HwK III

1

a) We have to compute $\langle (\Delta n_p)^2 \rangle$

n_p in the grand canonical are independent variables.

That is:

$$\Xi = \sum_{\{n_p\}} e^{-\beta \sum_p \varepsilon_p n_p + \beta\mu \sum_p n_p}$$

$$= \prod_p \Xi_p \quad \text{where} \quad \Xi_p = \sum_{n_p} e^{-\beta(\varepsilon_p - \mu) n_p}$$

For bosons $\Xi_p^B = \sum_{n_p=0}^{\infty} e^{-\beta(\varepsilon_p-\mu)n_p} = \dfrac{1}{1-e^{-\beta(\varepsilon_p-\mu)}}$

For fermions $\Xi_p^F = \sum_{n_p=0}^{1} e^{-\beta(\varepsilon_p-\mu)n_p} = 1 + e^{-\beta(\varepsilon_p-\mu)}$

and

$$\langle n_p^j \rangle = \frac{1}{\Xi} \sum_{\{n_{p'}\}} n_p^j \, e^{-\beta \sum_{p'} \varepsilon_{p'} n_{p'} + \beta\mu \sum_{p'} n_{p'}}$$

$$= \frac{1}{\Xi_p} \sum_{n_p} n_p^j \, e^{-\beta(\varepsilon_p-\mu)n_p} = $$

all $p' \neq p$ cancel

$$= \frac{1}{\Xi_p} \frac{1}{\beta^j} \frac{\partial^j}{\partial \mu^j} \Xi_p$$

So we can consider each n_p separately.

$$\langle n_p \rangle = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln \mathcal{Z}_p$$

$$\langle n_p^2\rangle - \langle n_p\rangle^2 = \frac{1}{\mathcal{Z}_p}\frac{1}{\beta^2}\frac{\partial^2}{\partial\mu^2}\mathcal{Z}_p - \frac{1}{\beta^2}\frac{1}{\mathcal{Z}_p^2}\left(\frac{\partial \mathcal{Z}_p}{\partial\mu}\right)^2$$

$$= \frac{1}{\beta^2}\frac{\partial^2}{\partial\mu^2}\ln\mathcal{Z}_p$$

$$\boxed{\langle(\Delta n_p)^2\rangle_{B,F}} = \langle n_p^2\rangle_{B,F} - \langle n_p\rangle_{B,F}^2 = \frac{1}{\beta^2}\frac{\partial^2}{\partial\mu^2}\ln\mathcal{Z}_p^{B,F}$$

$$= \frac{1}{\beta^2}(\mp)\frac{\partial^2}{\partial\mu^2}\ln\left(1\mp e^{-\beta(\varepsilon_p-\mu)}\right)$$

$$= \frac{\cancel{(\mp)}}{\beta^{\cancel{2}}}\frac{\partial}{\partial\mu}\frac{\cancel{(\mp)}\beta e^{-\beta(\varepsilon_p-\mu)}}{1\mp e^{-\beta(\varepsilon_p-\mu)}} = \frac{1}{\beta}\frac{\partial}{\partial\mu}\frac{1}{e^{\beta(\varepsilon_p-\mu)}\mp 1}$$

$$= \cancel{-}\frac{1}{\cancel{\beta}}\frac{\cancel{(-)}\cancel{\beta} e^{\beta(\varepsilon_p-\mu)}}{\left(e^{\beta(\varepsilon_p-\mu)}\mp 1\right)^2} = \boxed{\frac{e^{\beta(\varepsilon_p-\mu)}}{\left(e^{\beta(\varepsilon_p-\mu)}\mp 1\right)^2}}$$

Notice $\langle n_p\rangle_{B,F} = \dfrac{1}{e^{\beta(\varepsilon_p-\mu)}\mp 1}$

b

$$n(p,\Delta p) = \frac{V}{h^3} \int_{p<|\vec{p}|<p+\Delta p} d^3p \; n_p$$

this is a sum of indep. variables.

$$\boxed{\langle (\Delta n(p,\Delta p))^2 \rangle} = \frac{V}{h^3} \int_{p<|\vec{p}|<p+\Delta p} d^3p \; \langle (\Delta n_p)^2 \rangle$$

(or use direct calculation)

$\Delta p \ll |\vec{p}|$

$$\simeq \boxed{\frac{V}{h^3} (\Delta p)^3 \frac{e^{\beta(\varepsilon_p-\mu)}}{(e^{\beta(\varepsilon_p-\mu)} \mp 1)^2}}$$

c) $$\frac{\sqrt{\langle (\Delta n_p)^2 \rangle_{B,F}}}{\langle n_p \rangle_{B,F}} = \frac{e^{\beta(\varepsilon_p-\mu)}}{e^{\beta(\varepsilon_p-\mu)} \mp 1}$$

it is <u>not</u> small (generically)

$$\frac{\sqrt{\langle (\Delta n(p,\Delta p))^2 \rangle}}{\langle n(p,\Delta p) \rangle} = \frac{\frac{\sqrt{V}\,(\Delta p)^{3/2}}{h^{3/2}} \frac{e^{\beta(\varepsilon_p-\mu)}}{(e^{\beta(\varepsilon_p-\mu)} \mp 1)^{\cancel{2}}}}{\frac{V}{h^3} (\delta p)^3 \frac{1}{\cancel{e^{\beta(\varepsilon_p-\mu)} \mp 1}}}$$

$$= \frac{h^{3/2}}{\sqrt{V}\,(\delta p)^{3/2}} \frac{e^{\beta(\varepsilon_p-\mu)}}{e^{\beta(\varepsilon_p-\mu)} \mp 1} \xrightarrow[V\to\infty]{} 0$$

vanishes in thermodynamic limit!

②

For one harmonic oscillator. Canonical partition function is

$$Z = \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\hbar\omega_0} = \frac{e^{-\beta\frac{\hbar\omega_0}{2}}}{1-e^{-\beta\hbar\omega_0}}$$

$$= \frac{1}{2\,\mathrm{sh}\left(\beta\frac{\hbar\omega_0}{2}\right)}$$

For N harmonic osc.

$$Z_N = \frac{1}{2^N}\,\frac{1}{\mathrm{sh}^N\left(\beta\frac{\hbar\omega_0}{2}\right)}$$

grand canonical partition function

$$\Xi = \sum_N \frac{Z_N}{N!}\,e^{\beta\mu N} = \exp\left(\frac{e^{\beta\mu}}{2\,\mathrm{sh}\left(\beta\frac{\hbar\omega_0}{2}\right)}\right)$$

if identical

if not identical

$$\Xi = \sum_N Z_N\,e^{\beta\mu N} = \frac{1}{1-\frac{e^{\beta\mu}}{2\,\mathrm{sh}\left(\beta\frac{\hbar\omega_0}{2}\right)}}$$

For grand canonical we can do.

$$\Xi = \sum_\nu e^{-\beta E_\nu + \beta\mu N_\nu}$$

$$= \sum_{E,N} \underbrace{\Omega(E,N)}_{\#\text{ of states}} e^{-\beta E + \beta\mu N} \qquad S = k_B \ln \Omega$$

$$= \sum_{E,N} e^{\frac{S}{k_B} - \beta E + \beta\mu N}$$

In the thermodynamic limit, assuming small fluctuation we get

$$k_B \ln \Xi = S - \frac{E}{T} + \frac{\mu}{T} N$$

identical & non-identical differ by $N!$ in the number of states. which only affects the entropy. and free energy.

Consider case of identical

$$\ln \Xi = \frac{e^{\beta\mu}}{2\,\mathrm{sh}\left(\beta\frac{\hbar\omega_0}{2}\right)}$$

$$N = \frac{\partial \ln \Xi}{\beta\,\partial\mu} = \ln \Xi = \frac{e^{\beta\mu}}{2\,\mathrm{sh}\left(\beta\frac{\hbar\omega_0}{2}\right)}$$

$$\boxed{\mu = k_B T \ln N + k_B T \ln\left(2\,\mathrm{sh}\left(\beta\frac{\hbar\omega_0}{2}\right)\right)}$$

$$E = -\left.\frac{\partial \ln \Xi}{\partial\beta}\right|_{\beta\mu \text{ fixed}} = +\frac{e^{\beta\mu}}{2\,\mathrm{sh}^2\left(\beta\frac{\hbar\omega_0}{2}\right)}\,\mathrm{ch}\left(\beta\frac{\hbar\omega_0}{2}\right)\frac{\hbar\omega_0}{2}$$

$$\boxed{E = N \coth\left(\beta\frac{\hbar\omega_0}{2}\right)\frac{\hbar\omega_0}{2}}$$

$$S = k_B \ln \Xi + \frac{E}{T} - \frac{N\mu}{T}$$

$$= k_B N + \frac{E}{T} - N k_B \ln N - k_B N \ln\left(2\,\mathrm{sh}\left(\beta\frac{\hbar\omega_0}{2}\right)\right)$$

$$\boxed{S = \frac{N}{T}\coth\left(\beta\frac{\hbar\omega_0}{2}\right)\frac{\hbar\omega_0}{2} - k_B N \ln\left(2\,\mathrm{sh}\left(\beta\frac{\hbar\omega_0}{2}\right)\right) - k_B\,\underbrace{(N\ln N - N)}_{\simeq \ln N!}}$$

because of identical.

$$E - TS = A = \cancel{E} + k_B N T \ln\left(2\,\mathrm{sh}\left(\beta\frac{\hbar\omega_0}{2}\right)\right) - \cancel{E}$$

$$+ k_B T\,(N\ln N - N)$$

$$\boxed{A = k_B N T \ln\left(2\,\mathrm{sh}\left(\beta\frac{\hbar\omega_0}{2}\right)\right) + \underbrace{k_B T\,(N\ln N - N)}_{\text{for identical.}}}$$

$$C_v = \left.\frac{\partial E}{\partial T}\right)_N = N\frac{\hbar\omega_0}{2}\left(+\frac{1}{\mathrm{sh}^2\left(\beta\frac{\hbar\omega_0}{2}\right)}\right)\frac{\hbar\omega_0}{2}\left(+\frac{1}{k_B T^2}\right)$$

$$\boxed{C_v = \frac{N}{k_B}\left(\frac{\hbar\omega_0}{2}\right)^2\frac{1}{T^2}\,\frac{1}{\mathrm{sh}^2\left(\beta\frac{\hbar\omega_0}{2}\right)}}$$

P3

(a) Here they are not-identical.

We use canonical.

$$Z = \sum_r e^{-\beta E} = e^{-\beta A} \quad \begin{cases} Z_1 = e^{-\beta A_1} \\ Z_2 = e^{-\beta A_2} \end{cases}$$

$$A_1 = E - TS = E_1 + K_B N T \ln\left(2 \sinh\left(\beta \frac{\hbar\omega_1}{2}\right)\right)$$

$$A_2 = E_2 + K_B N T \ln\left(2 \sinh\left(\beta \frac{\hbar\omega_2}{2}\right)\right)$$

(b) low temp. 1 is preferred since $E_1 < E_2$

at large temp

$\omega_1 > \omega_2 \Rightarrow$ A_2 has smaller free energy

$\Rightarrow$ is preferred.

$\Rightarrow$ phase transition.

(c) Phase transition is when $A_1 = A_2$

$$E_1 + K_B N T_c \ln\left(2 \sinh\left(\beta_c \frac{\hbar\omega_1}{2}\right)\right) =$$

$$= E_2 + K_B N T_c \ln\left(2 \sinh\left(\beta_c \frac{\hbar\omega_2}{2}\right)\right)$$

$$|E_2 - E_1| = k_B N T_c \ln\left(\frac{\operatorname{sh}(\beta_c \frac{\hbar\omega_1}{2})}{\operatorname{sh}(\beta_c \frac{\hbar\omega_2}{2})}\right)$$

Non trivial eq. for T_c.

if $T \to \infty$ that is $\beta\hbar\omega_{1,2} \ll 1$.

and $\operatorname{sh} x \simeq x$

$$E_2 - E_1 = k_B N T_c \ln\left(\frac{\omega_1}{\omega_2}\right)$$

$$\boxed{T_c = \frac{E_2 - E_1}{k_B N \ln(\omega_1/\omega_2)}}$$

P4

$$\epsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{g v}\right)^{2/3} \qquad v = \frac{V}{N}$$

$g = 2s+1 = 2$ $s = 1/2$

for e^-, ~~Nucleons~~ & He^3

nucleons $g = 4$ spin 1/2 isospin 1/2

Take copper. $1e^-$ in last shell
$\rightarrow 1\, e^-$ per atom.

density ~ $9\,g/cm^3$

Atomic weight $\rightarrow 63.5 \times 1.66\times 10^{-24}\,g$.

$$v = 1.2\times 10^{-23}\,\frac{1}{cm^3}$$

$$\epsilon_F = \frac{\hbar^2}{2m} \times 1.85\times 10^{16}\,\frac{1}{cm^2}$$

$$\frac{\hbar^2 c^2}{2mc^2} = \frac{(197\,MeV\,fm)^2}{2\times 0.511\,MeV} = 3.8\times 10^4\,MeV\,fm^2$$

$$\epsilon_F = 3.8\times 10^4 \times 10^6\,eV \times 10^{-30}\,\cancel{m^2} \times \frac{1.85\times 10^{16}}{10^{-4}\,\cancel{m^2}}$$

$$= 7\times 10^{10-30+16+4}\,eV \simeq 7\,eV \quad (\text{agrees } \checkmark)$$

Heavy nucleus

Uranium 238

protons 92

$$n_0 \simeq 0.15 \frac{\text{nucleons}}{\text{fermi}^3}$$

$$\epsilon_F = \frac{\hbar^2 c^2}{2mc^2} \left(3\pi^2 \cdot 0.15 \frac{1}{fm^3} \right)^{2/3}$$

$$\frac{(197\,MeVfm)^2}{2 \times 940\,MeV} = 20.6\ MeVfm^2$$

$$\epsilon_F = 20.6\ MeV\cancel{fm^2} \times 2.7 \frac{1}{\cancel{fm^2}} = \boxed{56\ MeV}$$

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^3He

$$\frac{\hbar^2 c^2}{2mc^2} = \frac{20.6}{3}\ MeVfm^2 = 6.87\ MeVfm^2$$

$$\epsilon_F = 6.87\ MeVfm^2 \left(\frac{3\pi^2}{46\ \mathring{A}^3} \right)^{2/3}$$

$$= 5.12\,eV \times 10^6 \times 10^{-30}\ \cancel{m^2}\ \frac{1}{10^{-20}\ \cancel{m^2}} = 5.12 \times 10^{-4}\ eV$$

P.5

$$A = E - TS$$

$$k_B \ln \mathcal{Z} = S - \frac{E}{T} + \frac{\mu}{T} N$$

$$k_B T \ln \mathcal{Z} = TS - E + \mu N = \mu N - A$$

$$A = \mu N - k_B T \ln \mathcal{Z}$$

Fermi gas

$$\ln \mathcal{Z} = \frac{PV}{k_B T} = \frac{4\pi V}{h^3} \int_0^\infty dp\, p^2 \ln(1 + z e^{-\beta p^2/2m})$$

$$= \frac{V}{\lambda^3} f_{5/2}(z)$$

$$A = \mu N - \frac{V}{\lambda^3} f_{5/2}(z) k_B T \qquad v = \frac{\lambda^3}{f_{3/2}(z)}$$

$$\frac{A}{N} = \mu - \frac{1}{\lambda^3} \overbrace{\frac{V}{N}}^{v} f_{5/2}(z)\, k_B T$$

$$= \mu - \frac{f_{5/2}(z)}{f_{3/2}(z)} k_B T = k_B T \ln z - \frac{f_{5/2}(z)}{f_{3/2}(z)} k_B T$$

$$k_B T \ln z = \epsilon_F \left(1 - \frac{\pi^2}{12} \left(\frac{kT}{\epsilon_F} \right)^2 + \cdots \right)$$

$$f_{3/2}(z) \simeq \frac{4}{3\sqrt{\pi}} \left[(\ln z)^{3/2} + \frac{\pi^2}{8} (\ln z)^{-1/2} + \cdots \right]$$

$$\simeq \frac{4}{3\sqrt{\pi}} (\ln z)^{3/2} \left(1 + \frac{\pi^2}{8} (\ln z)^{-2} + \cdots \right)$$

$$f_{5/2}(z) \simeq \frac{(\ln z)^{5/2}}{\Gamma(7/2)} \left(1 + \frac{\pi^2}{6} \frac{5/2\,(5/2 - 1)}{(\ln z)^2} + \cdots \right)$$

$$f_a(z) \underset{z \to \infty}{\simeq} \frac{(\ln z)^a}{\Gamma(a+1)} \left(1 + \frac{\pi^2}{6} \frac{a(a-1)}{(\ln z)^2} + \cdots \right)$$

$$\Gamma(7/2) = 5/2 \;\; 3/2 \;\; 1/2 \sqrt{\pi} = \frac{15}{8} \sqrt{\pi} \qquad \frac{3 \times 5}{6 \times 4}$$

$$f_{5/2}(z) = \frac{8}{15\sqrt{\pi}} (\ln z)^{5/2} \left(1 + \frac{5}{8} \pi^2 \frac{1}{(\ln z)^2} + \cdots \right)$$

$$\frac{f_{5/2}}{f_{3/2}}(z) = \frac{\frac{8}{15\cancel{\sqrt{\pi}}} (\ln z)^{5/2}}{\frac{4}{3\cancel{\sqrt{\pi}}} (\ln z)^{3/2}} \left(1 + \frac{5}{8} \frac{\pi^2}{(\ln z)^2} - \frac{\pi^2}{8} \frac{1}{(\ln z)^2} + \cdots \right)$$

$$= \frac{2}{5} (\ln z) \left(1 + \frac{\pi^2}{2} \frac{1}{(\ln z)^2} + \cdots \right)$$

$$A = k_B T \ln z - \frac{2}{5} \ln z \left(1 + \frac{\pi^2}{2} \frac{1}{(\ln z)^2} + \ldots\right) k_B T$$

$$= \epsilon_F \left(1 - \frac{\pi^2}{12} \left(\frac{kT}{\epsilon_F}\right)^2 + \ldots\right) \left(\underbrace{1 - \frac{2}{5}}_{\frac{3}{5}} - \frac{\pi^2}{5} \frac{1}{(\ln z)^2} + \ldots\right)$$

$$= \frac{3}{5} \epsilon_F \left(1 - \frac{\pi^2}{12} \left(\frac{kT}{\epsilon_F}\right)^2 + \ldots\right) \left(1 - \frac{\pi^2}{3} \left(\frac{kT}{\epsilon_F}\right)^2 + \ldots\right)$$

$$\frac{1}{12} + \frac{1}{3} = \frac{1+4}{12} = \frac{5}{12}$$

$$\boxed{A = \frac{3}{5} \epsilon_F \left(1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{\epsilon_F}\right)^2 + \ldots\right)}$$

6

In class we estimated $T_D \approx 200°K$.

at room temp $C_V \sim 3 k_B N$

room temp $k_B T \sim 0.025\,eV \ll 7eV$.

$\underline{T \ll T_F}$ . .

$$C_V \simeq \frac{\pi^2}{12} \frac{k_B T}{\epsilon_F} N k_B$$

assuming $1 e^-$ per atom.

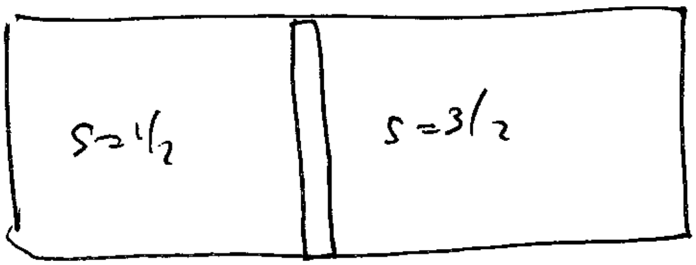
$$3 \cancel{k_B N} \ ? \ \frac{\pi^2}{12} \frac{k_B T}{e_F} \cancel{N k_B}$$

$$3 \ ? \ \frac{\pi^2}{12} \frac{0.025}{7}$$

$\underline{\gg}$

At low temp. is opposite since. $C_V \sim T^3$.

P7



same pressure & temp.

$$P_1 = k_B T \frac{(2s_1+1)}{\lambda^3} f_{5/2}(z_1)$$

$$\frac{1}{v_1} = \frac{(2s_1+1)}{\lambda^3} f_{3/2}(z_1)$$

$$(2s_1+1) f_{5/2}(z_1) = (2s_2+1) f_{5/2}(z_2)$$

$$\frac{\rho_1}{\rho_2} = \frac{1/v_1}{1/v_2} = \frac{(2s_1+1)}{(2s_2+1)} \frac{f_{3/2}(z_1)}{f_{3/2}(z_2)}$$

$$\frac{\rho_1}{\rho_2} = \frac{2}{4} \frac{f_{3/2}(z_1)}{f_{3/2}(z_2)}$$

$$2 f_{5/2}(z_1) = 4 f_{5/2}(z_2)$$

low temp. (lowest order)

$$2 \frac{\cancel{8}}{15\sqrt{\pi}} (\ln z_1)^{5/2} = 4 \frac{\cancel{8}}{\cancel{15}\cancel{\sqrt{\pi}}} (\ln z_2)^{5/2}$$

$$\ln z_1 = 2^{2/5} \ln z_2$$

$$\frac{\rho_1}{\rho_2} = \frac{1}{2} \frac{(\ln z_1)^{3/2}}{(\ln z_2)^{3/2}} = \frac{1}{2}\left(2^{2/5}\right)^{3/2} = \frac{2^{3/5}}{2}$$

$$\boxed{\frac{\rho_1}{\rho_2} = 2^{-2/5}} \qquad T \to 0$$

High temp. $z \to 0$

$$f_a(z) \simeq z$$

$$(2s_1+1) z_1 = (2s_2+1) z_2$$

$$\frac{\rho_1}{\rho_2} = \frac{(2s_1+1)}{(2s_2+1)} \frac{z_1}{z_2} = 1$$

same density.

$pV = Nk_B T$

ideal gas.. $\frac{N}{V} = \frac{P}{k_B T}$ ✓

P8

gas of photons.

$$\frac{E}{V} = \frac{\pi^2}{15} \frac{(k_B T)^4}{(\hbar c)^3}$$

$$\frac{S}{V} = \frac{4}{45} \pi^2 \frac{k_B^4 T^3}{(\hbar c)^3}$$

$$Z = \sum_{\{n_p\}}{}' e^{-\beta \sum_p n_p \varepsilon_p}$$

$$= \prod_p \frac{1}{1-e^{-\beta\varepsilon_p}}$$

$$\ln Z = -\sum_p \ln(1-e^{-\beta\varepsilon_p})$$

$$= -\frac{2V}{(2\pi\hbar)^3} \int d^3p \ln(1-e^{-\beta pc})$$

$$\langle N \rangle = \frac{1}{Z} \sum_{\{n_p\}} \left(\sum_p n_p\right) e^{-\beta \sum_p n_p \varepsilon_p}$$

$$= \frac{\overset{g}{2}V}{(2\pi\hbar)^3} \int d^3p \underbrace{\frac{1}{e^{\beta\varepsilon_p}-1}}_{n_p} = \frac{8\pi V}{(2\pi\hbar)^3} \underbrace{\int_0^\infty \frac{p^2\,dp}{e^{\beta pc}-1}}_{\text{finite}}$$

$$\langle N \rangle = \frac{8\pi V}{(2\pi\hbar)^3} \frac{1}{(\beta c)^3} \int_0^\infty \frac{x^2 dx}{e^x - 1}$$

$$= \frac{8\pi V}{(2\pi\hbar c)^3} (k_B T)^3 \, 2\zeta(3) \qquad (2\zeta(3) \simeq 2.4)$$

$$\frac{S}{\langle N \rangle} = \frac{4}{45} \frac{\pi^2 \cancel{V} (k_B^4 \cancel{T^3})}{\cancel{(\hbar c)^3}} \frac{(2\pi \cancel{\hbar c})^3}{8\pi \cancel{V} \cancel{(k_B T)^3} \, 2\zeta(3)}$$

$$\boxed{\frac{S/k_B}{\langle N \rangle} = \frac{\overset{2}{\cancel{4}}\pi^2 \times \cancel{8}\pi^3}{45 \times \cancel{8}\pi \times \cancel{2}\,\zeta(3)} = \frac{2\pi^4}{45\zeta(3)} \simeq 3.6}$$

$$\frac{E}{\langle N \rangle} = \frac{E}{S/k_B} \frac{S/k_B}{N} = \frac{3}{4} k_B T \times 3.6 = 2.7\, k_B T \, V$$

$$\frac{E}{S} = \frac{\cancel{\pi^2}}{15} \frac{\cancel{V}(k_B T)^4}{\cancel{(\hbar c)^3}} \frac{45}{4} \frac{1}{\cancel{\pi^2} \cancel{V}} \frac{\cancel{(\hbar c)^3}}{(k_B^4 T^3)} = \frac{3}{4} T$$

P9

area

$$\frac{pV}{k_B T} = \ln \mathcal{Z}_B = -\frac{V}{h^2}\int d^2p \ln\left(1 - e^{-\beta\varepsilon_p + \beta\mu}\right)$$

$$\boxed{p = -\frac{k_B T}{h^2} 2\pi \int_0^\infty p\,dp \ln\left(1 - e^{-\left(\frac{\beta p^2}{2m}\right) + \beta\mu}\right)}$$

pressure

$$x^2 = \frac{\beta p^2}{2m} \qquad p = \sqrt{\frac{2m}{\beta}}\, x \qquad z = e^{\beta\mu}$$

$$P = -\frac{k_B T}{h^2} 2\pi \left(\frac{2m}{\beta}\right) \int_0^\infty x\,dx \ln\left(1 - e^{-x^2 + \beta\mu}\right)$$

$y = x^2$

$$\boxed{P = -\frac{4\pi m}{h^2}\frac{(k_B T)^2}{2} \int_0^\infty dy \ln\left(1 - e^{-y + \beta\mu}\right)}$$

$$N_B = \frac{2\pi V}{h^2}\int_0^\infty p\,dp \frac{1}{e^{\frac{\beta p^2}{2m} - \beta\mu} - 1}$$

$$\boxed{N_B = \frac{\pi V}{h^2} 2m k_B T \int_0^\infty dy \frac{1}{e^{y - \beta\mu} - 1}}$$

$$E = -\frac{\partial \ln \mathcal{Z}}{\partial \beta}\bigg|_{z \text{ fixed}}$$

$$\ln \mathcal{Z}_B = -\frac{2\pi V}{h^2}\int_0^\infty p\,dp\, \ln\left(1-e^{-\frac{\beta p^2}{2m}+\beta\mu}\right)$$

$$= -\frac{2\pi V}{h^2}\frac{(\cancel{2}m k_B T)}{\cancel{2}}\int_0^\infty dy\, \ln\left(1-e^{-y+\beta\mu}\right)$$

$$= -\frac{2\pi m V}{h^2}\underbrace{k_B T}_{1/\beta}\int_0^\infty dy\, \ln\left(1-z e^{-y}\right)$$

$$E = \frac{2\pi m V}{h^2}\frac{1}{\beta^2}\int_0^\infty dy\, \ln\left(1-z e^{-y}\right)$$

$$E = \frac{2\pi m V}{h^2}(k_B T)^2\int_0^\infty dy\, \ln\left(1-z e^{-y}\right)$$

$$S = k_B \ln \mathcal{Z} + \frac{E}{T} - \frac{\mu}{T}N$$

$$\frac{S}{k_B} = \cancel{-\frac{2\pi m V}{h^2}k_B T\int_0^\infty dy\, \ln\left(1-z e^{-y}\right)} + \cancel{\frac{2\pi m V}{h^2}(k_B T)\int_0^\infty dy\, \ln\left(1-z e^{-y}\right)} - \frac{\mu}{\cancel{T}}\frac{\pi V}{h^2}2m\cancel{k_B T}\int_0^\infty \frac{dy}{e^{y-\beta\mu}-1}$$

$$\boxed{\frac{S}{k_B} = -\frac{2m\pi V}{h^2}\underbrace{\mu}_{\mu<0}\int_0^\infty \frac{dy}{e^{y-\beta\mu}-1}}$$

$$N = -\frac{S}{\cancel{k_B}}\frac{k_B T}{\mu}$$

Notice.

$$\boxed{N = \frac{\pi V}{h^2} 2m k_B T \int_0^\infty \frac{dy}{e^{y-\beta\mu}-1}}$$

$$S = -\frac{\mu N}{T}$$

when $T \to 0$

we have to take $\mu \to 0$

$$N \simeq \frac{\pi V 2m}{h^2} k_B T \int_0^\infty \frac{dy}{\frac{1}{z}(1+y)-1}$$

$$\simeq \frac{2m\pi V}{h^2} k_B T z \int_0^A \frac{dy}{1-z+y}$$

But the integral.

$$\int \frac{dy}{y} \quad \text{diverges at } 0.$$

So we can get any $N \rightarrow$ no Bose-Einstein's condensation.

$$\int_0^a \frac{dy}{(1-z)+y} = \frac{1}{(1-z)} \int_0^a \frac{dy}{1+ y/(1-z)}$$

$$= \frac{1}{\cancel{(1-z)}} = \int_0^{a/(1-z)} \frac{dx}{1+x}$$

$$= \int^{a/(1-z)} \frac{dx}{x} = \ln\left(\frac{a}{1-z}\right) \simeq -\ln(1-z)$$

$$N \simeq \frac{2m\pi V}{h^2} k_B T \left(-\ln(1-z)\right)$$

$z \rightarrow 1$