdots s'_n \rangle = \delta_{s_1 s'_1} \cdots \delta_{s_n s'_n} \cdot e^{\beta s_i s_{i+1}}$$

$$\boxed{V_2 = \prod_{\alpha=1}^{N} e^{\beta \sigma_3^{\alpha} \sigma_3^{\alpha+1}}}$$

$$V_1 = e^{\theta \sum_{\alpha=1}^{N} \sigma_1^{\alpha}}$$

$$\operatorname{th} \theta = e^{-2\beta}.$$

If $B=0$

$$\boxed{P = (2 \operatorname{sh} 2\beta)^{N/2} V_3 V_2 V_1}$$

$$V_3 = e^{\beta H \Sigma \sigma_3^{(\alpha)}}$$

Fermionic representation

$|0\,1\,1\,1\,0 - - 1\rangle$

$c_j^\dagger |0\rangle = |1\rangle$ $\qquad \{c_i, c_j^\dagger\} = \delta_{ij}$

$c_j |1\rangle = |0\rangle$ $\qquad$ anti-commute at different sites.

1-site

$$c^\dagger = \begin{array}{c|cc} & |0\rangle & |1\rangle \\ \hline |0\rangle & 0 & 0 \\ |1\rangle & 1 & 0 \end{array} \qquad c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

but at different sites they would ~~and~~ commute

take $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\qquad \left(\frac{1-\sigma_3}{2} = \text{fermion number}\right)$

$$\sigma_3 \cdot c^\dagger + c^\dagger \cdot \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$$

$$\sigma_3 \cdot c + c\,\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 0$$

$$C_j = \sigma_3 \otimes - - - \otimes \sigma_3 \otimes \overset{j}{C} \otimes \mathbb{1} - \otimes \mathbb{1}$$

$$\bar{C}_j = \sigma_3 \otimes - - \otimes \sigma_3 \otimes c^\dagger \otimes \mathbb{1} - - \otimes \mathbb{1}.$$

acting on each site.

$$V_1 = e^{\theta \sum_{\alpha=1}^{N} \sigma_1^\alpha} \xrightarrow[\text{(change of basis)}]{\text{rotation}} e^{\theta \sum_{\alpha=1}^{N} \sigma_3^\alpha}$$

$$V_2 = e^{\beta \sum_\alpha \sigma_3^\alpha \sigma_3^{\alpha+1}} \longrightarrow e^{\beta \sum_{\alpha=1}^{N-1} \sigma_1^\alpha \sigma_1^{\alpha+1}}$$

consider open b.c. instead of periodic.

$c c^{\dagger} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad c^{\dagger} c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$-c^{\dagger} c + c c^{\dagger} = \sigma_3$$

$$\mathbb{C}_j \mathbb{C}_j^{\dagger} - \mathbb{C}_j^{\dagger} \mathbb{C}_j = \mathbb{1} \otimes \cdots \otimes \overset{j}{\sigma_3} \otimes \mathbb{1} \cdots = \sigma_3^{(j)}$$

$$V_1 = e^{\theta \sum_j (\mathbb{C}_j^{\dagger} \mathbb{C}_j - \mathbb{C}_j \mathbb{C}_j^{\dagger})}$$

$i(c^{\dagger} - c) = \sigma_2 \qquad c + c^{\dagger} = \sigma_1 \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\mathbb{C}_{j+1} + \mathbb{C}_{j+1}^{\dagger} = \sigma_3 \otimes \cdots \otimes \overset{j}{\sigma_3} \otimes \overset{j+1}{\sigma_1} \otimes \mathbb{1} \cdots \mathbb{1}$$

$$i(\mathbb{C}_j^{\dagger} - \mathbb{C}_j) = \sigma_3 \otimes \cdots \otimes \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \cdots \mathbb{1}$$

$$i(\mathbb{C}_j^{\dagger} - \mathbb{C}_j)(\mathbb{C}_{j+1} + \mathbb{C}_{j+1}^{\dagger}) = \mathbb{1} \otimes \cdots \otimes \overset{j-1}{\mathbb{1}} \otimes \overset{j}{i\sigma_1} \otimes \overset{j+1}{\sigma_1} \otimes \overset{j+2}{\mathbb{1}} \cdots \otimes \mathbb{1}$$

$$= i\, \sigma_1^{(j)} . \sigma_1^{(j+1)}$$

$$V_2 = e^{+\beta \sum_{j=1}^{N-1} (\mathbb{C}_j^{\dagger} - \mathbb{C}_j)(\mathbb{C}_{j+1} + \mathbb{C}_{j+1}^{\dagger}) + \beta(\otimes \sigma_3)(c_N^{\dagger} - c_N)(c_1 + c_1^{\dagger})}$$

$$\prod_{j=1}^{N} S_j = (-1)^{N-N_F}$$

$\sigma_2 . \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1$ ✓

$\sigma_3 \otimes \sigma_3 \cdots \sigma_3 \otimes \sigma_2$

$\sigma_2 \otimes \mathbb{1} \cdots \otimes \mathbb{1}$

$-i\sigma_2 \otimes \sigma_3 \cdots \sigma_3 \otimes \sigma_2$

$\sigma_3 \otimes \sigma_3 \cdots \times \sigma_3$

$\sigma_3 . \sigma_1 = -i\sigma_2$

$(+i) i\sigma_1 \otimes \mathbb{1} \cdots \otimes \mathbb{1} \otimes i\sigma_1$

$= i\, \sigma_1 \otimes \mathbb{1} \cdots \otimes \mathbb{1} \otimes \sigma_1$

Fermionic representation

1 fermion in each site

$$P = (2\sinh 2\beta)^{N/2} \; V_2 V_1$$

$$V_2 = e^{\cancel{\theta}\beta \sum_j (c_j - c_j^\dagger)(c_{j+1} + c_{j+1}^\dagger)}$$

$$V_1 = e^{\theta \sum_j (c_j^\dagger c_j - c_j c_j^\dagger)}$$

Define $\{c_p, c_{p'}^\dagger\} = \delta(p-p')$ $\qquad -\pi \le p \le \pi$

$$c_j = \int_{-\pi}^{\pi} \frac{dp}{\sqrt{2\pi}} e^{ipj} c_p \quad ; \quad c_j^\dagger = \int_{-\pi}^{\pi} \frac{dp}{\sqrt{2\pi}} e^{-ipj} c_p^\dagger$$

$$\{c_j, c_{j'}^\dagger\} = \int_{-\pi}^{\pi} \frac{dp\,dp'}{2\pi} e^{ipj - ip'j'} \delta(p-p') =$$

$$= \int_{-\pi}^{\pi} \frac{dp}{2\pi} e^{ip(j-j')} = \begin{cases} j = j' & \to 1 \ \checkmark \\ j \neq j' & \left. \dfrac{e^{ip(j-j')}}{2\pi i (j-j')} \right|_{-\pi}^{\pi} = 0 \quad (j-j' \text{ integer}) \end{cases}$$

$N \int_{-\pi}^{\pi} \frac{dp}{2\pi} = 1$ density of states.

$$\boxed{\text{Also} \quad c_j = \sum_{n=0}^{N-1} \frac{e^{\frac{2\pi i}{N} nj}}{\sqrt{N}} c_n}$$

$$\sum_{j=-\infty}^{\infty} c_j^\dagger c_j - c_j c_j^\dagger = \int_{-\pi}^{\pi} \frac{dp\,dp'}{2\pi} \underbrace{\sum_j e^{-ipj+ip'j}}_{2\pi\,\delta_{\text{periodic}}(p-p')} (c_p^\dagger c_{p'} - c_p c_{p'}^\dagger)$$ (8)

$$= \int_{-\pi}^{\pi} dp\, (c_p^\dagger c_p - c_p c_p^\dagger)$$

$$\sum_j (c_j - c_j^\dagger)(c_{j+1} + c_{j+1}^\dagger) =$$

$$= \sum_j \int_{-\pi}^{\pi} \frac{dp\,dp'}{2\pi} \left(e^{ipj} c_p - e^{-ipj} c_p^\dagger\right)\left(e^{ip'(j+1)} c_{p'} + e^{-ip'(j+1)} c_{p'}^\dagger\right)$$

$$= \int_{-\pi}^{\pi} dp \left(e^{-ip} c_p c_{-p} + e^{-ip} c_p c_p^\dagger - e^{+ip} c_p^\dagger c_{+p} - e^{ip} c_p^\dagger c_{-p}^\dagger\right)$$

We can look at each p separately. $(p, -p)$

$|0\,0\rangle$ $|0\,1\rangle$ $|1\,0\rangle$ $|1\,1\rangle$
$p\ -p$ $\quad p\ -p$ $\quad p\ -p$ $\quad p\ -p$

fermionic parity.

But each term conserves even or odd # of fermions
since it conserves # of odd or subtracts 2.

(9)

$$V_1 = e^{\theta \int_0^{\pi} dp \, (c_p^\dagger c_p - c_p c_p^\dagger + c_{-p}^\dagger c_{-p} - c_{-p} c_{-p}^\dagger)}$$

$$V_2 = e^{\beta \int_0^{\pi} dp \, (e^{-ip} c_p c_{-p} + e^{ip} c_{-p} c_p + e^{-ip} c_p c_p^\dagger + e^{ip} c_{-p} c_{-p}^\dagger - e^{ip} c_p^\dagger c_p - e^{-ip} c_{-p}^\dagger c_{-p} - e^{ip} c_p^\dagger c_{-p}^\dagger - e^{-ip} c_{-p}^\dagger c_p^\dagger)}$$

Consider basis $|00\rangle \; |11\rangle \; |01\rangle \; |10\rangle$ for $p, -p$ states.

$$\begin{array}{c|cccc} & |00\rangle & |11\rangle & |01\rangle & |10\rangle \\ \hline \langle 00| & & & & \\ \langle 11| & & & & \\ \langle 01| & & & & \\ \langle 10| & & & & \end{array}$$

$$c_p = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix} \quad c_p^\dagger = \begin{pmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix} \quad c_{-p} = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&-1&0&0 \end{pmatrix} \quad c_{-p}^\dagger = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&-1 \\ 1&0&0&0 \\ 0&0&0&0 \end{pmatrix}$$

$$V_1 = e^{-2\theta \sum_p \begin{pmatrix} 1&0&0&0 \\ 0&-1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}} \qquad V_2 = e^{2\beta \sum_p \begin{pmatrix} cp & isp & 0 & 0 \\ -isp & -cp & 0 & 0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}}$$

looking at (2×2) part (the other 2 eigenvalues are 0).

$$V_1 = e^{-2\theta \sum_p \sigma_3} \qquad V_2 = e^{2\beta \sum_p (cp\,\sigma_3 - sp\,\sigma_2)}$$

$$e^{-2\theta\sigma_3} e^{2\beta(cp\,\sigma_3 - sp\,\sigma_2)} = (ch 2\theta - sh 2\theta\, \sigma_3) \cdot (ch 2\beta + sh 2\beta (cp\,\sigma_3 - sp\,\sigma_2))$$

$$= \big[ch 2\theta\, ch 2\beta + ch 2\theta\, sh 2\beta (cp\,\sigma_3 - sp\,\sigma_2) - sh 2\theta\, ch 2\beta\, \sigma_3 - sh 2\theta\, sh 2\beta\, cp + sh 2\theta\, sh 2\beta\, sp(-i\sigma_1) \big] = ch\mu + sh\mu \underbrace{(\hat{n}\cdot\sigma)}_{\text{squares to 1, eigenvalues } \pm 1}$$

$$e^{-2\theta\sigma_3}\, e^{2\beta(cp\,\sigma_3 - sp\,\sigma_2)} = e^{\mu(\hat{n}\cdot\sigma)}$$

eigenvalues $+\mu,\ -\mu,\ \sigma,\ 0$

symmetric $\beta \leftrightarrow \tilde{\beta}$.

$$\sum_p \mu(p) = \frac{N}{2\pi}\int_0^{\pi} dp\ \operatorname{arccosh}\left(\operatorname{ch}2\theta\,\operatorname{ch}2\beta - cp\right)$$

if $\mu(p) = 1$ we get N ✓.

$$\operatorname{sh}2\theta\,\operatorname{sh}2\beta = 1 \qquad \operatorname{ch}2\theta = \frac{\operatorname{ch}2\beta}{\operatorname{sh}2\beta}$$

$$\operatorname{Tr}P^N = e^{\frac{N^2}{2\pi}\int_0^{\pi} dp\ \operatorname{arccosh}\left(\frac{\operatorname{ch}^2 2\beta}{\operatorname{sh}2\beta} - cp\right)} \cdot (2\operatorname{sh}2\beta)^{N^2/2}$$

$$e^{-\beta A} = \operatorname{Tr}P^N$$

$$-\beta a = -\beta\frac{A}{N^2} = \frac{1}{2}\ln(2\operatorname{sh}2\beta) + \frac{1}{2\pi}\int_0^{\pi} dp\ \operatorname{arccosh}\left(\frac{\operatorname{ch}^2 2\beta}{\operatorname{sh}2\beta} - \cos p\right)$$

$$\beta a_{Ising} = -\frac{1}{2}\ln(2\operatorname{sh}2\beta) - \frac{1}{2\pi}\int_0^{\pi} dp\ \operatorname{arccosh}\left(\frac{\operatorname{ch}^2 2\beta}{\operatorname{sh}2\beta} - \cos p\right)$$

agree ✓

$$\beta a_{Ising} = -\ln(2\operatorname{ch}2\beta) - \frac{1}{2\pi}\int_0^{\pi} d\phi\ \ln\left(\frac{1}{2}\left(1 + \sqrt{1 - K^2 s^2\phi}\right)\right)$$

book

$$K = \frac{2\operatorname{sh}2\beta}{\operatorname{ch}^2 2\beta}$$