Renormalization group

(1d)

$$Z = \sum_{s_i = \pm 1} e^{\beta J \sum s_i s_{i+1} + \beta B \sum s_i}$$

$$= \sum_{s_i = \pm 1} A_{s_1 s_2} A_{s_2 s_3} A_{s_3 s_4} \cdots A_{s_N s_1} = \mathrm{Tr} A^N$$

$$A_{s_2 s_3} = e^{\beta J s_2 s_3 + \frac{1}{2}\beta B (s_2 + s_3)}$$

$$A = \begin{pmatrix} e^{\beta J + \beta B} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J - \beta B} \end{pmatrix}$$

$$Z = \mathrm{Tr}\,(A^2)^{N/2} \qquad \text{(Assume } N \text{ even).}$$

sum over even spins. gives interactions between spins separated by 2 lattice spacings

$$A^2 = \begin{pmatrix} e^{\beta J + \beta B} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J - \beta B} \end{pmatrix} \begin{pmatrix} e^{\beta J + \beta B} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J - \beta B} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{2\beta J + 2\beta B} + e^{-2\beta J} & e^{\beta B} + e^{-\beta B} \\ e^{\beta B} + e^{-\beta B} & e^{-2\beta J} + e^{2\beta J - 2\beta B} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\beta B}\, 2\,\mathrm{ch}(2\beta J + \beta B) & 2\,\mathrm{ch}\,\beta B \\ 2\,\mathrm{ch}\,\beta B & e^{-\beta B}\, 2\,\mathrm{ch}(2\beta J - \beta B) \end{pmatrix}$$

Can we write it as an "A" ?

3 eq. for 2 constant $\Rightarrow$ up to overall factor.

$$A^2 = 2\,\mathrm{ch}\,\beta B \begin{pmatrix} \frac{e^{\beta B}}{\cancel{2}\,\mathrm{ch}\,\beta B}\,\mathrm{ch}(2\beta J + \beta B) & 1 \\ 1 & \frac{e^{-\beta B}}{\mathrm{ch}\,\beta B}\,\mathrm{ch}(2\beta J - \beta B) \end{pmatrix}$$

$$= e^{-\tilde{\beta}\tilde{J}}\,\alpha \begin{pmatrix} e^{2\tilde{\beta}\tilde{J} + \tilde{\beta}\tilde{B}} & 1 \\ 1 & e^{2\tilde{\beta}\tilde{J} - \tilde{\beta}\tilde{B}} \end{pmatrix}$$

$$e^{4\tilde{\beta}\tilde{J}} = \frac{\mathrm{ch}(2\beta J + \beta B)\,\mathrm{ch}(2\beta J - \beta B)}{\mathrm{ch}^2 \beta B}$$

$$e^{2\tilde{\beta}\tilde{B}} = e^{2\beta B}\ \frac{\mathrm{ch}(2\beta J + \beta B)}{\mathrm{ch}(2\beta J - \beta B)}$$

$$\alpha\ e^{-\tilde{\beta}\tilde{J}} = 2\,\mathrm{ch}\,\beta B$$

$$\mathrm{ch}(2\beta J + \beta B)\,\mathrm{ch}(2\beta J - \beta B) = (\mathrm{ch}\,2\beta J\,\mathrm{ch}\,\beta B + \mathrm{sh}\,2\beta J\,\mathrm{sh}\,\beta B)\cdot$$
$$(\mathrm{ch}\,2\beta J\,\mathrm{ch}\,\beta B - \mathrm{sh}\,2\beta J\,\mathrm{sh}\,\beta B)$$

$$= \mathrm{ch}^2 2\beta J\,\mathrm{ch}^2\beta B - \mathrm{sh}^2 2\beta J\,\underbrace{\mathrm{sh}^2\beta B}_{\mathrm{ch}^2\beta B - 1}$$

$$= \mathrm{ch}^2\beta B + \mathrm{sh}^2 2\beta J$$

$$e^{4\tilde{\beta}\tilde{J}} = 1 + \frac{\mathrm{sh}^2 2\beta J}{\mathrm{ch}^2\beta B}$$

Define.

$$u = e^{-\beta J} \qquad A = \begin{pmatrix} \frac{1}{uV} & u \\ u & V/u \end{pmatrix}$$

$$V = e^{-\beta B}$$

$0 \le u \le 1$; $0 \le V \le 1$ $\qquad J > 0$, take $\underline{\beta > 0}$.

$$\frac{1}{\tilde{u}^4} = 1 + \frac{(1/u^2 - u^2)^2}{(1/V + V)^2}$$

$$\frac{1}{\tilde{V}^2} = \frac{1}{V^2} \frac{\frac{1}{u^2 V} + u^2 V}{\frac{V}{u^2} + u^2/V} = \frac{1}{V^2} \frac{1}{\frac{u^2 V}{u^2 V}} \frac{(1 + u^4 V^2)}{(V^2 + u^4)}$$

$$\tilde{V}^2 = \frac{V^2 + u^4}{u^4 + 1/V^2}$$

$$\alpha \tilde{u} = u + 1/u$$

$$\tilde{V} = \frac{\sqrt{V^2 + u^4}}{\sqrt{u^4 + 1/V^2}} \quad ; \quad \frac{1}{\tilde{u}^4} = \frac{(1/V + V)^2 + (1/U^2 - U^2)^2}{\frac{1}{V^2}(1 + V^2)^2}$$

$$\tilde{u}^4 = \frac{(1/V + V)^2}{(\frac{1}{V^2} + V^2 + \cancel{2} + \frac{1}{U^4} + U^4 \cancel{- 2})} \Rightarrow \tilde{u} = \frac{(1/V + V)^{1/2}}{(V^2 + \frac{1}{V^2} + U^4 + \frac{1}{U^4})^{1/4}}$$

$$\alpha = \frac{(V+1/V)}{\tilde{u}} = \frac{(V+1/V)}{(1/V+V)^{1/4}}\left(V^2+\frac{1}{V^2}+u^4+\frac{1}{u^4}\right)^{1/4}$$

$$\alpha = \sqrt{V+1/V}\left(V^2+\frac{1}{V^2}+u^4+\frac{1}{u^4}\right)^{1/4}$$

$$\tilde{V} = V\frac{\sqrt{V^2+u^4}}{\sqrt{1+V^2u^4}}$$

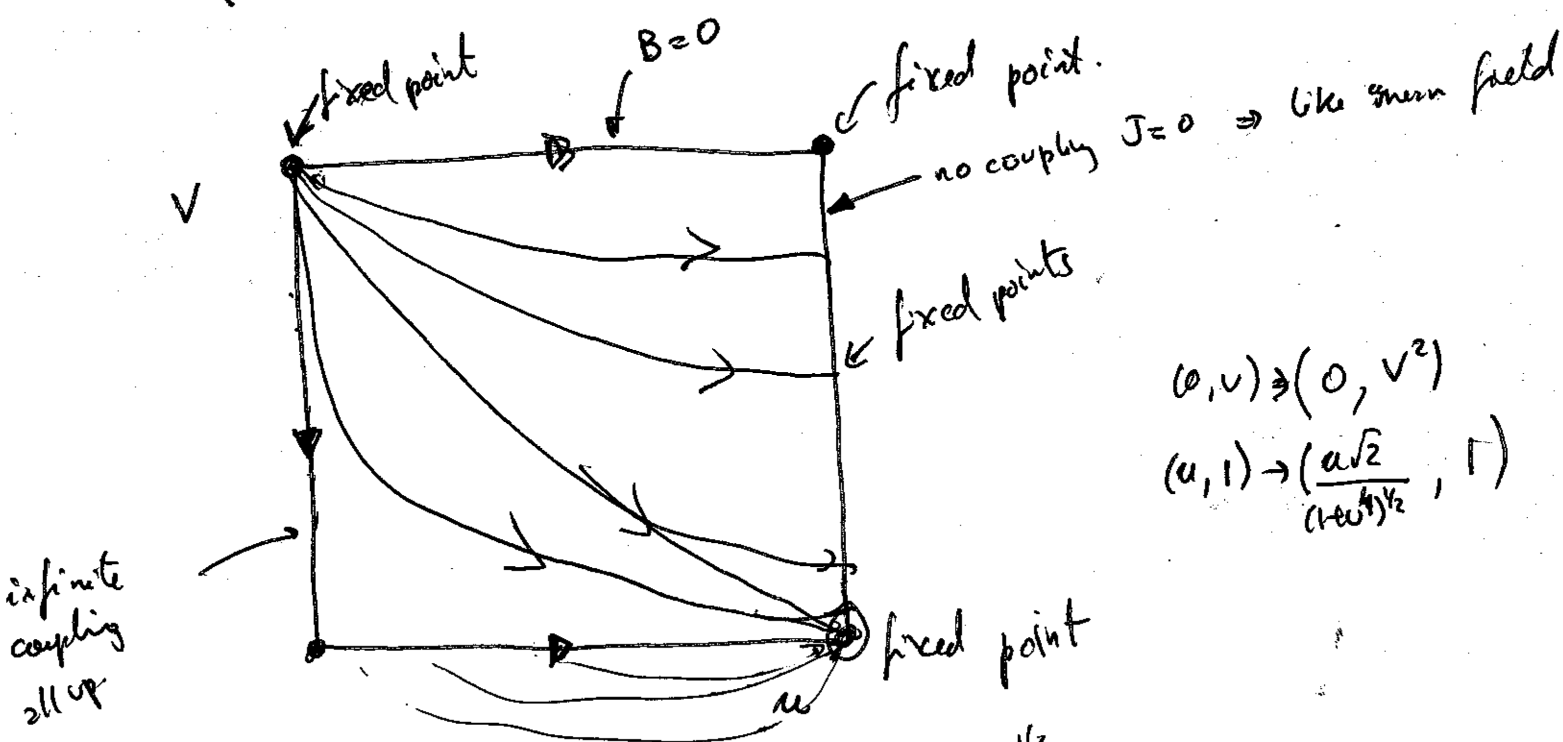
$$\tilde{u} = \frac{u(1+V^2)^{1/2}}{(u^4V^4+u^4+u^8V^2+V^2)^{1/4}}$$

$(1,1) \to (1,1)$

$\frac{\sqrt{2}}{4^{1/4}} = 1$

$(0,1) \to (0,1)$

$(1,0) \to (1,0)$



$(0,V) \to (0, V^2)$

$(u,1) \to \left(\frac{u\sqrt{2}}{(1+u^4)^{1/2}}, 1\right)$

$(u,0) \to (1,0)$ $\qquad$ $(1,V) \to \left(\frac{(1+V^2)^{1/2}}{(1+V^4+2V^2)^{1/4}} = 1, V\right)$

Check of $0 \le \tilde{u} \le 1$ $\quad 0 \le \tilde{v} \le 1$

both ≥ 0 ✓

$$v\,\frac{\sqrt{v^2+u^4}}{\sqrt{1+v^2u^4}} < 1$$

$$v^2(v^2+u^4) < 1+v^2u^4$$

$$v^4+\cancel{v^2u^4} < 1+\cancel{v^2u^4}$$

✓

$$\frac{u(1+v^2)^{1/2}}{(u^4v^4+u^4+u^8v^2+v^2)^{1/4}} < 1$$

$$u^4(1+v^2)^2 < u^4v^4+u^4+u^8v^2+v^2$$

$$\cancel{u^4}+2u^4v^2+\cancel{u^4v^4} < \cancel{u^4v^4}+\cancel{u^4}+u^8v^2+v^2$$

$$2u^4\cancel{v^2} < u^8\cancel{v^2}+\cancel{v^2}\,1$$

$$0 < 1+u^8-2u^4$$

$$0 < (1-u^4)^2$$

✓.

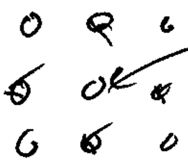
$v = e^{-\beta B}$ decreases $\rightarrow$ B increases

$u = e^{-\beta J}$ increases. $\rightarrow$ J decreases.

↑ more random ✓.

More generally.

In 2d



summing over one spin introduces more interactions involving 4 spins

$$E\{s\} = \sum_\alpha K_\alpha S_\alpha$$

all possible operators

couplings

K_α : parameterize the space of Hamiltonians.

If we sum over half the spins (or some other procedure like decimation) we get a new hamiltonian.

$$Z = \sum_{\{S_j = \pm 1\}} e^{-\beta E\{S_j\}} = \sum_{\underset{\text{remain}}{S_{\tilde{j}}},\ \underset{\text{subset } j'}{S_{j'}}} e^{-\beta E\{S_{j'}, S_{\tilde{j}}\}} =$$

$$= \sum_{S_{\tilde{j}}} \underbrace{e^{-\tilde{\beta} \tilde{E}\{S_{\tilde{j}}\}}}$$

any function of $\{S_{\tilde{j}}\}$ can be written like this by definition of K_α.

then we get (absorbing β in K)

$$\tilde{K}_a = f_a(K_1, K_2, \ldots K_\infty)$$

Define

$$e^{-Ng(K_a)} = \sum_{\{s\}} e^{-E[K_a]\{s\}}.$$

$$= \sum_{\{s'\}} e^{-E'[K'_a]\{s'\}} = e^{-N'g(K'_a)}$$

$$N' = b^{-d} N$$

↑ e.g. 2. if we do 2x2 blocks renormalizing.

$$N g(K_a) = b^{-d} N g(K'_a)$$

↑ same function

Fixed point

$$K_a^* = f_a(K_b^*)$$

$$K_a^{(n+1)} = f_a(K_b^{(n)})$$

if $K_a^{(n)} \simeq K_a^*$ $\Rightarrow$ $K_a^{(n+1)} \simeq K_a^*$ also.

$$K_a^{(n+1)} = K_a^* + \delta K_a^{(n+1)}$$

$$\cancel{K_a^*} + \delta K_a^{(n+1)} = f_a(K_b^* + \delta K_b^{(n)})$$

$$\simeq \underbrace{\cancel{f_a(K_b^*)}}_{\cancel{K_a^*}} + \partial_b f_a(K^*)\, \delta K_b^{(n)}$$

$$\delta K_a^{(n+1)} = \underbrace{\partial_b f_a(K^*)}_{\text{constant matrix}}\, \delta K_b^{(n)}$$

$$\delta K_a^{(n+1)} = W_{ab}\, \delta K_b^{(n)}$$

$$W_{ab} = \left.\frac{\partial f_a}{\partial K_b}\right|_{K=K^*}$$

left eigenvectors

$$\phi_a^{(j)} W_{ab} = \lambda^{(j)} \phi_b^{(j)}$$

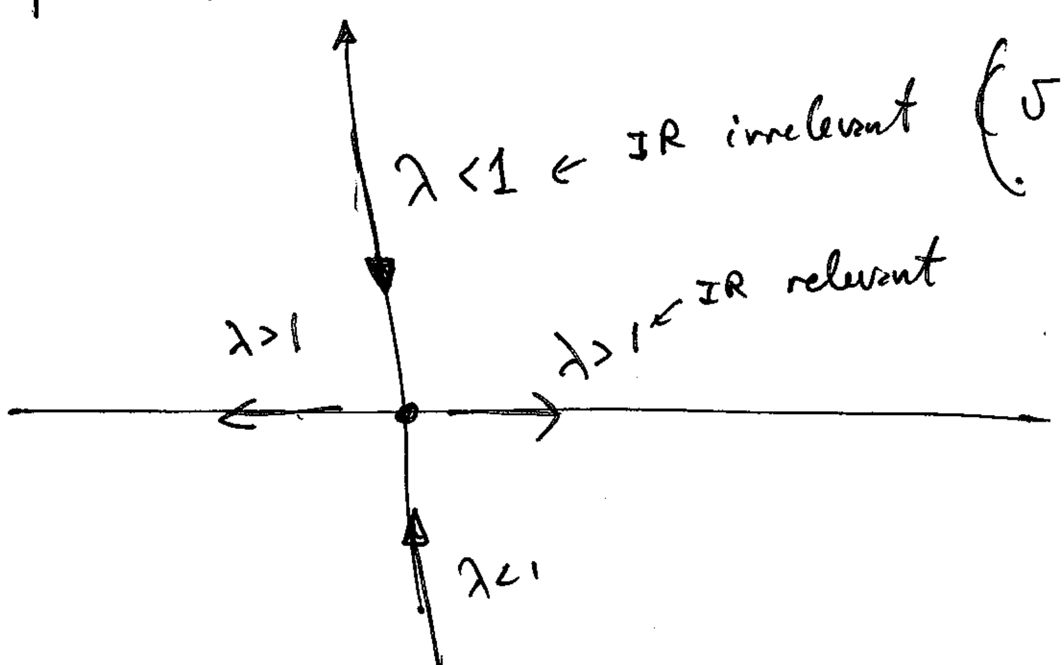
Define $v_n^{(j)} = \sum_a \phi_a^{(j)} \delta K_a^{(n)}$; new coordinates in Hamiltonian space.

$$v_{n+1}^{(j)} = \sum_a \phi_a^{(j)} W_{ab}\, \delta K_b^{(n)}$$

$$= \sum_a \lambda^{(j)} \phi_b^{(j)} \delta K_b^{(n)} = \lambda^{(j)} v_n^{(j)}.$$

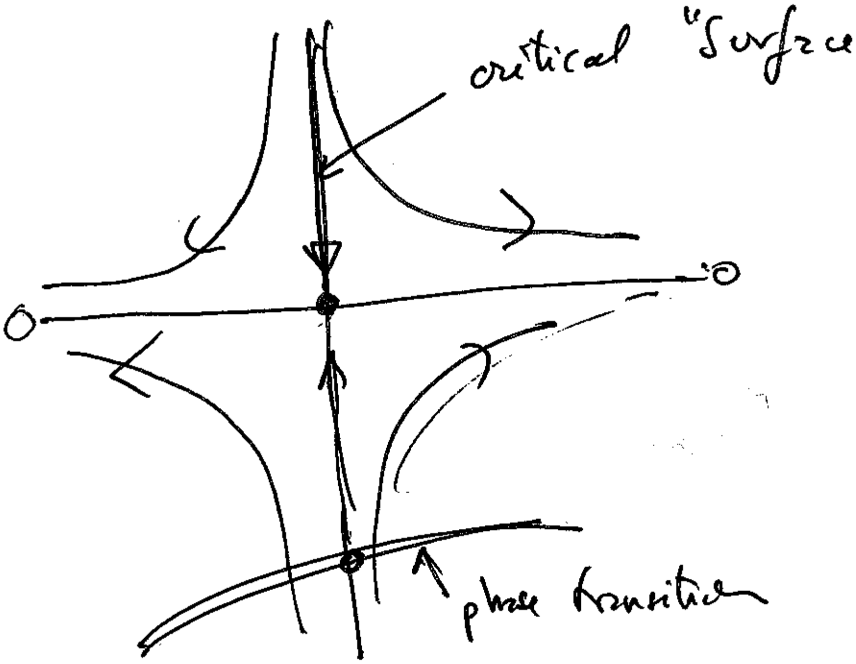
$$v_{n+p}^{(j)} = \left(\lambda^{(j)}\right)^p v_n^{(j)}$$

fixed point $v^{(j)} = 0$

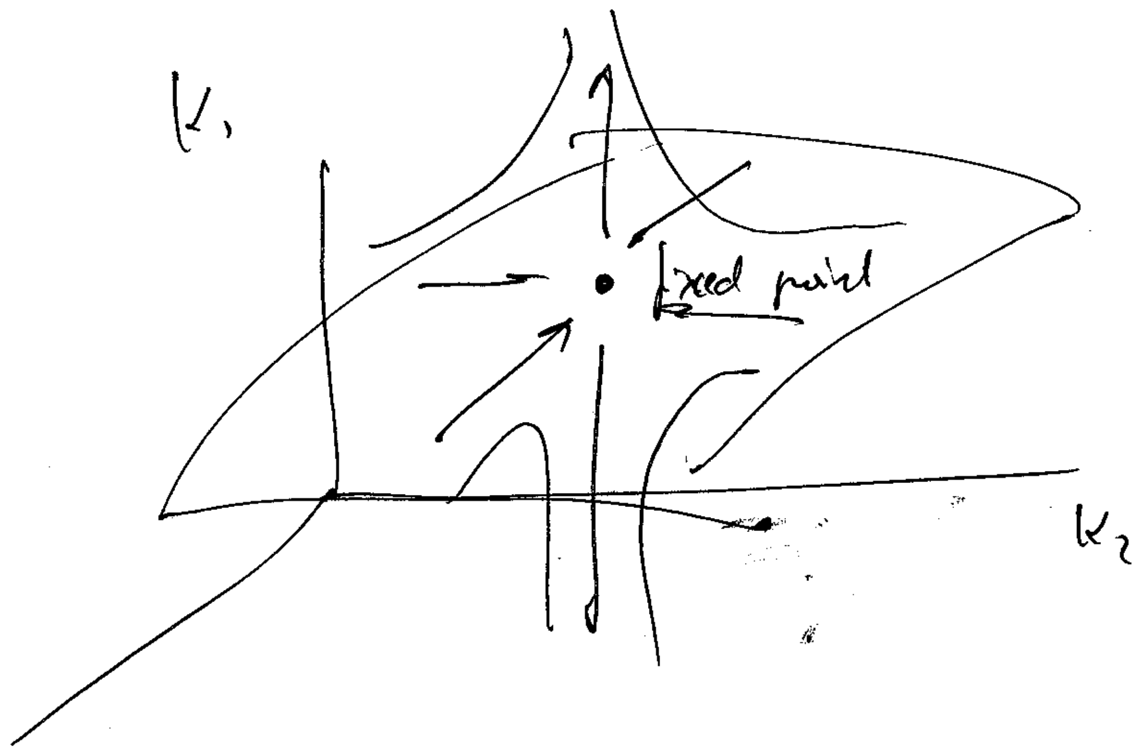


$\lambda = 1$ marginal. ⇒ many fix points (exactly marginal)
$\lambda = 1$ instead of $\lambda \simeq 1$.

RG trajectories



all relevant → 0.
all these points have the same IR limit



$$g(\underbrace{v_1, \dots v_n}) = b^{-d}\, g(\lambda_1 v_1, \dots \lambda_n v_n)$$

only keep relevant ones.

RG in momentum space.

$$E = \int d^d x \ \frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} r_0 \phi^2 - h\,\phi(x)$$

$|k| < \Lambda$ ← cut-off

$$E = \frac{1}{2} \int d^d k \ (k^2 + r_0)\, |\phi(k)|^2 - h\,\phi(0)$$

Step 1: Integrate out high energy ≡ high momentum modes

$$Z = \int d\phi(k) \ e^{-\frac{1}{2}\int^{\Lambda} d^d k \ (k^2+r_0)\,|\phi(k)|^2 - h\,\phi(0)}$$

$$= \int \prod_{k<\Lambda/b} d\phi(k) \int \prod_{\frac{\Lambda}{b}<k<\Lambda} d\phi(k) \ e^{-\frac{1}{2}\int^{\Lambda} d^d k \ (k^2+r_0)\,|\phi(k)|^2 - h\phi(0)}$$

$$= \prod_{\frac{\Lambda}{b}<k<\Lambda} \left(\sqrt{\frac{2\pi}{k^2+r_0^2}}\right) \int \prod_{k<\Lambda/b} d\phi(k) \ e^{-\frac{1}{2}\int^{\Lambda/b} d^3 k \ (k^2+r_0)\,|\phi(k)|^2 - h\phi(0)}$$

$k = k'/b$

Step 2: restore cut-off by changing variables of integration

$$Z = \mathcal{N}' \int \prod_{k'<\Lambda} d\phi(k') \ e^{-\frac{1}{2}\int^{\Lambda} \frac{d^3 k'}{b^3} \left(\frac{k'^2}{b^2} + r_0\right) |\phi(k')|^2 - h\phi(0)}$$

Step 3: Restore normalization of $\phi(k)$ (2)

$$\tilde{\phi} = \frac{1}{b^{5/2}} \phi$$

$$Z = \mathcal{N}'' \int_{k'<\Lambda} \prod d\phi(k') \, e^{-\frac{1}{2}\int^{\Lambda} d^3k \, (k^2 + r_0 b^2) |\phi(k)|^2 - h b^{5/2} \phi(0)}$$

$$r_0' = r_0 b^2$$
$$h' = h b^{\frac{d+2}{2}}$$

fixed point $r_0 = 0$ $h = 0$

$\underline{b>1}$ $\Rightarrow$ $\left.\begin{array}{l} r_0' \to \infty \\ h' \to \infty \end{array}\right\}$ relevant variables.

quartic coupling $\to$ phase transition!

$$E[\phi(x)] = \int d^dx \quad \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}r_0\phi^2 + u_0\phi^4$$

This will be very approximate.

$$\phi(x) = \underbrace{\bar{\phi}(x)}_{\text{slowly varying}} + \delta\phi(x)$$

$\uparrow$ fast varying. (short wavelength.)

+

$$\int \bar{\phi}(x)\,\delta\phi(x) = 0$$

small k. $\uparrow$ large k

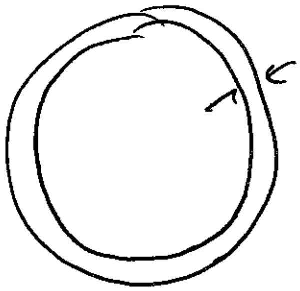
$$E[\phi(x)] = E[\bar{\phi}(x)] + \int d^dx \left\{ \frac{1}{2}(\nabla\delta\phi)^2 + \frac{1}{2}r_0(\delta\phi)^2 + u_0\left(\underbrace{4\underbrace{\bar{\phi}^3}_{\text{slowly}}\underbrace{\delta\phi}_{\text{fast}}}_{0} + 6\bar{\phi}^2\delta\phi^2 + 4\bar{\phi}\delta\phi^3 + \delta\phi^4\right)\right\}$$

ignore higher powers. (small fluctuations).

$$E[\phi(x)] \simeq E[\bar{\phi}(x)] + \int d^dx \left\{ \frac{1}{2}(\nabla\delta\phi)^2 + \frac{1}{2}r_0(\delta\phi)^2 + 6u_0\bar{\phi}^2\delta\phi^2 \right\}.$$

$\delta\phi$ has momenta $\Lambda/b < k < \Lambda$

$\simeq \Lambda(1-\ln b)$.



# of states $\frac{V}{(2\pi)^d} C_d \Lambda^d \ln b$

$C_d \Lambda^{d-1}$: area of S_d sphere

$$C_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$$

in shell $\quad k \simeq \Lambda$

$$Z \simeq \int \mathcal{D}\bar{\phi}(x) \int \mathcal{D}\delta\phi(k)\, e^{-E[\bar{\phi}] - \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{2}\Lambda^2 + \frac{1}{2} r_0 + 6u_0 \bar{\phi}^2\right) \delta\phi_k^2}$$

approx constant

$$\int \mathcal{D}\delta\phi(k) \to \left(\frac{\pi}{\frac{1}{2}\Lambda^2 + \frac{1}{2} r_0 + 6u_0\bar{\phi}^2}\right)^{\frac{1}{2}\frac{V}{(2\pi)^d} C_d \Lambda^d \ln b}$$

$$e^{-\frac{1}{2}\frac{V}{(2\pi)^d} C_d \Lambda^d \ln b \,\ln\left(\frac{1}{2}\Lambda^2 + \frac{1}{2} r_0 + 6u_0\bar{\phi}^2\right)}$$

$$\ln\frac{\Lambda^2}{2} + \ln\left(1 + \frac{r_0}{\Lambda^2} + \frac{12u_0\bar{\phi}^2}{\Lambda^2}\right)$$

$$\ln(1+\varepsilon) = \varepsilon - \varepsilon^2/2$$

$$\ln\left(1+\frac{r_0}{\Lambda^2}+\frac{12u_0\bar{\phi}^2}{\Lambda^2}\right) \simeq \frac{r_0}{\Lambda^2}+\frac{12u_0\bar{\phi}^2}{\Lambda^2}-\frac{1}{2\Lambda^4}\left(r_0+12u_0\bar{\phi}^2\right)^2$$

$\frac{144}{2}$ $\quad 72$

$$\simeq \frac{r_0}{\Lambda^2}+\frac{12u_0\bar{\phi}^2}{\Lambda^2}-\frac{r_0^2}{2\Lambda^4}-\frac{12r_0u_0\bar{\phi}^2}{\Lambda^4}-\frac{72u_0^2\bar{\phi}^4}{\Lambda^4}$$

$$e^{-\frac{1}{2}\frac{\overset{\int d^dx}{V}}{(2\pi)^d}C_d\Lambda^d\ln b\left(\frac{12u_0}{\Lambda^2}\bar{\phi}^2-\frac{12r_0u_0\bar{\phi}^2}{\Lambda^4}-\frac{72u_0^2}{\Lambda^4}\bar{\phi}^4\right)}$$

$$E = \int d^dx\left\{\frac{1}{2}(\nabla\bar{\phi})^2+\frac{1}{2}r_0\bar{\phi}^2+u_0\bar{\phi}^4+\right.$$

$$+\frac{1}{2}\left(\frac{C_d\Lambda^d}{(2\pi)^d}\right)\ln b\left(\frac{12u_0}{\Lambda^2}-\frac{12r_0u_0}{\Lambda^4}\right)\bar{\phi}^2$$

$$\left.-\left(\frac{C_d\Lambda^d}{(2\pi)^d}\right)\ln b\left(\frac{36u_0^2}{\Lambda^4}\right)\bar{\phi}^4\right\}$$

$$r_0 \to r_0+\frac{C_d}{(2\pi)^d}\ln b\ 12\left(u_0\Lambda^{d-2}-r_0u_0\Lambda^{d-4}\right)$$

$$u_0 \to u_0-\left(\frac{C_d}{(2\pi)^d}\right)\ln b\ 36u_0^2\Lambda^{d-4}$$

Change variable of integration $x = bx'$

$$\int d^d x' \; b^{d-2} \, (\nabla \bar{\phi})^2 \; ---$$

$$\bar{\phi} \to b^{\frac{-d+2}{2}} \bar{\phi} = b^{-\frac{d}{2}+1} \bar{\phi}$$

$$E = \int d^d x \left\{ \frac{1}{2} (\nabla \bar{\phi})^2 + \frac{1}{2} r_0 \, b^{\cancel{d}-\cancel{d}+2} \bar{\phi}^2 + \right.$$

$$+ u_0 \, b^{d-2d+4} \bar{\phi}^4 + \underbrace{---}_{\text{same.}}$$

$$b^2 = (1+\varepsilon)^2 = 1+2\varepsilon = 1+2\ln b$$

$$b^{-d+4} = 1 + (4-d)\ln b$$

$$r_0' \simeq r_0 + 2\ln b \; r_0 + \frac{c_d}{(2\pi)^d} \ln b \; 12 (u_0 \Lambda^{d-2} - r_0 u_0 \Lambda^{d-4})$$

$$u_0' \simeq u_0 + (4-d)\ln b \, u_0 - \left(\frac{c_d}{(2\pi)^d}\right) \ln b \; 36 u_0^2 \Lambda^{d-4}$$

$\tau = \ln b$ small.

$$\frac{r_0' - r_0}{\tau} \to \frac{d r_0}{d\tau} = 2 r_0 + \frac{12 c_d}{(2\pi)^d} (u_0 \Lambda^{d-2} - u_0 r_0 \Lambda^{d-4})$$

$$\frac{d u_0}{d\tau} = (4-d) u_0 - \left(\frac{c_d}{(2\pi)^d}\right) 36 u_0^2 \Lambda^{d-4}$$

$x = \alpha r_0 \qquad y = \beta u_0$

$$\frac{dx}{d\tau} = 2x + 12\frac{c_d}{(2\pi)^d}\left(\alpha \frac{y}{\beta}\Lambda^{d-2} - \frac{y}{\beta} x \Lambda^{d-4}\right)$$

$$\frac{dy}{d\tau} = (4-d)\,y - \frac{c_d}{(2\pi)^d}\frac{36}{\beta}\beta\, y^2 \Lambda^{d-4}$$

$$\beta = \frac{c_d \Lambda^{d-4}}{(2\pi)^d} \quad ; \quad \alpha = 1/\Lambda^2$$

$$\frac{dx}{d\tau} = 2x + \frac{12\cancel{c_d}}{\cancel{(2\pi)^d}}\frac{\cancel{(2\pi)^d}}{\cancel{c_d}\Lambda^{d-4}}\left(\alpha y \Lambda^{d-2} - xy\Lambda^{d-4}\right)$$

$\alpha y \Lambda^2 \qquad \alpha = 1/\Lambda^2$

$$\frac{dy}{d\tau} = (4-d)\,y - 36y^2$$

$$\frac{dx}{d\tau} = 2x + 12\,(y - xy)$$

$4-d=\epsilon$ small

$$\begin{cases} \dfrac{dx}{dz} = 2x + 12y - 12xy \\ \dfrac{dy}{dz} = \epsilon y - 36y^2 \end{cases}$$

fixed points. ; $x = y = 0$.

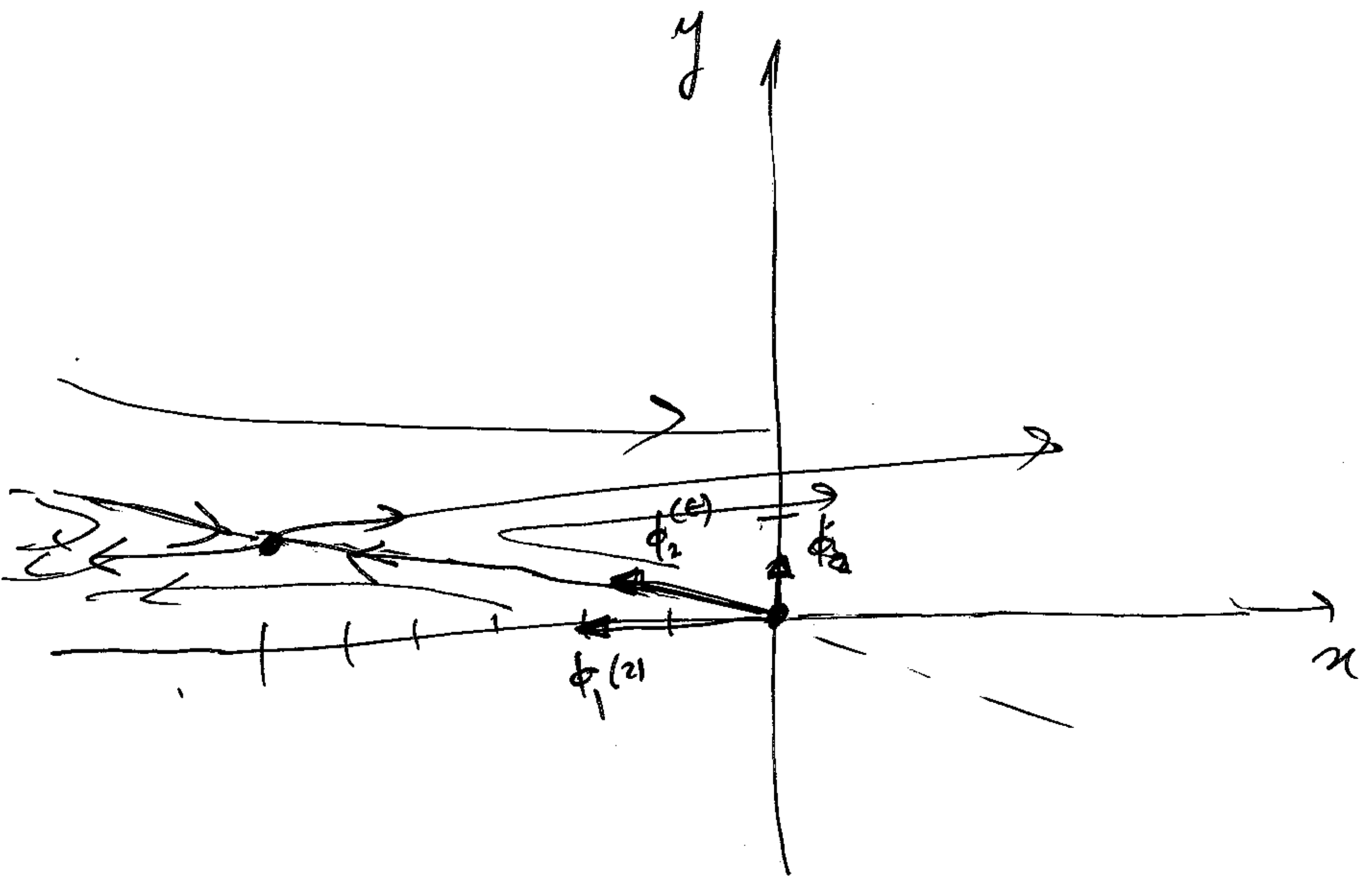
and $y(\epsilon - 36y)$ $\boxed{y^* = \epsilon/36}$

$$\cancel{2}x^* + \overset{6}{\cancel{12}}(1-x^*)\,y^* = 0$$

$$x^* + 6y^* - 6x^*y^* = 0$$

$$x^*(1-6y^*) = -6y^* \qquad x^* = -\frac{6y^*}{1-6y^*} \simeq -6y^*$$

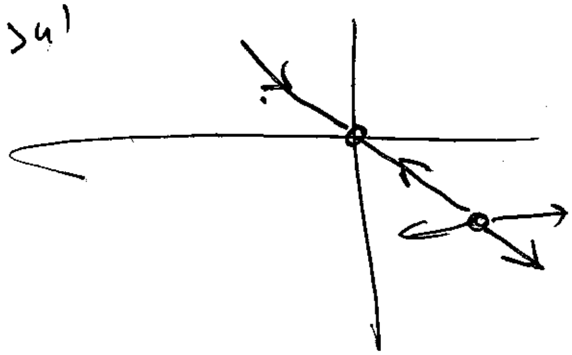
$$x^* = -\epsilon/6$$



$$\frac{dy}{d\tau} = y\,(\epsilon - 36y)$$

$$\frac{dx}{d\tau} = 2x\,(1-6y) + 12y$$

If $\epsilon < 0$ $(d > 4)$



near $x = y = 0$

$$\frac{dy}{dz} = \epsilon y \qquad \frac{dx}{dz} = 2x + 12y$$

$$\frac{d}{dz}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 12 \\ 0 & \epsilon \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$(v_1 \quad v_2)\begin{pmatrix} 2 & 12 \\ 0 & \epsilon \end{pmatrix} = \lambda (v_1 \quad v_2)$$

$$(2v_1 \quad 12v_1 + \epsilon v_2) = (\lambda v_1 \quad \lambda v_2)$$

$$\begin{cases} (2-\lambda) v_1 = 0 \\ 12 v_1 + (\epsilon - \lambda) v_2 = 0 \end{cases}$$

$\lambda = 2 \qquad v_2 = -\dfrac{12 v_1}{\epsilon - 2} \simeq 6 v_1$

$(1, 6)$

$\lambda \neq 2 \qquad v_1 = 0 \qquad \underline{\lambda = \epsilon} \qquad (0, 1)$

$$\phi_1 = (1, 6)\begin{pmatrix} x \\ y \end{pmatrix} = x + 6y$$

$$\phi_2 = y$$

$$\frac{d\phi_1}{dz} = 2x + 12y + 6\epsilon y \simeq 2(x + 6y) = 2\phi_1$$

$$\frac{d\phi_2}{dz} = \epsilon \phi_2$$

Near $x^* = -6y^*$ $\quad y^* = \epsilon/36$ ⑪

$x = x^* + \delta x$

$y = y^* + \delta y$

$$\frac{d\,\delta y}{d\tau} = y^* (\cancel{\epsilon - 36 y^*} - 36\delta y)$$

$$= -36 y^* \delta y = -\epsilon \delta y$$

$$\frac{d\,\delta x}{d\tau} = 2(x^* + \delta x)(1 - 6y^* - 6\delta y) + 12 y^* + 12\delta y$$

$$= 2(\cancel{-6y^*} + \delta x)(1 - 6y^* - 6\delta y) + \cancel{12y^*} + 12\delta y$$

$$= \cancel{2\delta x (1 - 6y^*) - 12 y^*}$$

$$= (-12 y^* + 2\delta x)(1 - 6y^* - 6\delta y) + 12 y^* + 12\delta y$$

$$= -12 y^* (1 - 6y^*) + 72\,\delta y\, y^* + 2\delta x (1 - 6y^*)$$

$$+ 12 y^* + 12\delta y$$

$$= -\cancel{12y^*} + \cancel{72 y^{*2}}^{\epsilon^2} + \cancel{72 y^* \delta y} + 2(1 - \cancel{6y^*})\delta x$$

$$+ \cancel{12y^*} + 12\delta y$$

$$= 2\delta x + 12\delta y = 2(\delta x + 6\delta y)$$

$$\frac{d\delta x}{dz} = 2(\delta x + 6\delta y)$$

$$\frac{d\delta y}{dz} = -\epsilon\,\delta y$$

same w/ $\epsilon \to -\epsilon$

$$\phi_1 = \delta x + 6\delta y \qquad \frac{d\phi_1}{dz} = 2\phi_1$$

$2-\epsilon/3$

$$\phi_2 = \delta y \qquad \frac{d\phi_2}{dz} = -\epsilon\,\phi_2$$

$$\frac{d\phi_1}{dz} = \cancel{2(\delta x + 6\delta y) - 6\epsilon\,\delta y}$$