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Notes on string theory

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Abstract: These are introductory notes to string theory.

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2.1 Action and equations of motion

A convenient method to describe a physical system is through a principle of minimal action. Given an initial and final state for the system, a number, the action, is assigned to every possible trajectory. The classical trajectory that the system follows is the one which has minimal (or extremal) action. For example Newton's equation for a particle in an external potential follows from the action:

$$S = \int dt \, \frac{1}{2} m \left(\frac{d\vec{x}}{dt}\right)^2 - V(\vec{x}) \tag{2.1}$$

and the equations of motion for a relativistic particle in an external electromagnetic field A_μ from the action:

$$S = m \int ds + q \int A_\mu(x) dx^\mu \tag{2.2}$$

Exercise Verify that extremizing eq.(2.1) gives Newton's equation

$$m\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}V(x) \tag{2.3}$$

and that extremization of eq.(2.2) gives the (relativistic) equation of motion for a particle (see appendix for notation) in an electromagnetic field

$$\frac{d}{dt}\frac{m\vec{v}}{\sqrt{1-v^2}} = q\left(\vec{E} + v \times \vec{B}\right), \tag{2.4}$$

where

$$\vec{E} = -\vec{\nabla}\phi + \frac{\partial \vec{A}}{\partial t}, \qquad B = \vec{\nabla} \times \vec{A}. \tag{2.5}$$

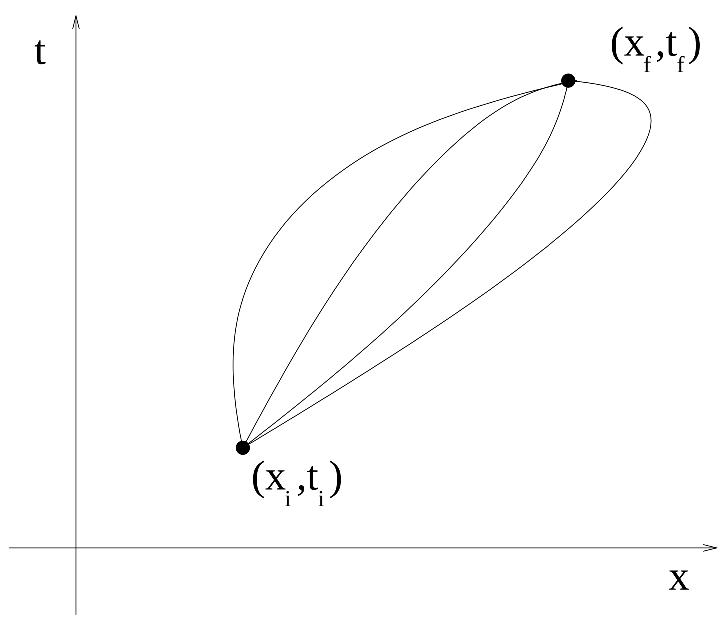

Figure 1: From all possible trajectories, the particle follows the one with minimal action.

Now we would like to find a principle of minimal action for a string. Given an initial and final shape for the string, all possible trajectories are surfaces that describe the motion of the initial shape into the final one. Topologically these surfaces are cylinders. To each of them we want to assign a number. A natural suggestion that generalizes the action of a particle is to assign to each trajectory, the area of the corresponding surface. A way to understand the physical meaning of this is to divide the string in portions of fixed length and consider each portion as a particle of mass proportional to the length. Using eq.(2.2) (with no external field) we would get the total action as the area of the surface.

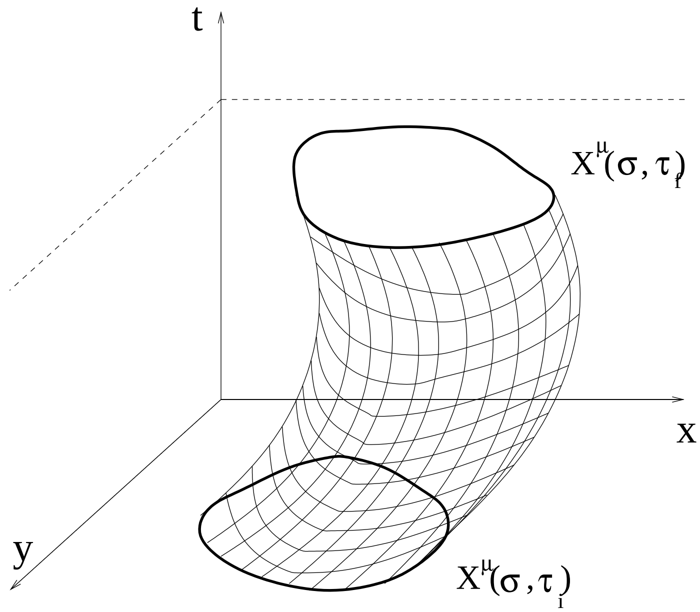

Figure 2: A string also follows the path of minimal action.

We should now get a concrete formula for the area. The surface is parameterized by two coordinates that are usually called $(\sigma,\ \tau)$. The surface is given by functions $X^\mu(\sigma,\tau)$. The coordinate σ runs between 0 and 2π and the functions X^μ are periodic in σ. The parameter τ runs between and initial τ_i and a final τ_f value. The functions $X^\mu(\sigma,\tau)$ satisfy:

$$X^\mu(\sigma,\tau_i) = X^\mu_i(\sigma), \quad X^\mu(\sigma,\tau_f) = X^\mu_f(\sigma), \quad X^\mu(\sigma+2\pi,\tau) = X^\mu(\sigma,\tau) \tag{2.6}$$

where $X^\mu_i(\sigma)$ and $X^\mu_f(\sigma)$ are the (arbitrary) initial and final shape of the string.

To compute the area of the surface determined by the functions $X^\mu(\sigma,\tau)$ we use standard analysis with the caveat that the space time metric is Minkowski. To do that consider first the standard Euclidean case, where we embed a surface in R^n. In that case, we compute the area by making a grid in σ, τ, sum the areas of all the small rectangles and then taking the limit of the size of the grid going to zero. If we are at a given point $X^\mu(\sigma,\tau)$ and change σ by $d\sigma$, the position is going to change by $dX^\mu_1 = \partial_\sigma X^\mu d\sigma$ and if we move in τ by $dX^\mu_2 = \partial_\tau X^\mu d\tau$. The area of the corresponding parallelogram (see fig.3) is

$$dA = |dX_1||dX_2| \sin\theta_{12} \tag{2.7}$$

where θ_{12} is the angle between the two vectors dX_1 and dX_2. The simplest way to compute the angle is to use the scalar product:

$$dX_1.dX_2 = |dX_1||dX_2| \cos\theta_{12} \tag{2.8}$$

which then gives

$$(dA)^2 = |dX_1|^2|dX_2|^2 - (dX_1.dX_2)^2 \tag{2.9}$$

The area can then be computed as

$$A = \int d\sigma\, d\tau \sqrt{(\partial_\sigma X)^2(\partial_\tau X)^2 - (\partial_\sigma X.\partial_\tau X)^2} \tag{2.10}$$

Exercise Verify that for a sphere parameterized as:

$$X_1 = R\sin\theta\cos\phi \tag{2.11}$$
$$X_2 = R\sin\theta\sin\phi \tag{2.12}$$
$$X_3 = R\cos\theta \tag{2.13}$$

the previous formula gives the standard result for the area.

The generalization to Minkowski space is simply to consider the scalar products with the Minkowski metric, namely

$$\partial_\sigma X.\partial_\tau X = -\partial_\sigma X_0\partial_\tau X_0 + \partial_\sigma X_1\partial_\tau X_1 + \partial_\sigma X_2\partial_\tau X_2 + \partial_\sigma X_3\partial_\tau X_3 \tag{2.14}$$

and the same with the other ones. The string action is then

$$S = \frac{1}{2\pi\alpha'} \int d\sigma\, d\tau \sqrt{(\partial_\sigma X.\partial_\tau X)^2 - (\partial_\sigma X)^2(\partial_\tau X)^2} \tag{2.15}$$

where we changed signs inside the square root because otherwise, in Minkowski space, the area would be imaginary. Also we included a constant α' with units of length squared to make the action adimensional. Anticipating that we are going to quantize the string, we take $\hbar = 1$. Usually the action has the same units as $\hbar$ and since $\hbar = 1$ now is adimensional. The physical interpretation of $1/\alpha'$, as we see later, is the tension of the string. Namely, a string of length L has a mass $L/alpha'$.

After having obtained the action we have to find the equations of motion that describe the trajectory that minimizes it. Let us derive those equations in general. Suppose we have an action

$$S = \int d\sigma d\tau \mathcal{L}(X^\mu, \partial_\sigma X^\mu, \partial_\tau X^\mu) \tag{2.16}$$

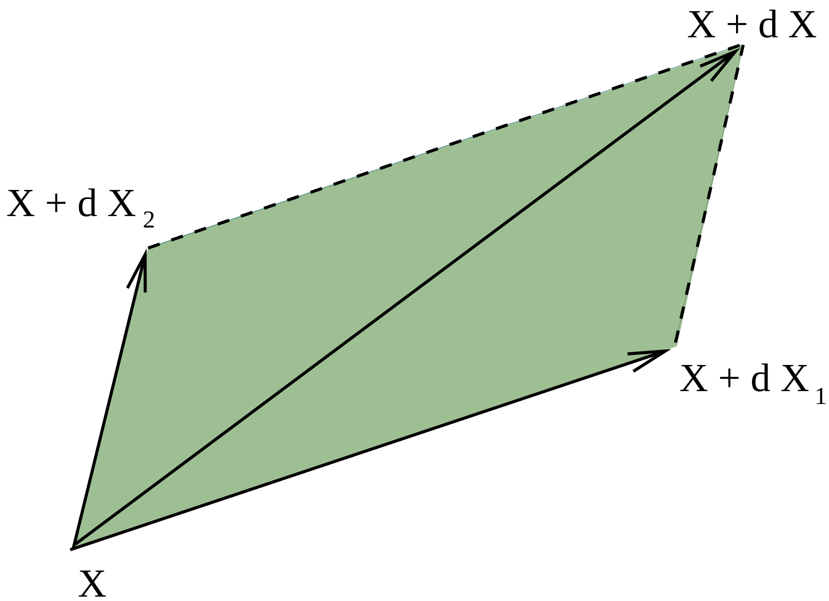

Figure 3: Computation of area element.

where $\mathcal{L}$, the Lagrangian is an arbitrary function of X^μ and its first derivatives. Let us say now that $\bar{X}^\mu(\sigma,\tau)$ are the, as yet undetermined functions, that minimize the action. The statement means that, if we perform a first order variation $\bar{X} \to \bar{X} + \delta X$ then, at first order the action remains stationary. This assumes that the variations respect the boundary conditions, namely:

$$\delta X^\mu(\sigma,\tau_i) = 0, \quad \delta X^\mu(\sigma,\tau_f) = 0, \quad \delta X^\mu(\sigma+2\pi,\tau) = \delta X^\mu(\sigma,\tau) \tag{2.17}$$

Formally, we have

$$\begin{aligned} S = \bar{S} &+ \int d\sigma\, d\tau\, \frac{\delta S}{\delta X^\mu(\sigma,\tau)} \delta X^\mu(\sigma,\tau) \\ &+ \frac{1}{2}\int d\sigma_1\, d\tau_1 \int d\sigma_2\, d\tau_2\, \frac{\delta^2 S}{\delta X^\mu(\sigma_1,\tau_1)\delta X^\nu(\sigma_2,\tau_2)} \delta X^\mu(\sigma_1,\tau_1)\delta X^\nu(\sigma_2,\tau_2) + \ldots \end{aligned} \tag{2.18}$$

If $\frac{\delta S}{\delta X^\mu}$ is not zero then performing different variations δX^μ we can get the action to increase or decrease at will, namely we cannot be at a minimum. The first variation is computed from

$$\delta S = \int d\sigma d\tau \mathcal{L}(\bar{X}^\mu + \delta X^\mu, \partial_\sigma \bar{X}^\mu + \partial_\sigma \delta X^\mu, \partial_\tau \bar{X}^\mu + \partial_\tau \delta X^\mu) - \int d\sigma d\tau \mathcal{L}(\bar{X}^\mu, \partial_\sigma \bar{X}^\mu, \partial_\tau \bar{X}^\mu) \tag{2.19}$$

where only first order terms in the variation are to be kept. A little algebra leads to

$$\delta S = \int d\sigma d\tau \frac{\partial \mathcal{L}}{\partial X^\mu}\delta X^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\sigma X^\mu)}\partial_\sigma \delta X^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)}\partial_\tau \delta X^\mu \tag{2.20}$$

In the last two terms it is convenient to integrate by parts. This gives rise to a boundary term in the integral in τ (not in σ since that coordinate is periodic). In total we get

$$\delta S = \int d\sigma d\tau \left\{ \frac{\partial \mathcal{L}}{\partial X^\mu} - \partial_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\sigma X^\mu)} - \partial_\tau \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)} \right\} \delta X^\mu + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)}\delta X^\mu \right]\Bigg|_i^f \tag{2.21}$$

Since the position of the string at the initial and final times is fixed we consider, as mentioned, only variations such that $\delta X^\mu(\sigma, \tau_{i,f}) = 0$ which eliminates the boundary term. We get in the end:

$$\frac{\delta S}{\delta X^\mu} = \frac{\partial \mathcal{L}}{\partial X^\mu} - \partial_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\sigma X^\mu)} - \partial_\tau \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)} = 0 \tag{2.22}$$

with

$$\mathcal{L}(\bar{X}_\mu, \partial_\sigma \bar{X}^\mu, \partial_\tau \bar{X}^\mu) = \sqrt{(\partial_\sigma X.\partial_\tau X)^2 - (\partial_\sigma X)^2(\partial_\tau X)^2} \tag{2.23}$$

These gives the equations of motion:

$$\partial_\sigma \left\{ \frac{(\partial_\sigma X.\partial_\tau X)\partial_\tau X^\mu - (\partial_\tau X)^2 \partial_\sigma X^\mu}{\sqrt{(\partial_\sigma X.\partial_\tau X)^2 - (\partial_\sigma X)^2(\partial_\tau X)^2}} \right\} + \partial_\tau \left\{ \frac{(\partial_\sigma X.\partial_\tau X)\partial_\sigma X^\mu - (\partial_\sigma X)^2 \partial_\tau X^\mu}{\sqrt{(\partial_\tau X.\partial_\sigma X)^2 - (\partial_\tau X)^2(\partial_\sigma X)^2}} \right\} = 0 \tag{2.24}$$

one for each value of μ.

These equations determine the surface of minimal area. Consider now an example of a solution to these equations. Consider a string moving in a plane (x, y) and propose a solution[1]:

$$t = \kappa\tau \tag{2.25}$$

$$x = \kappa \sin\sigma \cos\tau \tag{2.26}$$

$$y = \kappa \sin\sigma \sin\tau \tag{2.27}$$

To understand the shape of the string it is convenient to consider the complex coordinate $x + iy = \kappa \sin\sigma e^{i\tau}$ which shows that the string extends in the radial direction to a distance κ of the center and is folded over itself (remember $0 \le \sigma \le 2\pi$). The

[1]Here we give the solution, later we are going to learn how to find such solutions

dependence in τ (which is identified with time up to a constant κ) is simply a rotation in the (x, y) plane. We can compute now:

$$X^\mu = (t, x, y) \tag{2.28}$$
$$\partial_\tau X^\mu = (\kappa, -\kappa \sin\sigma \sin\tau, \kappa \sin\sigma \cos\tau) \tag{2.29}$$
$$\partial_\sigma X^\mu = (0, \kappa \cos\sigma \cos\tau, \kappa \cos\sigma \sin\tau) \tag{2.30}$$
$$(\partial_\tau X)^2 = -\kappa^2 + \kappa^2 \sin^2\sigma = -\kappa^2 \cos^2\sigma \tag{2.31}$$
$$(\partial_\sigma X)^2 = \kappa^2 \cos^2\sigma \tag{2.32}$$
$$(\partial_\tau X.\partial_\sigma X) = 0 \tag{2.33}$$

which implies

$$\sqrt{(\partial_\tau X.\partial_\sigma X)^2 - (\partial_\sigma X)^2(\partial_\tau X)^2} = \kappa^2 \cos^2\sigma \tag{2.34}$$

The equations of motion (2.24) reduce to

$$(\partial_\sigma^2 - \partial_\tau^2) X^\alpha = 0 \tag{2.35}$$

for $\alpha = 0, 1, 2$. This is just the usual wave equation and it can easily be seen that the functions in (2.27) satisfy it.

Another example appears when we consider one coordinate to be periodic:

$$x \equiv x + 2\pi R \tag{2.36}$$

for some radius R. This is an example of compactification of a spatial dimension, something that we are going to use later. Here we can see it as a trick to get a simple solution. In that case we can take

$$t = \kappa\tau \tag{2.37}$$
$$x = \sigma R \tag{2.38}$$

and is easy to verify that all equations are satisfied. Note that we have to take the coordinate x to be periodic to respect the periodicity in σ.

To finalize let us mention that one possible but unrelated application is to find the shape of a film of soap attached to a given contour. In that case, in a static configuration, the film of soap minimizes the energy which is the area times the tension of the film (as opposed to the string where the surface is in space-time and we minimize the action).

Exercise Consider two circles $x^2 + y^2 = R^2, z = -a$ and $x^2 + y^2 = R^2, z = a$ and suppose they represent two rings which are the boundary of a film of soap. Find the shape of the film assuming it is the surface of minimal area. **Hint**: by rotational symmetry parameterize the surface as: $x = f(\tau)\cos\sigma$, $y = f(\tau)\sin\sigma$, $z = \tau$ and use the equations of motion that we derived to obtain $f(\tau)$.

2.2 Noether's theorem and conserved quantities

For a free string, we expect that the total energy and momentum are conserved. We can now derive this by using a general procedure due to E. Noether. We start first by noticing that, although we consider $\mathcal{L}$ to be a generic function, in reality, from (2.15) we see that it is independent of X^μ, namely it only depends on the derivatives. This means that the equations of motion are:

$$\partial_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\sigma X^\mu)} + \partial_\tau \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)} = 0 \tag{2.39}$$

Integrating in σ and dropping the boundary terms we get that

$$\partial_\tau \int d\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)} = 0 \tag{2.40}$$

namely

$$P_\mu = \int d\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)} \tag{2.41}$$

are conserved quantities: $\partial_\tau P_\mu = 0$. The component P_0 is naturally identified with the energy and the spatial component P_i with the total center of mass momentum. Using the action (2.15) we get

$$P_\mu = \frac{1}{2\pi\alpha'} \int d\sigma \frac{(\partial_\sigma X.\partial_\tau X)\eta_{\mu\alpha}\partial_\sigma X^\alpha - (\partial_\sigma X)^2 \eta_{\mu\alpha}\partial_\tau X^\alpha}{\sqrt{(\partial_\tau X.\partial_\sigma X)^2 - (\partial_\tau X)^2(\partial_\sigma X)^2}} \tag{2.42}$$

Example Consider the solution (2.27) we checked in the previous section. Using formula (2.42) we can compute its energy and momentum resulting in

$$P_\mu = -\eta_{\mu\alpha}\frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\, \partial_\tau X^\alpha \tag{2.43}$$

The only non-vanishing integral is for $\mu = 0$ (energy) and gives:

$$E = P_0 = \frac{\kappa}{\alpha'} \tag{2.44}$$

That the momentum (P_i) is zero, is not surprising since the center of mass is at rest.

The second example (2.38) gives

$$P_\mu = -\frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \partial_\tau X^\alpha \tag{2.45}$$

Again, only P_0 is non-zero and its value is

$$P_0 = \frac{R}{\alpha'} \tag{2.46}$$

That means that the energy of a stretched string is proportional to the length. The proportionality constant is the tension $1/\alpha'$.

The action is also invariant under Lorentz transformations, namely linear transformations of the form

$$\tilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu \tag{2.47}$$

that leave the scalar product invariant. Consider an infinitesimal Lorentz transformation given by

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon\, \eta^{\mu\alpha}\, \omega_{\alpha\nu}. \tag{2.48}$$

where ω is antisymmetric (see appendix). Since the Lagrangian is written in terms of scalar products it is invariant under these transformations, namely:

$$\mathcal{L}(X_\mu, \partial_\sigma X^\mu, \partial_\tau X^\mu) = \mathcal{L}(\tilde{X}_\mu, \partial_\sigma \tilde{X}^\mu, \partial_\tau \tilde{X}^\mu) \tag{2.49}$$

At first order in ϵ we have:

$$\tilde{X}^\mu = X^\mu + \epsilon \eta^{\mu\alpha} \omega_{\alpha\nu} X^\nu \tag{2.50}$$

and the same for $\partial_{\sigma,\tau} X^\mu$ since ω is independent of σ and τ. The fact that the Lagrangian is invariant implies

$$\frac{\partial \mathcal{L}}{\partial X^\mu} \eta^{\mu\alpha} \omega_{\alpha\nu} X^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\sigma X^\mu)} \eta^{\mu\alpha} \omega_{\alpha\nu} \partial_\sigma X^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)} \eta^{\mu\alpha} \omega_{\alpha\nu} \partial_\tau X^\nu = 0 \tag{2.51}$$

Using the equations of motion we obtain

$$\partial_\sigma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\sigma X^\mu)} \eta^{\mu\alpha} \omega_{\alpha\nu} X^\nu \right) + \partial_\tau \left(\frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)} \eta^{\mu\alpha} \omega_{\alpha\nu} X^\nu \right) = 0 \tag{2.52}$$

which, as before, implies the conservation of:

$$M_\omega = \int d\sigma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)} \eta^{\mu\alpha} \omega_{\alpha\nu} X^\nu \right) \tag{2.53}$$

This is true for any antisymmetric omega. Looking at each independent component of ω we obtain the conserved quantities:

$$M^{\mu\nu} = \frac{1}{2} \int d\sigma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\alpha)} \eta^{\alpha\mu} X^\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\alpha)} \eta^{\alpha\nu} X^\mu \right) \tag{2.54}$$

which is the angular momentum of the string.

Example Going back to our example of the rotating string, we get, after all the simplifications due to the form of the solution:

$$M^{\mu\nu} = \frac{1}{4\pi\alpha'} \int d\sigma \, (X^\mu \partial_\tau X^\nu - X^\nu \partial_\tau X^\mu) \tag{2.55}$$

The integral is zero for M^{01} and M^{02} since the integrands are proportional to $\sin\sigma$ or $\cos\sigma$. The only non-zero one is

$$M_{12} = J = \frac{\kappa^2}{2\pi\alpha'}\int_0^{2\pi}\sin^2\sigma d\sigma = \frac{\kappa^2}{4\alpha'} \tag{2.56}$$

which is conventionally denoted as J. Using the result we already have for the energy (2.44) we obtain the important relation

$$P_0 = E = \frac{2}{\sqrt{\alpha'}}\sqrt{J} \tag{2.57}$$

that is, a linear relation between the energy squared and the angular momentum . It turns out that mesons obey, to a certain approximation, such relation between their mass squared and their spin (a law known as Regge trajectories). This was one of the origins of string theory as a model for hadrons.

We can obtain other two conservation laws from the fact that the Lagrangian does not depend explicitly on σ and τ. Namely, given a solution of the equations of motion $X^\mu(\sigma,\tau)$, after replacing in the Lagrangian we get $\mathcal{L}$ as a function of (σ,τ). We have:

$$\frac{d\mathcal{L}}{d\sigma} = \frac{\partial\mathcal{L}}{\partial\sigma} + \frac{\partial\mathcal{L}}{\partial X^\mu}\partial_\sigma X^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\sigma X^\mu)}\partial_\sigma^2 X^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\tau X^\mu)}\partial_{\sigma\tau}X^\mu \tag{2.58}$$

The first term is zero $\frac{\partial\mathcal{L}}{\partial\sigma} = 0$ and the others can be simplified using the equations of motion to give:

$$\frac{d\mathcal{L}}{d\sigma} = \partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial(\partial_\sigma X^\mu)}\partial_\sigma X^\mu\right) + \partial_\tau\left(\frac{\partial\mathcal{L}}{\partial(\partial_\tau X^\mu)}\partial_\sigma X^\mu\right) \tag{2.59}$$

Integrating in σ we obtain the conservation law:

$$\partial_\tau P_\sigma = \partial_\tau\int d\sigma\left(\frac{\partial\mathcal{L}}{\partial(\partial_\sigma X^\mu)}\partial_\sigma X^\mu\right) = 0 \tag{2.60}$$

Doing the same for τ we obtain:

$$\frac{d\mathcal{L}}{d\tau} = \partial_\tau\left(\frac{\partial\mathcal{L}}{\partial(\partial_\tau X^\mu)}\partial_\tau X^\mu\right) + \partial_\sigma\left(\frac{\partial\mathcal{L}}{\partial(\partial_\sigma X^\mu)}\partial_\tau X^\mu\right) \tag{2.61}$$

Integrating in σ we get now

$$\partial_\tau P_\tau = \partial_\tau\int d\sigma\left(\frac{\partial\mathcal{L}}{\partial(\partial_\tau X^\mu)}\partial_\tau X^\mu - \mathcal{L}\right) = 0 \tag{2.62}$$

Since we already obtained the conservation of energy, momentum and angular momentum, it is not clear what these new conserved quantities $P_{\sigma,\tau}$ could be. To find out we replace the Lagrangian (2.15) that we had and actually find that

$$\frac{\partial \mathcal{L}}{\partial(\partial_\sigma X^\mu)}\partial_\sigma X^\mu = 0 \tag{2.63}$$

and

$$\frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)}\partial_\tau X^\mu - \mathcal{L} = 0 \tag{2.64}$$

so $P_{\sigma,\tau} = 0$ and no new conservation laws appear. This is actually very important and is related to the fact that the action is invariant under reparameterizations of σ and τ. That is, the area of the surface does not depend on how we parameterize it.

2.3 Static, conformal and light-cone gauges

The equation of motion as they stand are rather complicated. However, as we mentioned, the action is invariant under reparameterizations of (σ, τ). A judicious choice of coordinates can simplify the equations. Now we are going to see several such choices. In string theory, a choice of coordinates on the world-sheet is usually called a gauge choice and hence the name of this section.

The first choice is static gauge. In this gauge we identify two space time coordinates with σ and τ. For example we can choose:

$$X^0 = \tau, \quad X^1 = \sigma \tag{2.65}$$

This reduces the number of equations that we need to solve. However, we notice that the solutions are not general, for example already, to choose such a gauge we need to have that X^1 is a periodic coordinate, otherwise the string will not be closed. A more generic choice is to use other space-time coordinates, for example spherical or cylindrical and then identify one of those coordinates with σ. We will see examples of that below.

Let us consider another common choice called conformal gauge. First compute the distance between two points on the world-sheet which are very close to each other. Let us say we have point $X(\sigma, \tau)$ and $X(\sigma + d\sigma, \tau + d\tau)$. The distance between those two points is:

$$ds^2 = dX^\mu dX^\nu \eta_{\mu\nu} = (\partial_\sigma X^\mu d\sigma + \partial_\tau X^\mu d\tau)(\partial_\sigma X^\nu d\sigma + \partial_\tau X^\nu d\tau)\eta_{\mu\nu} \tag{2.66}$$

$$= (\partial_\sigma X)^2 d\sigma^2 + (\partial_\tau X)^2 d\tau^2 + 2(\partial_\sigma X . \partial_\tau X) d\sigma d\tau \tag{2.67}$$

This distance is called the induced metric on the world sheet. It can be written in the generic form:

$$ds^2 = h_{\sigma\sigma} d\sigma^2 + h_{\tau\tau} d\tau^2 + 2h_{\sigma\tau} d\sigma d\tau \tag{2.68}$$

where

$$h_{\sigma\sigma} = (\partial_\sigma X)^2, \quad h_{\tau\tau} = (\partial_\tau X)^2, \quad h_{\sigma\tau} = (\partial_\sigma X.\partial_\tau X) \tag{2.69}$$

If we redefine (σ, τ) then, although the distance is the same, the components of the metric $h_{\sigma,\sigma}$, $h_{\tau\tau}$ and $h_{\sigma\tau}$ change. Since we have two functions to choose, namely the two new coordinates as a function of the old ones, we can put the metric in a form which contains only one arbitrary function. In fact it is possible to prove that the metric can always be put in the form:

$$ds^2 = e^\phi \left(d\sigma^2 - d\tau^2\right) \tag{2.70}$$

In such gauge we have

$$h_{\sigma\tau} = (\partial_\sigma X.\partial_\tau X) = 0 \tag{2.71}$$

$$h_{\sigma\sigma} + h_{\tau\tau} = (\partial_\sigma X)^2 + (\partial_\tau X)^2 = 0 \tag{2.72}$$

This simplifies the equations of motion enormously because, from (2.24) they reduce to:

$$(\partial_\sigma^2 - \partial_\tau^2)X^\mu = 0 \tag{2.73}$$

The most generic solution to these equations is

$$X^\mu = X_L(\sigma + \tau) + X_R(\sigma - \tau) \tag{2.74}$$

with $X_{L,R}$ two arbitrary functions describing left and right moving waves. It appears that we have solved the problem completely but that is not the case. For our purpose these functions are not arbitrary, they have to satisfy the constraints (2.72). This makes the problem complicated again but in a different way. Which gauge, static or conformal is more convenient depends on which problem we have to solve.

There is a further refinement of conformal gauge which is the light-cone gauge. Notice that we can introduce world-sheet coordinates $\sigma_\pm$ defined as

$$\sigma_\pm = \sigma \pm \tau \tag{2.75}$$

in terms of which the reference metric can be written as

$$ds^2 = e^{\phi(\sigma,\tau)} d\sigma_+ d\sigma_- \tag{2.76}$$

It is obvious now that if we make a coordinate change

$$\tilde{\sigma}_+ = \tilde{\sigma}_+(\sigma_+), \qquad \tilde{\sigma}_- = \tilde{\sigma}_-(\sigma_-) \tag{2.77}$$

the metric transforms as

$$ds^2 = e^\phi \frac{d\sigma_+}{d\tilde{\sigma}_+}\frac{d\sigma_-}{d\tilde{\sigma}_-} d\tilde{\sigma}_+ d\tilde{\sigma}_- \tag{2.78}$$

which has the same form. This means that our choice of gauge does not fix the coordinates completely. However we are now allowed to choose two functions of one variable (as opposed to two functions of two variables as we had before). It is convenient at this point to also choose light-cone coordinates in space time by defining:

$$X^{\pm} = X^0 \pm X^1 \tag{2.79}$$

In the world-sheet there is only one spacial coordinate to do this, but in space time we need to single out one particular coordinate (in this case X^1) making Lorentz invariance less explicit. The equations of motion for these coordinates is the same as before:

$$(\partial_\sigma^2 - \partial_\tau^2)X^{\pm} = 0 \tag{2.80}$$

Consider X^+. The generic solution of the equation of motion is

$$X^+ = X_L^+(\sigma_+) + X_R^+(\sigma_-) \tag{2.81}$$

We can define now new coordinates $\tilde{\sigma}_+ = X_L^+(\sigma_+)$, $\tilde{\sigma}_- = -X_R^+(\sigma_r)$ such that

$$X^+ = \tilde{\sigma}_+ - \tilde{\sigma}_- = 2\tilde{\tau} \tag{2.82}$$

This means that we can fix the last ambiguity by choosing

$$X^+ = 2\tau \tag{2.83}$$

This gauge is called light cone gauge. It is usually more convenient when studying the quantum theory as we will see later.

2.4 Strings in curved space

We saw that the metric on the world-sheet is determined by its embedding in space time. Suppose now that space time itself has a non-trivial metric. For example we can consider that the string is constrained so stay in the surface of a two-sphere parameterized by polar angles (θ, ϕ) as:

$$x = R\sin\theta\cos\phi \tag{2.84}$$

$$y = R\sin\theta\sin\phi \tag{2.85}$$

$$z = R\cos\theta \tag{2.86}$$

The distance between two points at (θ, ϕ) and $(\theta + d\theta, \phi + d\phi)$ is

$$dx^2 + dy^2 + dz^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.87}$$

If we include time the total space time metric is given by:

$$ds^2 = -dt^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.88}$$

Generically we can have a space time metric given by

$$ds^2 = G_{\mu\nu}(X)dX^\mu dX^\nu \tag{2.89}$$

where $G_{\mu\nu}$ are given functions of the coordinates. If we embed a string in this space we can again compute the action as the area of the surface that the string describes when it moves. Suppose we are in the case we mentioned before, namely that the string is moving on the surface of a sphere. What we can do is consider the world-sheet of the string as embedded on ordinary space time and compute the area there which gives the result we already know (2.15). Now we can compute the same scalar products in coordinates (θ, ϕ). For example:

$$\begin{aligned}(\partial_\sigma X)^2 &= -(\partial_\sigma t)^2 + (\partial_\sigma x)^2 + (\partial_\sigma y)^2 + (\partial_\sigma z)^2 && (2.90)\\ &= -(\partial_\sigma t)^2 + R^2\left[(\partial_\sigma\theta)^2 + \sin^2\theta(\partial_\sigma\phi)^2\right] && (2.91)\\ &= G_{\mu\nu}\partial_\sigma X^\mu \partial_\sigma X^\nu && (2.92)\end{aligned}$$

So, we see that the action is the same, we only need to replace $\eta_{\mu\nu}$ by $G_{\mu\nu}$ in all scalar products. The only difference is that, since $G_{\mu\nu}$ are functions of the coordinates then it is no longer true that $\partial\mathcal{L}/\partial X^\mu = 0$. In particular this implies that momentum and energy are not necessarily conserved. The equations of motion are now

$$\partial_\sigma\left\{\frac{(\partial_\sigma X.\partial_\tau X)\partial_\tau X^\mu - (\partial_\tau X)^2 G_{\mu\alpha}\partial_\sigma X^\alpha}{\sqrt{(\partial_\sigma X.\partial_\tau X)^2 - (\partial_\sigma X)^2(\partial_\tau X)^2}}\right\} + \partial_\tau\left\{\frac{(\partial_\sigma X.\partial_\tau X)\partial_\sigma X^\mu - (\partial_\sigma X)^2\partial_\tau G_{\mu\alpha}X^\alpha}{\sqrt{(\partial_\tau X.\partial_\sigma X)^2 - (\partial_\tau X)^2(\partial_\sigma X)^2}}\right\}$$
$$= \frac{2\partial_\mu G_{\alpha\beta}\partial_\tau X^\alpha \partial_\sigma X^\beta(\partial_\sigma X.\partial_\tau X) - \partial_\mu G_{\alpha\beta}\partial_\tau X^\alpha\partial_\tau X^\beta(\partial_\sigma X)^2 - \partial_\mu G_{\alpha\beta}\partial_\sigma X^\alpha\partial_\sigma X^\beta(\partial_\tau X)^2}{2\sqrt{(\partial_\sigma X.\partial_\tau X)^2 - (\partial_\sigma X)^2(\partial_\tau X)^2}}$$

A final comment is that one can also use the formalism of curved space to study strings in flat space when using non-cartesian coordinates. For example in spherical coordinates we have that the metric is:

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{2.93}$$

and therefore

$$G = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{2.94}$$

For some string configurations it is useful to work with a metric like this one instead of the Cartesian one.

3. Quantum strings and string spectrum

Now we are going to consider the quantum mechanics of a string. Before going into details let us describe the results that we are going to obtain since their general features can be easily understood. First, we expect the string to have a continuous value of the center of mass momentum. On the other hand the internal motion of the string is bounded so, for that part, we expect a discrete spectrum. As we discuss below, the discrete set of internal states is labeled by two infinite sets of non-negative integers which are usually denoted as $(N^i_{m=1\ldots\infty}, \tilde{N}^i_{m=1\ldots\infty})$. A generic state of the string is then given by:

$$|\psi\rangle = |p^\mu, \{N^i_m, \tilde{N}^i_m\}\rangle \tag{3.1}$$

where, as we said p^μ is the space time momentum, and $\{N^i_m, \tilde{N}^i_m\}$ are non-negative integers describing the internal motion of the string. The index i in N^i_m refers to the transverse directions in light-cone gauge and runs from 1 to $(D-2)$ where D is the number of space time dimensions. For reasons that will become apparent later we leave the number of dimensions D arbitrary. The total energy is a function of the momentum and internal energy. It turns out that we find:

$$M^2 = p_0^2 - \vec{p}^2 = \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right) \tag{3.2}$$

where

$$N = \sum_{i=1}^{D-2}\sum_{m=1}^{\infty} m N^i_m, \qquad \tilde{N} = \sum_{i=1}^{D-2}\sum_{m=1}^{\infty} m\tilde{N}_m, \tag{3.3}$$

This means that the internal motion leads to a discrete spectrum of mass. We can therefore think a string as an infinite set of particles, each with a mass given by the formula (3.3)[2]. In the rest of the section we show how to derive the spectrum and the physical meaning of the quantum numbers N^i_m. In order to do that, we have to quantize the string, namely we have to replace the classical quantities by operators obeying canonical commutation relations. In the case of strings, a straight-forward quantization is possible in light-cone gauge since there the variables X^i are independent and their equations of motion are linear. All quantum states are physical states of the string. This is not that case if we do not fix completely the reparameterization symmetry. For example in conformal gauge, many states describe the same physical state of the string. In light-cone gauge, the only problem we face is when studying Lorentz invariance. Since we single out a spacial coordinate, the operators that mix $X^\pm$ with the transverse coordinates X^i have complicated expressions and it is not

[2]As we see later there is a constraint $N = \tilde{N}$ on the states

straight-forward to verify that they behave as they should. In particular, it turns out that Lorentz symmetry is preserved only if the string moves in 26 dimensions!. This is a rather peculiar and fundamental property of strings, namely that they determine the dimension of space-time. Of course it is a bit odd that the result is 26 but nevertheless quite remarkable that only a certain dimension is allowed.

Having discussed the general results, let us now concentrate on their detailed derivation.

3.1 Quantization in light-cone gauge and free string spectrum

As mentioned, quantization is straight-forward in light-cone gauge, so let us revise some formulas. If the space-time dimension is D we have coordinates X^0, ,X^1,..., X^{D-1}. First we introduce two new space-time coordinates:

$$X^{\pm} = X^0 \pm X^{D-1} \tag{3.4}$$

The metric becomes

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu = -dX^+ dX^- + dX^i dX^i, \quad i = 1\ldots(D-2) \tag{3.5}$$

From here we see that, in these coordinates, $\eta_{+-} = \eta_{-+} = -\frac{1}{2}$, $\eta_{ii} = 1$ and the inverse $\eta^{+-}, \eta^{-+} = -2$, $\eta^{ii} = 1$. We now consider conformal gauge. In this gauge there is a residual symmetry that allows us to choose:

$$X^+ = x^+ + \alpha' p^+ \tau \tag{3.6}$$

where we introduced the constant p^+. From the conformal constraints we obtain

$$\partial_\sigma X^- = \frac{2}{\alpha' p^+} \partial_\sigma X^i \partial_\tau X^i \tag{3.7}$$

$$\partial_\tau X^- = \frac{1}{\alpha' p^+} \left[\partial_\sigma X^i \partial_\sigma X^i + \partial_\tau X^i \partial_\tau X^i\right] \tag{3.8}$$

From here we derive a constraint on the X^i:

$$\int_0^{2\pi} d\sigma\, \partial_\sigma X^i \partial_\tau X^i = \frac{\alpha' p^+}{2} \int_0^{2\pi} d\sigma\, \partial_\sigma X^- = 0 \tag{3.9}$$

Otherwise, the X^i are independent and determine completely the dynamics, since X^+ is fixed and X^- is derived from (3.8). Now we use the formulas in conformal gauge to find the momenta:

$$P^\mu = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma\, \partial_\tau X^\mu \tag{3.10}$$

Of particular interest are

$$P^+ = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\,\partial_\tau X^+ = p^+ \tag{3.11}$$

$$P^- = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\,\partial_\tau X^- = \frac{1}{2\pi(\alpha')^2}\int_0^{2\pi} d\sigma\,\left[\partial_\sigma X^i\partial_\sigma X^i + \partial_\tau X^i\partial_\tau X^i\right] \tag{3.12}$$

We should remember that P^- is the conjugate of X^+. Since X^+ and τ are proportional we can also think P^- as the world-sheet Hamiltonian that generates translations in τ. We should also note that

$$P^\pm = P^0 \pm P^1 \tag{3.13}$$

which implies that they are both positive. Finally, we can compute the mass of the string as:

$$M^2 = P^+P^- - P^iP^i = \frac{1}{2\pi(\alpha')^2}\int_0^{2\pi} d\sigma\,\left[\partial_\sigma X^i\partial_\sigma X^i + \partial_\tau X^i\partial_\tau X^i\right] - P^iP^i \tag{3.14}$$

Since P^μ are conserved, so is M^2. Not only that, it is also Lorentz invariant and therefore an important quantity to characterize the motion of the string.

Before discussing the quantization we are going to write the dynamic in terms of normal modes which are then easy to quantize.

3.1.1 Normal modes

The X^i satisfy the equations of motion

$$(\partial_\sigma^2 - \partial_\tau^2)X^i = 0 \tag{3.15}$$

which are solved by

$$X^i = X_L^i(\sigma+\tau) + X_R^i(\sigma-\tau) \tag{3.16}$$

Taking into account the periodicity in σ we can write X^i as;

$$X^i = x^i + p^i\tau\alpha' + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\left(\frac{1}{n}\alpha_n^i e^{-in(\sigma+\tau)} + \frac{1}{n}\tilde{\alpha}_n^i e^{in(\sigma-\tau)}\right) \tag{3.17}$$

The periodicity in σ rules out a term linear in σ and the rest is simply Fourier analysis in σ. The coefficients follow some peculiar conventions that are standard in string theory and facilitate some of the calculations. The factors of α' are necessary for dimensional reasons. Since X^i is real we should have

$$\alpha_{-n}^i = (\alpha_n^i)^* \tag{3.18}$$

The interpretation of the expansion is that x^i, p^i are the center of mass position and momentum. The rest simply describes oscillations of the string around its center of mass and the α_n^i, $\tilde{\alpha}_n^i$ are the normal coordinates, corresponding to independent oscillations. We can now write everything in terms of them. Using

$$\partial_\sigma X^i = \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \left(\alpha_n^i e^{-in(\sigma+\tau)} - \tilde{\alpha}_n^i e^{in(\sigma-\tau)}\right) \tag{3.19}$$

$$\partial_\tau X^i = p^i\alpha' + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \left(\alpha_n^i e^{-in(\sigma+\tau)} + \tilde{\alpha}_n^i e^{in(\sigma-\tau)}\right) \tag{3.20}$$

$$\tag{3.21}$$

we compute

$$0 = \int_0^{2\pi} d\sigma\, \partial_\sigma X^i \partial_\tau X^i = \pi\alpha' \sum_{n\neq 0} \left(\alpha_n^i \alpha_{-n}^i - \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i\right) \tag{3.22}$$

which gives the only constraint among the α_n^i, $\tilde{\alpha}_n^i$. The momenta follow as

$$P^i = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma\, p^i\alpha' = p^i \tag{3.23}$$

$$P^- = \frac{p^i p^i}{p^+} + \frac{1}{\alpha' p^+} \sum_{n\neq 0} \left(\alpha_n^i \alpha_{-n}^i + \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i\right) \tag{3.24}$$

which justifies calling p^i the center of mass momentum. The mass follows as:

$$M^2 = \frac{1}{\alpha'} \sum_{n\neq 0} \left(\alpha_n^i \alpha_{-n}^i + \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i\right) \tag{3.25}$$

Finally we would like to obtain the coordinate X^-. In conformal gauge, X^- satisfies the same equation as the X^i so we can also write

$$X^- = x^- + p^-\tau\alpha' + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \left(\frac{1}{n}\alpha_n^- e^{-in(\sigma+\tau)} + \frac{1}{n}\tilde{\alpha}_n^- e^{in(\sigma-\tau)}\right) \tag{3.26}$$

However the α_n^- are not independent variables nor is p^-. On the other hand x^- is independent, since the constraints involve derivatives of X^-. In fact, x^- is conjugate to the other independent variable p^+. To find the α_m^- we use that:

$$\partial_\sigma X^- = \frac{2}{p^+\alpha'} \partial_\sigma X^i \partial_\sigma X^i \tag{3.27}$$

$$= \sqrt{2\alpha'} \frac{p^i}{p^+} \sum_{n\neq 0} \left(\alpha_n^i e^{-in(\sigma+\tau)} - \tilde{\alpha}_n^i e^{in(\sigma-\tau)}\right) \tag{3.28}$$

$$+ \frac{1}{p^+} \sum_{n\neq 0, m\neq 0} \left(\alpha_n^i \alpha_m^i e^{-i(n+m)(\sigma+\tau)} - \tilde{\alpha}_n^i \tilde{\alpha}_m^i e^{i(n+m)(\sigma-\tau)}\right) \tag{3.29}$$

Introducing the expansion of X^- and identifying coefficients we obtain

$$\alpha_n^- = \frac{2p^i}{p^+}\alpha_n^i + \sqrt{\frac{2}{\alpha'}}\frac{1}{p^+}\sum_{m\neq 0, m\neq n} \alpha_{n-m}^i \alpha_m^i \tag{3.30}$$

and the same formula replacing the α's by $\tilde{\alpha}$'s. From here we see that it is convenient to define:

$$\alpha_0^i = \sqrt{\frac{\alpha'}{2}}p^i \tag{3.31}$$

and write

$$\alpha_n^- = \sqrt{\frac{2}{\alpha'}}\frac{1}{p^+}\sum_{m=-\infty}^{\infty} \alpha_{n-m}^i \alpha_m^i \tag{3.32}$$

What we have just done is reproduce the usual result that the Fourier components of the product are given by the convolution (defined by the sum over m) between the Fourier components of the terms.

The last calculation is to compute the angular momentum:

$$\begin{aligned} M^{\mu\nu} &= \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma \left(X^\mu \partial_\tau X^\nu - X^\nu \partial_\tau X^\mu \right) \\ &= x^\mu p^\nu - x^\nu p^\mu - \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu\right) - \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\left(\tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu\right) \end{aligned} \tag{3.33}$$

which can be interpreted as the sum of an orbital angular momentum plus an internal spin. Note that we can use this formula for components such as M^{i-} but we should remember then to replace α_n^- by its value (3.30). It should be noted that in the calculation of P^μ and $M^{\mu\nu}$ the dependence on τ canceled as it should since they are conserved quantities.

This concludes our discussion of the classical dynamics in terms of normal modes. We are ready to go to the quantum theory.

3.1.2 Quantization

We managed to write the dynamics of the string in terms of simple variables $x^\pm$, p^+, x^i, p^i, α_n^i, $\tilde{\alpha}_n^i$. Formally, quantizing means that we replace these variables by operators with some particular commutation relations. The operators are then represented as linear operators acting on a Hilbert space, the space of all possible states of the string.

The canonical commutation relations are that $[p, x] = -i$ for canonically conjugated variables. The momentum p is defined as the derivative of the Lagrangian with respect to the velocity: $p = \frac{\partial L}{\partial \partial_t x}$. In our case we have as coordinates $X_i(\sigma, \tau)$, and the Lagrangian is $L = \int_0^{2\pi} d\sigma\, \mathcal{L}$. The derivative gives, in conformal gauge,

$$\Pi_i(\sigma, \tau) = \frac{\partial \mathcal{L}}{\partial(\partial_\tau X)} = \frac{1}{2\pi\alpha'} \partial_\tau X^i(\sigma, \tau) \tag{3.34}$$

where we denote with $\Pi_i(\sigma, \tau)$ the momentum conjugate to $X^i(\sigma, \tau)$. Note that we have

$$P^i = \int_0^{2\pi} d\sigma\, \Pi_i(\sigma, \tau) \tag{3.35}$$

namely, the zero mode of Π_i is the center of mass momentum. The canonical commutation relations read now:

$$\left[\Pi_i(\sigma, \tau), X^j(\sigma', \tau)\right] = -i\delta_i^j \delta(\sigma - \sigma') \tag{3.36}$$

and all other commutators vanish. Notice that the commutator is taken between fields evaluated at the same value of τ. The function $\delta(\sigma - \sigma')$ is the Dirac delta function[3]. The commutation relations should be considered as a definition of the quantum theory but they are natural since they express the fact that $\Pi_i(\sigma)$ is canonically conjugated to $X^i(\sigma)$, namely at the same value of σ and with the same index i. If $\sigma \neq \sigma'$ or $i \neq j$ the commutator vanishes.

We can now use that, from (3.17), we have

$$\alpha_n^i = \frac{1}{4\pi} \sqrt{\frac{2}{\alpha'}} e^{in\tau} \int_0^{2\pi} d\sigma\, e^{in\sigma} \left(-inX^i + \partial_\tau X^i\right) \tag{3.37}$$

Using (3.36) we can compute

$$\left[\alpha_n^i, \alpha_m^j\right] = n\delta^{ij} \delta_{m+n} \tag{3.38}$$

where $\delta_0 = 1$ and $\delta_{n \neq 0} = 0$. Similarly, from

$$\tilde{\alpha}_n^i = \frac{1}{4\pi} \sqrt{\frac{2}{\alpha'}} e^{in\tau} \int_0^{2\pi} d\sigma\, e^{-in\sigma} \left(-inX^i + \partial_\tau X^i\right) \tag{3.39}$$

we obtain

$$\left[\tilde{\alpha}_n^i, \tilde{\alpha}_m^j\right] = n\delta^{ij} \delta_{m+n} \tag{3.40}$$

$$\left[\alpha_n^i, \tilde{\alpha}_m^j\right] = 0 \tag{3.41}$$

[3]Technically it is called a distribution and is defined by the equation $\int_0^{2\pi} d\sigma\, f(\sigma)\delta(\sigma - \sigma') = f(\sigma')$

That means that the only non-zero commutators are

$$\left[\alpha^i_n, \tilde{\alpha}^i_{-n}\right] = n \tag{3.42}$$
$$\left[\tilde{\alpha}^i_n, \tilde{\alpha}^i_{-n}\right] = n \tag{3.43}$$

and we can consider $n > 0$ (since $n < 0$ is the same commutator). Moreover, since the relation (3.18) becomes

$$\alpha^i_{-n} = (\alpha^i_n)^\dagger \tag{3.44}$$

we see that the commutation relations are the standard commutation relations of the harmonic oscillator up to a rescaling. In fact

$$a^i_n = \frac{1}{\sqrt{n}}\alpha^i_n, \qquad \tilde{a}^i_n = \frac{1}{\sqrt{n}}\tilde{\alpha}^i_n, \qquad (n > 0) \tag{3.45}$$

obey the usual relations:

$$\left[a^i_n, (a^i_n)^\dagger\right] = 1, \tag{3.46}$$

and the same for $\tilde{a}^i_n$. That is, α^i_n with positive index is understood as a lowering or annihilation operator and α^i_n with negative subindex as raising or creation operator. We can then represent each a^i_n on a space of states labeled by occupation numbers N^i_n such that

$$a^i_n|N^i_n\rangle = \sqrt{N^i_n}|N^i_n\rangle, \quad (a^i_n)^\dagger = \sqrt{N^i_n+1}|N^i_n\rangle, \quad n = 1\ldots\infty, \tag{3.47}$$

and the same with $\tilde{a}^i_n$. In fact, N^i_n is the eigenvalue of the number operator:

$$N^i_n = (a^i_n)^\dagger\, a^i_n = \frac{1}{n}\alpha^i_{-n}\,\alpha^i_n \tag{3.48}$$

The space of states of the string is the product of all the possible states of the oscillators and therefore is labeled by the set of non-negative integers $N^i_m, \tilde{N}^i_m$. An important state is the vacuum state $|0\rangle$ where all $N^i_n = \tilde{N}^j_m = 0$ which satisfies

$$\alpha^i_n|0\rangle = 0, \quad \text{for all } n > 0. \tag{3.49}$$

We still have to consider the zero modes:

$$x^i = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\, X^i - \tau\int_0^{2\pi} d\sigma\,\partial_\tau X^i, \qquad p^i = \int_0^{2\pi} d\sigma\,\partial_\tau X^i \tag{3.50}$$

Again from (3.36) we obtain

$$\left[x^i, p^j\right] = i\delta^{ij} \tag{3.51}$$

as expected. They can be represented on the space of wave functions $\psi(x^i)$ where x^i acts by multiplication and $p^i = -i\frac{\partial}{\partial x^i}$. Of particular importance are the eigenstates of momentum $|p^i\rangle$ with wave-function $\psi_{p^i} = e^{ix^ip^i}$. We complement this by defining

$$\left[x^-, p^+\right] = i\eta^{+-} = -2i \tag{3.52}$$

to be consistent with Lorentz invariance (namely $[x^\mu, p^\nu] = i\eta^{\mu\nu}$). A generic state of the string is then determined as

$$|\psi\rangle = |p^+, p^i, N_n^i, \tilde{N}_m^j\rangle \tag{3.53}$$

where p^+, p^i are real numbers and N_n^i, $\tilde{N}_m^j$ non-zero integers.

Now we have to extend this to all operators which, generically, are sums of terms, each of which is a product of α_n^i, $\tilde{\alpha}_n^i$. When doing that one has to face the problem that the α_n^i do not commute. For that reason it is important to define normal ordered operators. Those are operators such that all annihilation operators appear to the right of the creation operators. Since creation operators commute among themselves and so do the annihilation ones, that defines uniquely the order in which to multiply them. The important property of normal ordered operators is that their expectation value in the vacuum is finite. In fact it is given by whatever c-number term one has since all terms containing operators have zero expectation value in the vacuum. For example, for the operator

$$A = \sum_{n\neq 0} \alpha_n^i \alpha_{-n}^i \tag{3.54}$$

we have

$$\langle 0|A|0\rangle = \langle 0|\sum_{n>0} \alpha_n^i \alpha_{-n}^i |0\rangle = \sum_{n>0} n = \text{ divergent} \tag{3.55}$$

On the other hand, the normal ordered operator that we denote as $: A :$ is

$$: A :=: \sum_{n\neq 0} \alpha_n^i \alpha_{-n}^i := 2\sum_{n>0} \alpha_{-n}^i \alpha_n^i \tag{3.56}$$

and satisfies

$$\langle 0| : A : |0\rangle = 0 \tag{3.57}$$

Of course A and $: A :$ are not the same, they differ in commutators which in this case is an infinite constant. We should always work with normal ordered operators which are well defined (as opposed to for example A in the previous example).

After this digression we are in position of writing all momenta and angular momenta in terms of the oscillators. The momenta p^+ and p^i are trivial but P^- is precisely of the form of the operator A we discussed in our example. We define it to be equal to

$$P^- = \frac{p^i p^i}{p^+} + \frac{2}{\alpha' p^+} \left(\sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n + \sum_{n=1}^{\infty} \tilde{\alpha}^i_{-n} \tilde{\alpha}^i_n - 2a \right) \tag{3.58}$$

When going from the classical to the quantum expression there is an order ambiguity that we resolved by writing the operators in normal ordered form. However one can think of other orderings that differ by a commutator which in this case is just a number. For that reason we introduce the (for now) arbitrary constant a. If we introduce the notation

$$N = \sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n = \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} m N^i_m, \quad \tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}^i_{-n} \tilde{\alpha}^i_n = \sum_{i=1}^{D-2} \sum_{n=1}^{\infty} m \tilde{N}^i_m, \tag{3.59}$$

we have

$$P^- = \frac{p^i p^i}{p^+} + \frac{2}{\alpha' p^+} \left(N + \tilde{N} - 2a \right) \tag{3.60}$$

and the condition (3.22) upon quantization becomes

$$N = \tilde{N} \tag{3.61}$$

which is usually called the level matching condition (since N is sometimes called the level). That means that the total contribution to P^- from left and right moving oscillators is the same but the states can be different.

With all this in mind we find the mass spectrum to be

$$M^2 = \frac{2}{\alpha'} \left(N + \tilde{N} - 2a \right) \tag{3.62}$$

The degeneracy of each mass level is determined by different ways in which we can choose the N^i_n and $\tilde{N}^j_m$ such that $N = \tilde{N}$ is fixed.

The lowest levels are:

vacuum: $N = \tilde{N} = 0$. Mass: $M^2 = -\frac{4a}{\alpha'}$. One state: $|0\rangle$.

first level: $N = \tilde{N} = 1$. Mass $M^2 = \frac{4(1-a)}{\alpha'}$. $(D-2)^2$ states: $\alpha^i_{-1} \tilde{\alpha}^j_{-1} |0\rangle$.

second level: $N = \tilde{N} = 2$. Mass $M^2 = \frac{4(2-a)}{\alpha'}$. $\frac{1}{4}(D-2)^2(D+1)^2$ states:

$$\begin{array}{rc} (D-2)^2 \text{ states:} & \alpha^i_{-2} \tilde{\alpha}^j_{-2} |0\rangle, \\ \frac{1}{2}(D-2)^2(D-1) \text{ states:} & \alpha^i_{-1} \alpha^j_{-1} \tilde{\alpha}^k_{-2} |0\rangle, \\ \frac{1}{2}(D-2)^2(D-1) \text{ states:} & \alpha^i_{-2} \tilde{\alpha}^j_{-1} \tilde{\alpha}^k_{-1} |0\rangle, \\ \frac{1}{4}(D-2)^2(D-1)^2 \text{ states:} & \alpha^i_{-1} \alpha_j{-1} \tilde{\alpha}^k_{-1} \tilde{\alpha}^l_{-1} |0\rangle. \end{array} \tag{3.63}$$

To understand the spectrum we should remember that massive particles transform in representations of $SO(D-1)$ and massless ones in representations of $SO(D-2)$. At level two we have $(\frac{1}{2}(D-2)(D+1))^2$ states. Since the number of states in the traceless symmetric representation of $SO(D-1)$ is precisely $\frac{1}{2}(D-2)(D+1)$ we seem to have two copies of such representation. In fact such representation splits into the singlet, traceless symmetric and vector representations of $SO(D-2)$ which is what we have in the left and right moving sides.

However, at level one we have a representation of $SO(D-2)$ which cannot be lifted to a representation of $SO(D-1)$. We expect then this level to be massless and therefore $a=1$. In that case we have massless particles in the symmetric traceless, antisymmetric and singlet representations of $SO(D-1)$. These states are the graviton, the B-field and the dilaton. From here is where the idea of string theory as a theory of quantum gravity arose. However, if $a=1$ we have that the vacuum has $M^2=-\frac{4}{\alpha'}$ which means that there is a tachyon. The potential for such field is an upside down quadratic potential and the theory is unstable. In later section we find a solution to this problem but for the moment we are going to study the bosonic string a little further.

3.2 Massive and massless particles in D dimensions

In this subsection we briefly recall some facts about massive and massless particles in arbitrary dimension D.

3.2.1 Massive particles

The different states of a particle are labeled by their momentum and polarization. The momentum p is such that $p^2=-m^2$ where m is the mass of the particle. For a fixed momentum, there is always a frame where the momentum is of the form:

$$p=(m,0,0,\ldots,0) \tag{3.64}$$

that is where the particle is at rest. After fixing the momentum, a particle still has a discrete set of possible states which are its different polarization states. If we perform an $SO(D-1)$ rotation in the spacial directions, the momentum does not change, *i.e.* the particle is still at rest. However the different polarization states of the particle transform into each other filling some representation of $SO(D-1)$. Examples are:

Scalar particle Corresponds to the identity representation, namely a single state invariant under rotations.

Vector particle The states transform in the vector representation of $SO(D-1)$, therefore there are $D-1$ states.

Two index antisymmetric representation The states are organized into an antisymmetric tensor of two indices. Therefore it has $\frac{(D-1)(D-2)}{2}$ components.

Two index traceless symmetric The states transform as a two index traceless symmetric tensor. Therefore it has $\frac{D(D-1)}{2} - 1$ states.

3.2.2 Massless particles

A massless particle moves at the speed of light and there is no frame of reference where it is at rest. However we can always choose our axis such that one of them is parallel to the direction of motion. Namely, we can take the momentum to be of the form

$$p = (k, k, 0, 0, \ldots, 0), \qquad \Rightarrow \qquad p^2 = 0 \tag{3.65}$$

for some k. Now, we can perform an $SO(D-2)$ rotation that leaves the momentum invariant and therefore only transform the different polarization states among themselves. Thus, the polarizations of a massless particle fit in representations of $SO(D-2)$. Examples are:

Scalar particle Corresponds to the identity representation, namely a single state invariant under rotations. It is the same as in the massive case.

Vector particle (gauge boson, ***e.g.*** **photon)** The states transform in the vector representation of $SO(D-2)$, therefore there are $D-2$ states.

Two index antisymmetric representation (B-field) The states are organized into an antisymmetric tensor of two indices. Therefore it has $\frac{(D-2)(D-3)}{2}$ components.

Two index traceless symmetric (graviton) The states transform as a two index traceless symmetric tensor. Therefore it has $\frac{(D-1)(D-2)}{2} - 1$ states.

Notice that, in four dimensions, both, the photon and the graviton have two polarizations but in higher dimensions that is not the case any more.

To write a Lorentz invariant equation of motion for a massless field we need to add unphysical components to fill a finite dimensional representation of the Lorentz group. For example a photon is represented by a vector field A_μ with D components of which only $D-2$ should be physical. The fact that there are extra components which are not physical means that there is a large symmetry because changing the value of the non-physical components at any point of space time should not change the physics. The symmetries that arise are local, namely depending on parameters which are arbitrary functions of space-time, and are called gauge symmetries. We proceed in the next subsection to see how the equation of motion and the gauge symmetry eliminate the unwanted components reducing the field to its physical components.

3.3 Massless vectors

A massless vector field is represented by a field A_μ. We have to impose a gauge invariance, compatible with the Lorentz symmetry, to eliminate the unwanted components. The only possibility is:

$$\tilde{A}_\mu = A_\mu + \partial_\mu \lambda \tag{3.66}$$

where λ is an arbitrary function of the position and the statement is that A_μ and $\tilde{A}_\mu$ describe the same physical situation. Consider now an equation of motion for A_μ which we are going to take to be up to second order in partial derivatives. Since we need D equations for D variables, we need to construct a vector out of A_μ, $\partial_\alpha A_\mu$ and $\partial_{\alpha\beta} A_\mu$. The most general possibility is:

$$a\partial_{\alpha\alpha} A_\mu + b\partial_{\mu\alpha} A_\alpha + cA_\mu = 0 \tag{3.67}$$

where repeated indices are contracted with the Minkowski metric $\eta^{\mu\nu}$ and a, b, c are arbitrary constant coefficients. If we write the equation in terms of $\tilde{A}_\mu = A_\mu + \partial_\mu \lambda$, we get

$$a\partial_{\alpha\alpha} \tilde{A}_\mu + b\partial_{\mu\alpha} \tilde{A}_\alpha + c\tilde{A}_\mu - a\partial_{\alpha\alpha\mu}\lambda - b\partial_{\mu\alpha\alpha}\lambda - c\partial_\mu \lambda = 0 \tag{3.68}$$

If we impose gauge invariance, namely that, for any λ, the equation for $\tilde{A}$ were the same as for A_μ we need to have

$$a = -b, \quad c = 0 \tag{3.69}$$

and the equation of motion is then

$$\partial_{\alpha\alpha} A_\mu - \partial_{\mu\alpha} A_\alpha = 0 \tag{3.70}$$

which is the Maxwell equation for the vector potential.

Now we want to see that the equation of motion and gauge invariance determine that, for a wave with given momentum p there are only $D-2$ physical polarizations. We start by noticing that we can always do a gauge transformation to an $\tilde{A}_\mu$ such that $\partial_\alpha \tilde{A}_\alpha = 0$. Indeed we just need to choose λ such that

$$\partial_{\alpha\alpha}\lambda = -\partial_\alpha A_\alpha \tag{3.71}$$

which can always be done. In fact this is just a wave equation with a source that can be solved for example by the method of Green functions. In this gauge, known as the Lorentz gauge, the equations of motion simplify to

$$\partial_{\alpha\alpha} A_\mu = 0, \qquad \partial_\alpha A_\alpha = 0 \tag{3.72}$$

The solutions are plane waves. For a given momentum $p = (k, k, 0 \ldots, 0)$ we have

$$A_\mu = a_\mu e^{ipx} \tag{3.73}$$

which solve $\partial_{\alpha\alpha} A_\mu = 0$ if $p^2 = 0$ as we have. The vector a_μ is a constant vector called the polarization. It has D components but they are not independent since the gauge condition $\partial_\alpha A_\alpha = 0$ implies

$$\eta^{\mu\nu} p_\mu a_\nu = 0, \quad \Rightarrow \quad -ka_0 + ka_1 = 0, \quad \Rightarrow \quad a_0 = a_1 \tag{3.74}$$

So it has only $D-1$ independent components, still one more than the expected $D-2$. Now we notice that we have not fixed the gauge completely with the condition $\partial_\alpha A_\alpha = 0$ since a gauge transformation

$$\tilde{A}_\mu = A_\mu + \partial_\mu \lambda \tag{3.75}$$

leaves this condition invariant if

$$\partial_{\alpha\alpha} \lambda = 0 \tag{3.76}$$

So we still have a freedom that can allow us to eliminate one more component. Consider then a gauge transformation generated by

$$\lambda = \bar{\lambda} e^{ipx} \tag{3.77}$$

where $\bar{\lambda}$ is a constant. The vector $\tilde{A}_\mu = \tilde{a}_\mu e^{ipx}$ has components

$$\tilde{a}_\mu = a_\mu + ip_\mu \bar{\lambda}, \quad i.e. \quad \tilde{a}_0 = \tilde{a}_1 = a_0 + ik\bar{\lambda}, \quad \tilde{a}_i = a_i, \quad i = 2, \ldots (D-1) \tag{3.78}$$

If we choose $\bar{\lambda}$ as

$$\bar{\lambda} = -\frac{a_0}{ik} \tag{3.79}$$

then the only non-vanishing components of $\tilde{A}$ are $\tilde{A}_i$ with $i = 2 \ldots (D-1)$, namely $D-2$ independent components as expected.

In summary, we can impose a gauge symmetry and write an equation of motion in a Lorentz invariant way for a vector field A_μ such that only $D-2$ components are physical. The same number we obtained by group theory considerations for a massless vector representation.

Finally, if we want to put a source to the equation, namely couple the photon to charged matter we can do so by constructing, out of the matter fields, a vector j_μ, the current, and inserting it in the right hand side of eq.(3.70):

$$\partial_{\alpha\alpha} A_\mu - \partial_{\mu\alpha} A_\alpha = j_\mu \tag{3.80}$$

Taking the derivative ∂_μ on both sides we obtain

$$\partial_\mu j_\mu = \partial_\mu \left(\partial_{\alpha\alpha} A_\mu - \partial_{\mu\alpha} A_\alpha \right) = 0 \tag{3.81}$$

namely j_μ has to be a *conserved* current.

3.4 Graviton: massless two index traceless symmetric representation

In general relativity the gravitational interaction is described by variations in the space time metric. Thus, the (square) distance between two points whose coordinates differ by dx^μ is given by

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu \tag{3.82}$$

where $g_{\mu\nu}$ is a symmetric tensor function of the position. For our purpose here we only need to consider small fluctuations around the Minkowski metric, namely:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x) \tag{3.83}$$

where the components of h are much smaller than 1. Since gravity is a long range force the fluctuations $h_{\mu\nu}$ should describe a massless particle. An obvious candidate is the traceless symmetric representation which has $\frac{(D-1)(D-2)}{2} - 1$ components. On the other hand $h_{\mu\nu}$ has $\frac{D(D-1)}{2}$ components so the equation of motion together with a local symmetry should eliminate some of them. The correct equation of motion follows from general relativity and is the linearized Einstein equation. We can derive it here doing the same procedure than in the previous section for the photon. The most general local symmetry we can impose is generated by a vector ξ_μ:

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{3.84}$$

If the equation has up to two partial derivatives, it has to be a combination of $\partial_{\alpha\beta} h_{\mu\nu}$, $\partial_\alpha h_{\mu\nu}$, and $h_{\mu\nu}$. It has to be a two index tensor, so the most general equation is

$$a\partial_{\alpha\alpha} h_{\mu\nu} + b\partial_{\mu\nu} h_{\alpha\alpha} + c\partial_{\alpha\mu} h_{\alpha\nu} + d\partial_{\alpha\nu} h_{\alpha\mu} + e h_{\mu\nu} = 0 \tag{3.85}$$

Imposing gauge invariance as before determines:

$$a = b = -c = -d, \quad e = 0, \tag{3.86}$$

and the equation becomes

$$\partial_{\alpha\alpha} h_{\mu\nu} + \partial_{\mu\nu} h_{\alpha\alpha} - \partial_{\alpha\mu} h_{\alpha\nu} - \partial_{\alpha\nu} h_{\alpha\mu} = 0 \tag{3.87}$$

Now we have to use the equation of motion and the gauge symmetry to determine how many physical components we have. First notice that we can rewrite the equation of motion as:

$$\partial_{\alpha\alpha} h_{\mu\nu} - \partial_\mu \left(\partial_\alpha h_{\alpha\nu} - \frac{1}{2} \partial_\nu h_{\alpha\alpha} \right) - \partial_\nu \left(\partial_\alpha h_{\alpha\mu} - \frac{1}{2} \partial_\mu h_{\alpha\alpha} \right) = 0 \tag{3.88}$$

Thus, it seems natural to choose the gauge condition

$$\partial_\alpha h_{\alpha\nu} - \frac{1}{2}\partial_\nu h_{\alpha\alpha} = 0 \tag{3.89}$$

To see if that is possible consider an h which does not satisfy that and perform a gauge transformation of parameters ξ_ν to $\tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. It is easy to see that $\tilde{h}$ satisfies the gauge condition if

$$\partial_{\alpha\alpha}\xi_\mu = -\left[\partial_\alpha h_{\alpha\nu} - \frac{1}{2}\partial_\nu h_{\alpha\alpha}\right] \tag{3.90}$$

which is again a wave equation with source for the ξ_μ and which can be solved. In fact, we can do still slightly more since we have

$$\tilde{h}_{\alpha\alpha} = h_{\alpha\alpha} + \partial_\alpha\xi_\alpha \tag{3.91}$$

we can always choose ξ_μ such that

$$\partial_\alpha\xi_\alpha = -h_{\alpha\alpha} \tag{3.92}$$

which implies $\tilde{h}_{\alpha\alpha} = 0$. Putting everything together, we find that the equation of motion is reduced to

$$\partial_{\alpha\alpha}h_{\mu\nu} = 0 \tag{3.93}$$

$$\partial_\alpha h_{\alpha\mu} = 0 \tag{3.94}$$

$$h_{\alpha\alpha} = 0 \tag{3.95}$$

The solutions of the wave equation with given momentum $p = (k, k, 0\ldots, 0)$ are of the form

$$h_{\mu\nu} = \bar{h}_{\mu\nu}e^{ipx} \tag{3.96}$$

where $\bar{h}_{\mu\nu}$ are constants related by

$$\partial_\alpha h_{\alpha\mu} = 0 \quad \Rightarrow \quad \eta^{\alpha\beta}p_\alpha\bar{h}_{\beta\mu} = 0, \quad -k\bar{h}_{0\mu} + k\bar{h}_{1\mu} = 0 \tag{3.97}$$

$$h_{\alpha\alpha} = 0, \quad \Rightarrow \quad -\bar{h}_{00} + \bar{h}_{11} + \sum_{i=2}^{D-1}\bar{h}_{ii} = 0 \tag{3.98}$$

From the first equation we get, taking $\mu = 0$ and $\mu = 1$, that $\bar{h}_{00} = \bar{h}_{01} = \bar{h}_{11}$ and taking $\mu = i$ that $\bar{h}_{0i} = \bar{h}_{1i}$. We conclude that, in this gauge the matrix $\bar{h}_{\mu\nu}$ is of the form:

$$\bar{h}_{\mu\nu} = \begin{pmatrix} \bar{h}_{00} & \bar{h}_{00} & \bar{h}_{02} & \cdots & \bar{h}_{0(D-1)} \\ \bar{h}_{00} & \bar{h}_{00} & \bar{h}_{02} & \cdots & \bar{h}_{0(D-1)} \\ \bar{h}_{02} & \bar{h}_{02} & \bar{h}_{22} & \cdots & \bar{h}_{2(D-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{h}_{0(D-1)} & \bar{h}_{0(D-1)} & \bar{h}_{2(D-1)} & \cdots & \bar{h}_{(D-1)(D-1)} \end{pmatrix} \tag{3.99}$$

and we have to impose $-h_{00}+h_{11}+\sum_{i=2}^{D-1}\bar{h}_{ii}=\sum_{i=2}^{D-1}\bar{h}_{ii}=0$. As before, we still have a gauge invariance generated by ξ_μ's that satisfy

$$\partial_{\alpha\alpha}\xi_\mu=0, \quad \partial_\alpha\xi_\alpha=0 \tag{3.100}$$

This means that they are of the form

$$\xi_\mu=\bar{\xi}_\mu e^{ipx}, \quad p.\bar{\xi}=0, \quad \Rightarrow \quad -k\xi_0+k\xi_1=0, \quad \Rightarrow \quad \xi_0=\xi_1 \tag{3.101}$$

The gauge transformation that they generate is

$$\tilde{\bar{h}}_{\mu\nu}=\bar{h}_{\mu\nu}+ip_\mu\xi_\nu+ip_\nu\xi_\mu \tag{3.102}$$

It is clear that such gauge transformation with $\xi_0=\xi_1$ preserves the form (3.99) of the matrix $\bar{h}$ which had to be the case since they preserve the gauge conditions. The new independent components are:

$$\tilde{\bar{h}}_{00}=\bar{h}_{00}+2ik\xi_0 \tag{3.103}$$
$$\tilde{\bar{h}}_{0i}=\bar{h}_{0i}+ik\xi_i, \quad i=2\ldots(D-1) \tag{3.104}$$
$$\tilde{\bar{h}}_{ij}=\bar{h}_{ij} \quad i,j=2\ldots(D-1) \tag{3.105}$$

Taking

$$\xi_0=-\frac{\bar{h}_{00}}{2ik}, \quad \xi_i=-\frac{\bar{h}_{0i}}{ik} \tag{3.106}$$

we get that the new matrix $\tilde{\bar{h}}$ is of the form:

$$\bar{h}\mu\nu=\begin{pmatrix} 0\ 0 & 0 & \cdots & 0 \\ 0\ 0 & 0 & \cdots & 0 \\ 0\ 0 & \bar{h}_{22} & \cdots & \bar{h}_{2(D-1)} \\ \vdots\ \vdots & \vdots & \ddots & \vdots \\ 0\ 0 & \bar{h}_{2(D-1)} & \cdots & \bar{h}_{(D-1)(D-1)} \end{pmatrix} \tag{3.107}$$

and we still have $\sum_{i=2}^{D-1}\bar{h}_{ii}=0$. Therefore we see that the physical degrees of freedom are precisely those of a traceless symmetric matrix of $(D-2)\times(D-2)$ as we expected.

Summarizing we were able to write a gauge and Lorentz invariant wave equation for the graviton that left only $\frac{(D-1)(D-2)}{2}-1$ physical components. General relativity is in fact a non-linear theory and the equations we got are valid only for small fluctuations. However these small fluctuations are the gravitational waves whose number of independent polarizations we wanted to count.

Finally, if we want to put a source to equation (3.87) we need a two index symmetric tensor $S_{\mu\nu}$ to write

$$\partial_{\alpha\alpha}h_{\mu\nu} + \partial_{\mu\nu}h_{\alpha\alpha} - \partial_{\alpha\mu}h_{\alpha\nu} - \partial_{\alpha\nu}h_{\alpha\mu} = S_{\mu\nu} \tag{3.108}$$

Taking the derivative ∂_μ on both sides we get:

$$\partial_\mu S_{\mu\nu} = \partial_{\mu\mu\nu}h_{\alpha\alpha} - \partial_{\mu\nu\alpha}h_{\alpha\mu} = \partial_\nu\left(\partial_{\mu\mu}h_{\alpha\alpha} - \partial_{\alpha\mu}h_{\alpha\mu}\right) \tag{3.109}$$

At first sight, it seems that $S_{\mu\nu}$ does not need to satisfy any equation, but, if we compute $S_{\alpha\alpha}$ from (3.108) we get

$$\frac{1}{2}S_{\alpha\alpha} = \partial_{\mu\mu}h_{\alpha\alpha} - \partial_{\alpha\mu}h_{\alpha\mu} \tag{3.110}$$

so that $S_{\mu\nu}$ in fact has to satisfy the equation:

$$\partial_\mu S_{\mu\nu} - \frac{1}{2}\partial_\nu S_{\alpha\alpha} = 0 \tag{3.111}$$

If we define the tensor

$$T_{\mu\nu} = S_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}S_{\alpha\alpha} \tag{3.112}$$

we have

$$\partial_\mu T_{\mu\nu} = 0 \tag{3.113}$$

namely $T_{\mu\nu}$ is a conserved tensor which can be identified with the energy momentum tensor. From the definition of $T_{\mu\nu}$ in terms of $S_{\mu\nu}$ and from (3.108) we can write and equation for $h_{\mu\nu}$ with $T_{\mu\nu}$ as a source:

$$\left[\partial_{\alpha\alpha}h_{\mu\nu} + \partial_{\mu\nu}h_{\alpha\alpha} - \partial_{\alpha\mu}h_{\alpha\nu} - \partial_{\alpha\nu}h_{\alpha\mu}\right] - \eta_{\mu\nu}\left[\partial_{\alpha\alpha}h_{\beta\beta} - \partial_{\alpha\beta}h_{\alpha\beta}\right] = T_{\mu\nu} \tag{3.114}$$

If there is no source $T_{\mu\nu} = 0$ we get an equation for $h_{\mu\nu}$ that is exactly equivalent to (3.87).

3.5 Lorentz symmetry and the critical dimension

To see if the theory is Lorentz invariant we have to consider the Lorentz generators which are the components of the angular momentum:

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\right) - i\sum_{n=1}^{\infty}\frac{1}{n}\left(\tilde{\alpha}^\mu_{-n}\tilde{\alpha}^\nu_n - \tilde{\alpha}^\nu_{-n}\tilde{\alpha}^\mu_n\right) \tag{3.115}$$

We should now check the commutation relations:

$$\left[M^{\mu\nu}, M^{\alpha\beta}\right] = -i\eta^{\nu\alpha}M^{\mu\beta} + i\eta^{\mu\alpha}M^{\nu\beta} + i\eta^{\nu\beta}M^{\mu\alpha} - i\eta^{\mu\beta}M^{\nu\alpha} \tag{3.116}$$

A particular case is

$$\left[M^{i-},M^{j-}\right]=0 \tag{3.117}$$

If the commutator does not vanish, it is problematic since it implies that the Lorentz group is not a symmetry of the string. It turns out that quantum mechanically the commutator vanishes only if the number of dimensions is 26 and $a=1$. To see that let us write:

$$M^{i-}=x^ip^- - x^-p^i - 2i\sqrt{\frac{2}{\alpha'}}\frac{1}{p^+}E^i - 2i\sqrt{\frac{2}{\alpha'}}\frac{1}{p^+}\tilde{E}^i \tag{3.118}$$

where

$$E^i=\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha^i_{-n}\mathcal{L}_n-\mathcal{L}_{-n}\alpha^i_n\right) \tag{3.119}$$

with

$$\mathcal{L}_n=\frac{1}{2}\sqrt{\frac{\alpha'}{2}}p^+\alpha^-_n \tag{3.120}$$

and the same for $\tilde{E}^i$ and $\tilde{\mathcal{L}}_n$.

Before continuing it is instructive to compute the commutator of the operators $\mathcal{L}_n$. Using the value of α^-_n that we found in (3.30) we get:

$$\mathcal{L}_n=\frac{1}{2}\sum_{\bar{n}=-\infty}^{\infty}:\alpha^i_{n-\bar{n}}\alpha^i_{\bar{n}}: \tag{3.121}$$

Since $n\neq 0$, $[\alpha^i_{n-\bar{n}},\alpha^i_{\bar{n}}]=0$. Therefore we can drop the normal order and compute

$$\begin{aligned}
[\mathcal{L}_n,\mathcal{L}_m] &= \frac{1}{4}\left[\sum_{\bar{n}}\alpha^i_{n-\bar{n}}\alpha^i_{\bar{n}},\sum_{\bar{m}}\alpha^i_{m-\bar{m}}\alpha^i_{\bar{m}}\right] && (3.122)\\
&= \frac{1}{2}\sum_{\bar{n}}\left(\bar{n}\alpha^i_{n-\bar{n}}\alpha^i_{\bar{n}+m}+(n-\bar{n})\alpha^i_{n-\bar{n}+m}\alpha^i_{\bar{n}}\right) && (3.123)\\
&= (n-m)\mathcal{L}_{n+m} && (3.124)
\end{aligned}$$

where in the first step we used $[AB,CD]=AC[B,D]+A[B,C]D+C[A,D]B+[A,C]DB$ and in the last step we shifted $\bar{n}\to\bar{n}-m$ in the first sum. We see from here that, if $n+m=0$ we get $\mathcal{L}_0$ which we have actually not defined. In fact, $\mathcal{L}_0$ is related to p^- and has normal ordering ambiguities. Let us compute that special commutator. When doing so we have to be extremely careful and always use normal ordered expressions. Consider now $n>0$. We have

$$\mathcal{L}_n=\sqrt{\frac{\alpha'}{2}}p^i\alpha^i_n+\frac{1}{2}\sum_{\bar{n}=1}^{n-1}\alpha^i_{n-\bar{n}}\alpha^i_{\bar{n}}+\sum_{\bar{n}=1}^{\infty}\alpha^i_{-\bar{n}}\alpha^i_{n+\bar{n}} \tag{3.125}$$

$$\mathcal{L}_{-n}=\sqrt{\frac{\alpha'}{2}}p^j\alpha^j_{-n}+\frac{1}{2}\sum_{\bar{n}=1}^{n-1}\alpha^j_{-n+\bar{n}}\alpha^j_{-\bar{n}}+\sum_{\bar{n}=1}^{\infty}\alpha^j_{-\bar{n}-n}\alpha^j_{\bar{n}} \tag{3.126}$$

For example in $\mathcal{L}_n$ the first term is an annihilation operators, the second term has two annihilations ops. and the third, one creation and one annihilation. We can now compute:

$$[\mathcal{L}_n, \mathcal{L}_{-n}] = \frac{\alpha'}{2} n p^i p^i + \frac{1}{2}\sum_{\bar{n}=1}^{n-1} \bar{n}\alpha^i_{n-\bar{n}}\alpha^i_{\bar{n}-n} + \frac{1}{2}\sum_{\bar{n}=1}^{n-1}(n-\bar{n})\alpha^i_{-\bar{n}}\alpha^i_{\bar{n}} \tag{3.127}$$

$$+\sum_{\bar{n}=1}^{\infty}(\bar{n}+n)\alpha^i_{-\bar{n}}\alpha^i_{\bar{n}} - \sum_{\bar{n}=1}^{\infty}\bar{n}\alpha^i_{-n-\bar{n}}\alpha^i_{\bar{n}+n} \tag{3.128}$$

$$= \frac{\alpha'}{2} n p^i p^i + 2n\sum_{m=1}^{\infty}\alpha^i_{-m}\alpha^i_m + \frac{1}{2}\sum_{\bar{n}=1}^{n-1}(n-\bar{n})\left[\alpha^i_{\bar{n}}, \alpha^i_{-\bar{n}}\right] \tag{3.129}$$

where the commutator appears when we normal order the terms and the rest is similar to the calculation of $[\mathcal{L}_n, \mathcal{L}_m]$. We can now compute:

$$\frac{1}{2}\sum_{\bar{n}=1}^{n-1}(n-\bar{n})\left[\alpha^i_{\bar{n}}, \alpha^i_{-\bar{n}}\right] = \frac{1}{2}\sum_{\bar{n}=1}^{n-1}(n-\bar{n})\,\bar{n}\,\delta^{ii} = \frac{D-2}{12}(n^3-n) \tag{3.130}$$

If we now define

$$\mathcal{L}_0 = \frac{\alpha'}{4}p^i p^i + N - a \tag{3.131}$$

we obtain

$$[\mathcal{L}_n, \mathcal{L}_{-n}] = 2n\mathcal{L}_0 + \frac{D-2}{12}(n^3-n) + 2na \tag{3.132}$$

If we do the same for the left moving modes we see that the definition of $\mathcal{L}_0$ is chosen so that we can write

$$p^+p^- = \frac{2}{\alpha'}\left(\mathcal{L}_0 + \tilde{\mathcal{L}}_0\right) \tag{3.133}$$

In total, it turns out that the $\mathcal{L}_n$ obey an algebra:

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \left(\frac{D-2}{12}(n^3-n) + 2na\right)\delta_{m+n} \tag{3.134}$$

which is called a Virasoro algebra (with central extension). Classically one can see that the second term is absent. This phenomenon is called an anomaly and means that a classical symmetry is not present quantum mechanically.

This lengthy calculation is a preliminary step to understand the computation of the commutator $[M^{i-}, M^{j-}]$. As a first stage of that calculation we use $[x^-, p^+] = -2i$ and the definition of p^- to obtain

$$[x^-, p^-] = \frac{2i}{p^+}p^-, \quad [x^i, p^-] = \frac{2i}{p^+}p^i \tag{3.135}$$

from where we find

$$[x^i p^- - x^- p^i, x^j p^- - x^- p^j] = 0 \tag{3.136}$$

The second stage requires computing

$$[x^i, \mathcal{L}_n] = i\sqrt{\frac{\alpha'}{2}}\alpha^i_n \tag{3.137}$$

$$[x^i, E^j] = -i\sqrt{\frac{\alpha'}{2}}E^{ij} = -i\sqrt{\frac{\alpha'}{2}}\sum_{n=1}^{\infty}(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n) \tag{3.138}$$

and using again $[x^-, p^+] = -2i$ to get

$$[M^{i-}, M^{j-}] = -\frac{4}{(p^+)^2}\sqrt{\frac{2}{\alpha'}}\left(p^i E^j - p^j E^i\right) - \frac{4}{p^+}E^{ij}p^- - \frac{8}{\alpha'(p^+)^2}[E^i, E^j] \tag{3.139}$$

$$-\frac{4}{(p^+)^2}\sqrt{\frac{2}{\alpha'}}\left(p^i \tilde{E}^j - p^j \tilde{E}^i\right) - \frac{4}{p^+}\tilde{E}^{ij}p^- - \frac{8}{\alpha'(p^+)^2}[\tilde{E}^i, \tilde{E}^j] \tag{3.140}$$

At this stage we could use the same techniques than when we computed $[\mathcal{L}_n, \mathcal{L}_m]$. However we can reason in the following way: after evaluating the commutator $[E^i, E^j]$ the right hand side will be a sum of terms at most quartic in creation and annihilation operators. As we saw in the previous computation, a quantum mechanical anomaly can appear from normal ordering the terms. The anomaly can therefore have at most two oscillators. One can easily see that the indices of the oscillators should add up to zero which leaves only quadratic or constant terms. By rotational invariance any constant term should be of the form δ^{ij} which is impossible since the result should by antisymmetric in ij. In fact we can take $i \neq j$ since for $i = j$ the commutator is obviously zero.

We conclude that, quantum mechanically, we can have contributions quadratic in oscillators, namely

$$[M^{i-}, M^{j-}] = \sum_{m=1}^{\infty} c_m(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m) \tag{3.141}$$

and a similar contribution from the right moving modes. We can obtain such contribution from evaluating

$$\langle 0|\alpha^k_m \tilde{\alpha}^{\tilde{k}}_m \; [M^{i-}, M^{j-}] \; \alpha^l_{-m}\tilde{\alpha}^{\tilde{l}}_{-m}|0\rangle \tag{3.142}$$

where we put the same oscillator number in the left and right moving modes as required by the level matching condition. We can also concentrate on the left moving modes since the other terms in the commutator give the same result. We start by computing

$$\langle 0|\mathcal{L}_m \alpha^j_{-m}|0\rangle = \sqrt{\frac{\alpha'}{2}}mp^j, \qquad \langle 0|\alpha^i_m \mathcal{L}_{-m}|0\rangle = \sqrt{\frac{\alpha'}{2}}mp^i \tag{3.143}$$

which helps us to find that

$$\langle 0|\alpha_m^k E^j \alpha_{-m}^l|0\rangle = m\sqrt{\frac{\alpha'}{2}}\left(\delta^{jk}p^l - \delta^{jl}p^k\right) \tag{3.144}$$

It is also simple to see that

$$\langle 0|\alpha_m^k E^{ij} p^- \alpha_{-m}^l|0\rangle = \left(\frac{p^i p^i}{p^+} + \frac{2}{\alpha' p^+}(N+\tilde{N}-2a)\right) m\left(\delta^{ik}\delta^{jl} - \delta^{jk}\delta^{il}\right) \tag{3.145}$$

where $N = \tilde{N} = m$ (although we are ignoring the right moving modes we have to remember that they contribute to $\tilde{N}$). Finally we can use that

$$\langle 0|\alpha_m^k \sum_{n=1}^{\infty}\frac{1}{n}\mathcal{L}_{-n}\alpha_n^i = \langle 0|\left[\sqrt{\frac{\alpha'}{2}}p^k\alpha_m^i + \sum_{n=1}^{m-1}\frac{m}{n}\alpha_{m-n}^k\alpha_n^i\right] \tag{3.146}$$

to simplify

$$\langle 0|\alpha_m^k[E^i, E^j]\alpha_{-m}^l|0\rangle = \langle 0|\left(\delta^{ik}\mathcal{L}_m - \sqrt{\frac{\alpha'}{2}}p^k\alpha_m^i - \sum_{n=1}^{m-1}\frac{m}{n}\alpha_{m-n}^k\alpha_n^i\right) \tag{3.147}$$

$$\left(\sqrt{\frac{\alpha'}{2}}\alpha_{-m}^j p^l + \sum_{n=1}^{m-1}\frac{m}{n}\alpha_{-n}^j\alpha_{n-m}^l - \delta^{jl}\mathcal{L}_{-m}\right)|0\rangle \tag{3.148}$$

We can now compute

$$\langle 0|\mathcal{L}_m\mathcal{L}_{-m}|0\rangle = \frac{\alpha'}{2}mp^ip^i + \frac{D-2}{12}(m^3-m) \tag{3.149}$$

$$\langle 0|\mathcal{L}_m\sum_{n=1}^{m-1}\frac{m}{n}\alpha_{-n}^j\alpha_{n-m}^l|0\rangle = \frac{1}{2}m^2(m-1)\delta^{jl} \tag{3.150}$$

$$\langle 0|\sum_{n=1}^{m-1}\frac{m}{n}\alpha_{m-n}^k\alpha_n^i\mathcal{L}_{-m}|0\rangle = \frac{1}{2}m^2(m-1)\delta^{ik} \tag{3.151}$$

$$m^2\sum_{n=1}^{m-1}\sum_{\bar{n}=1}^{m-1}\frac{1}{n\bar{n}}\langle 0|\alpha_{m-n}^k\alpha_n^i\alpha_{-\bar{n}}^j\alpha_{\bar{n}-m}^l|0\rangle = m^2(m-1)\delta^{jk}\delta^{il}, \qquad (i\neq j) \tag{3.152}$$

Using this we find

$$\begin{aligned}\langle 0|\alpha_m^k[E^i, E^j]\alpha_{-m}^l|0\rangle &= \frac{m\alpha'}{2}\left(\delta^{ik}p^jp^l + \delta^{jl}p^ip^k - \delta^{jk}p^ip^l - \delta^{il}p^jp^k\right) \\ &+ \left(\delta^{ik}\delta^{jl} - \delta^{jk}\delta^{il}\right)\left(-\frac{m\alpha'}{2}p^ip^i + 2m^3 - 2m^2 - \frac{D-2}{12}(m^3-m)\right)\end{aligned} \tag{3.153}$$

We can now put everything together and compute:

$$
\begin{aligned}
&\langle 0|\alpha_m^k \tilde{\alpha}_m^{\tilde{k}} \;\; [M^{i-}, M^{j-}] \;\; \alpha_{-m}^l \tilde{\alpha}_{-m}^{\tilde{l}}|0\rangle = \\
&\quad m\delta^{\tilde{k}\tilde{l}} \Bigg\{ -\frac{4m}{(p^+)^2}\left(\delta^{jk}p^ip^l - \delta^{jl}p^ip^k - \delta^{ik}p^jp^l + \delta^{il}p^jp^k\right) \\
&\qquad -\frac{4m}{(p^+)^2}\left(p_\perp^2 + \frac{2}{\alpha'}\left(N + \tilde{N} - 2a\right)\right)\left(\delta^{ik}\delta^{jl} - \delta^{jk}\delta^{il}\right) \\
&\qquad -\frac{8}{\alpha'(p^+)^2}\frac{m\alpha'}{2}\left(\delta^{ik}p^jp^l + \delta^{jl}p^ip^k - \delta^{jk}p^ip^l - \delta^{il}p^jp^k\right) \\
&\qquad +\frac{8}{\alpha'(p^+)^2}\left(\delta^{ik}\delta^{jl} - \delta^{jk}\delta^{il}\right)\left(\frac{m\alpha'}{2}p_\perp^2 - 2m^3 + 2m^2 + \frac{D-2}{12}(m^3 - m)\right)\Bigg\}
\end{aligned} \tag{3.154}
$$

plus a similar contribution from the right moving modes. Although many terms cancel, after we use $N = \tilde{N} = m$ we get

$$
\langle 0|\alpha_m^k \tilde{\alpha}_m^{\tilde{k}} \;\; [M^{i-}, M^{j-}] \;\; \alpha_{-m}^l \tilde{\alpha}_{-m}^{\tilde{l}}|0\rangle = \tag{3.155}
$$

$$
-\frac{8}{\alpha'(p^+)^2} m\delta^{\tilde{k}\tilde{l}}\left(\delta^{ik}\delta^{jl} - \delta^{jk}\delta^{il}\right)\left[\frac{26-D}{12}m^3 + m\frac{D-2-24a}{12}\right] \tag{3.156}
$$

This can only vanish for all m if $D = 26$ and $a = 1$. We managed to quantize the theory but at the price of obtaining a tachyon and living in 26 dimensions.

We conclude this section by studying the dependence between mass and spin which classically was found to be $M = \sqrt{\frac{2}{\alpha'}}\sqrt{J}$. To do that define the operator:

$$
\alpha_{-1}^z = \alpha_{-1}^1 + i\alpha_{-1}^2 \tag{3.157}
$$

We can easily compute

$$
[M^{12}, \alpha_{-1}^z] = -i\left[\left(\alpha_{-1}^1\alpha_1^2 - \alpha_{-1}^2\alpha_1^1\right), \, \alpha_{-1}^z\right] = \alpha_{-1}^z \tag{3.158}
$$

which implies that

$$
M^{12}(\alpha_{-1}^z)^J|0\rangle = J(\alpha_{-1}^z)^J|0\rangle \tag{3.159}
$$

Therefore the state $|J\rangle = (\alpha_{-1}^z)^J|0\rangle$ has angular momentum $J = 0\ldots\infty$. Its level is clearly J since we have J oscillators each of wave-number one. We can put the same state in the right moving part which then gives angular momentum $2J$. The mass is

$$
M^2 = \frac{2}{\alpha'}\left(N + \tilde{N} - 2a\right) = \frac{2}{\alpha'}(2J - 2) \tag{3.160}
$$

and then

$$
M^2 = \sqrt{\frac{2}{\alpha'}}\sqrt{2J - 2} \tag{3.161}
$$

which is equal to the classical one for large J except that the total angular momentum is quantized. More generically, we see that

$$\frac{\alpha'}{2}M^2 + 2 = \text{integer} \tag{3.162}$$

For a given spectrum of particles one can in principle observe this behavior even if the integer is not associated with angular momentum.

4. Superstrings

The superstring generalizes the bosonic string by incorporating fermions propagating on the world-sheet. The first important difference is that the superstring does not have a tachyon thus avoiding a severe problem of the bosonic case. Besides that, the critical dimension is now 10 and the theory has space-time fermions. These fermions, together with the bosonic states form, at each mass level, representations of supersymmetry, a larger symmetry than the Lorentz symmetry. Whereas the different states making up a representation of the Lorentz symmetry are associated with polarizations of the same particle, supersymmetry relates the states of different particles with the same mass (although of course they can be interpreted as different states of the same "superparticle").

To understand the space-time symmetry, the first thing to do is to generalize spinor representations to higher dimensions. Since superstrings live in 10 dimensions, the appropriate Lorentz group is $SO(9,1)$. The different polarization states of massive particles fill representations of $SO(9)$ and those of massless ones fill representations of $SO(8)$. Moreover, since we are going to work exclusively in light-cone gauge, the only manifest symmetry of the theory will actually be $SO(8)$, although, of course, one can construct all the Lorentz generators as in the bosonic case. For that reason it seems appropriate to start our study by considering the rotational group in arbitrary dimension.

4.1 Spinor representations of $SO(n)$

The group of rotations in n dimensions can be represented by orthogonal matrices A of $n \times n$, namely satisfying

$$A^t A = 1 \tag{4.1}$$

These matrices act on an n-dimensional space of vectors v by multiplication:

$$v \to A\,v \tag{4.2}$$

and the representation is irreducible, namely there is no subspace invariant under *all* rotations. We can always write A as

$$A = e^M \tag{4.3}$$

where, if A is orthogonal, M is antisymmetric, *i.e.*

$$M = -M^t \quad \Rightarrow \quad A^{-1} = A^t \tag{4.4}$$

as can be easily seen. An $n \times n$ antisymmetric matrix is determined by $\frac{1}{2}n(n-1)$ parameters. For example defining the matrices M^{ij} as

$$\left(M^{ij}\right)_{pq} = \delta^i_p \delta^j_q - \delta^i_q \delta^j_p \tag{4.5}$$

we can write a generic matrix M as

$$M = \theta_{ij} M^{ij} \quad \Rightarrow \quad A = e^{\theta_{ij} M^{ij}} \tag{4.6}$$

where i, j are summed from 1 to n. In the previous expression, θ_{ij} are numbers with $\theta_{ij} = -\theta_{ij}$. Notice that the indices ij in M^{ij} indicate which matrix we are dealing with. For example M^{12} is the matrix

$$M^{12} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{4.7}$$

and the same for all the M^{ij}. The matrix M^{ij} represents an infinitesimal rotation in the $(i,\, j)$ plane and commutes with another M^{kl} as

$$\left[M^{ij}, M^{kl}\right] = \delta^{il} M^{jk} + \delta^{jk} M^{il} - \delta^{ik} M^{jl} - \delta^{jl} M^{ik} \tag{4.8}$$

as can be found by explicitly computing the matrix elements of both sides using the definition (4.5). Sometimes it is convenient to define hermitian operators as

$$J^{ij} = i M^{ij} \tag{4.9}$$

which commute according to;

$$\left[J^{ij}, J^{kl}\right] = i\left(\delta^{il} J^{jk} + \delta^{jk} J^{il} - \delta^{ik} J^{jl} - \delta^{jl} J^{ik}\right) \tag{4.10}$$

The representation we discussed is the fundamental or defining representation of $SO(n)$ and of its Lie algebra $so(n)$. In the following we try to find other representations of the Lie algebra, namely of the generators J^{ij}. This simply means finding a set of $\frac{1}{2}n(n-1)$ matrices obeying (4.10). If the matrices are of, say, $m \times m$ then the representation is m-dimensional, namely, the generators act on m-dimensional vectors. After that, by exponentiation as in (4.3) we get an m-dimensional representation of the full rotational group. In previous sections we saw some examples. Indeed, given the fundamental representation we saw that one can construct tensor representations by direct product. Although reducible, these representations can be easily decomposed into irreducible ones. For example, the two-index representation splits into the two-index antisymmetric representation, the two-index traceless symmetric and the singlet or identity representation given by the trace. However, now we want to look for other representations called spinor representations which can be found by generalizing the well known construction of $SO(3)$ spinors.

As we mentioned, after obtaining a representation of the Lie algebra we get a representation of the group by exponentiation as in (4.3). Formally, a representation of dimension m is a function ρ from a group G into the space of invertible matrices of $m \times m$ such that

$$\rho(g_1.g_2) = \rho(g_1).\rho(g_2), \quad \forall\ g_{1,2} \in G \tag{4.11}$$

where the dot on the left-hand side represents the product in the group and on the right hand side the usual matrix product. Notice that given a representation ρ we can write four representations $\rho_{1,2,3,4}$:

$$\rho_1(g) = \rho(g), \quad \rho_2(g) = \left[\rho^t(g)\right]^{-1}, \quad \rho_3(g) = \rho^*(g), \quad \rho_4(g) = \left[\rho^\dagger(g)\right]^{-1}, \tag{4.12}$$

which are the original representation, the inverse transpose, and their conjugates. In principle they can be equivalent to the original one, namely they can be the same representation in a different basis. If there is a fixed matrix S such that

$$\rho(g) = S\rho^*(g)S^{-1}, \quad \forall \quad g \in G \tag{4.13}$$

then the representation is self-conjugate and the same in the other cases. For example the fundamental representation does not give any new representation in this way since $A = A^* = [A^t]^{-1}$. In terms of the Lie algebra what this means is that, if we find matrices J^{ij} satisfying (4.10) then the following matrices all satisfy the same algebra:

$$J^{ij}, \quad -\left(J^{ij}\right)^t, \quad -\left(J^{ij}\right)^*, \quad \left(J^{ij}\right)^\dagger \tag{4.14}$$

If we look at unitary representations then $(J^{ij})^\dagger = J^{ij}$ and the only possible new representation is the inverse transpose (or conjugate) one. The bottom line is that, if we find a representation, we should look at the other ones that can be constructed in this way to see if they are the same or not.

With all this in mind we try now to generalize the spinor representations. In the case of $SO(3)$ the spinor representations are constructed by using the Pauli matrices that obey:

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c, \qquad \{\sigma_a, \sigma_b\} = 2\delta_{ab} \tag{4.15}$$

where $a = 1, 2, 3$, $\{A, B\} = AB + BA$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.16}$$

The Pauli matrices obey the $SO(3)$ algebra if we take $J^{12}=\frac{1}{2}\sigma_3$, $J^{13}=-\frac{1}{2}\sigma_2$, $J^{23}=\frac{1}{2}\sigma_1$ and, for that reason, define a two dimensional complex representation of $SO(3)$. Technically it is not a representation of $SO(3)$ since to each rotation correspond two matrices, for example, the identity is represented by 1 and -1. However, physically this is actually correct since a 2π rotation does not restore an electron to its original state, but to minus the state, a physically relevant effect.

We should note at this point that another way to construct spinors is to use that $SO(3)\sim SU(2)$ in which case the fundamental representation of $SU(2)$ are the spinors. In higher dimension this only works for $SO(4)\sim SU(2)\times SU(2)$ and $SO(6)\sim SU(4)$. so we need to generalize the Pauli matrices to higher dimension. As shown by Dirac, it turns out that it is convenient to generalize the anticommutation relations of the Pauli matrices and find n matrices γ^i such that

$$\{\gamma^i,\gamma^j\}=\gamma^i\gamma^j+\gamma^j\gamma^i=2\delta^{ij} \tag{4.17}$$

Later we are going to see a concrete construction of these matrices and which dimension the have but, for the moment, let us just assume that we have the γ^i which square to one and anticommute with each other. Consider now the commutator

$$\begin{aligned}
[\gamma^i\gamma^j,\gamma^k\gamma^l] &= \gamma^i\gamma^j\gamma^k\gamma^l-\gamma^k\gamma^l\gamma^i\gamma^j & (4.18)\\
&= -\gamma^i\gamma^k\gamma^j\gamma^l+2\delta^{jk}\gamma^i\gamma^l-\gamma^k\gamma^l\gamma^i\gamma^j & (4.19)\\
&= \gamma^k\gamma^i\gamma^j\gamma^l-2\delta^{ik}\gamma^j\gamma^l+2\delta^{jk}\gamma^i\gamma^l-\gamma^k\gamma^l\gamma^i\gamma^j & (4.20)\\
&= -\gamma^k\gamma^i\gamma^l\gamma^j+2\delta^{jl}\gamma^k\gamma^i-2\delta^{ik}\gamma^j\gamma^l+2\delta^{jk}\gamma^i\gamma^l-\gamma^k\gamma^l\gamma^i\gamma^j & (4.21)\\
&= -2\delta^{il}\gamma^k\gamma^j+2\delta^{jl}\gamma^k\gamma^i-2\delta^{ik}\gamma^j\gamma^l+2\delta^{jk}\gamma^i\gamma^l & (4.22)
\end{aligned}$$

where we used (4.17) repeatedly. This has a similar flavor to the commutation relations of the J^{ij} but since the J^{ij} are antisymmetric in the indices ij we should antisymmetrize the product $\gamma^i\gamma^j$. After doing that one can see that a normalization constant is needed and that

$$\Sigma^{ij}=\frac{i}{4}[\gamma^i,\gamma^j] \tag{4.23}$$

obey the same algebra as the J^{ij} and therefore provide a new representation of $so(n)$. This representation is the spinor representation we were looking for. With slightly more algebra we can find that

$$[\Sigma^{ij},\gamma^k]=i\left(\delta^{jk}\gamma^i-\delta^{ik}\gamma^j\right) \tag{4.24}$$

which has a very nice interpretation. Expanding in indices we find that the last equality can be understood as

$$\left(\Sigma^{ij}\right)_{\alpha\beta}\gamma^k_{\beta\delta}-(\Sigma^{ij})^t_{\delta\beta}\gamma^k_{\alpha\beta}+(J^{ij})_{kl}\gamma^l_{\alpha\delta}=0 \tag{4.25}$$

which is interpreted as saying that $\gamma^k_{\alpha\beta}$ is a symbol with three indices such that, if we rotate the index α in the spinor representation, the index β in the inverse transpose of the spinor representation (which we later see to be the same as the spinor representation) and the index k in the fundamental or vector representation, the symbol $\gamma^k_{\alpha\beta}$ is invariant. In that sense it is completely analogous to the Clebsch-Gordan coefficients or the $3j$-symbols and they determine the appropriate way to compose two spinor representations into a vector representation.

Now we should find a concrete representation of the gamma matrices whose properties, as we will see, depend on the particular dimension n in which we are working. We start by considering $so(n)$ for the case where n is an even number. In that case it turns out to be convenient to define an auxiliary space of $\frac{n}{2}$ fermions created an annihilated by anticommuting operators c_a, $c^\dagger_a$, $a = 1 \dots \frac{n}{2}$. For the moment we are going to work toward finding a concrete representation for these operators and later see their relation to gamma matrices. The anticommutation relations we want to represent are:

$$\{c^\dagger_a, c_b\} = \delta_{ab}, \qquad \{c_a, c_b\} = 0, \qquad \{c^\dagger_a, c^\dagger_b\} = 0 \tag{4.26}$$

The operators c_a and $c^\dagger_a$ act on a $2^{\frac{n}{2}}$ dimensional space, since each fermionic state can be empty or full. A basis on this space is given by

$$|\psi\rangle = |m_1, \dots, m_{\frac{n}{2}}\rangle, \qquad m_a = 0, 1 \tag{4.27}$$

On this basis the operators act as

$$c^\dagger_a|m_1, \dots, m_a = 0, \dots, m_{\frac{n}{2}}\rangle = (-1)^{\sum_{b<a} m_a} \; |m_1, \dots, m_a = 1, \dots, m_{\frac{n}{2}}\rangle \tag{4.28}$$
$$c_a|m_1, \dots, m_a = 1, \dots, m_{\frac{n}{2}}\rangle = (-1)^{\sum_{b<a} m_a} \; |m_1, \dots, m_a = 0, \dots, m_{\frac{n}{2}}\rangle \tag{4.29}$$

and zero otherwise, namely if we want to create a fermion on a site which is occupied or destroy one that is empty. Note that there is a sign given by the number of fermions occupying states to the left of the one we create or destroy. This is crucial for the operators at different sites to anticommute and reflects the fact that, for fermions, there is an overall sign on the state depending in which particular order they are created. If, in each site a we define a two dimensional space with basis $|m_a = 0\rangle$, $|m_b = 1\rangle$, then the operators can be written as direct product of two by two matrices acting on each site:

$$c^\dagger_a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{a} \otimes 1 \otimes \dots \otimes 1 \tag{4.30}$$

$$\tag{4.31}$$

$$c_a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{a} \otimes 1 \otimes \ldots \otimes 1 \tag{4.32}$$

where 1 represents the identity matrix, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the operator $(-1)^{n_b}$ at each site and the matrices $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ represent the act of creating or destroying a fermion.

Now we can proceed to define the following hermitian operators:

$$\gamma^a = c_a + c_a^\dagger \tag{4.33}$$

$$\gamma^{\frac{n}{2}+n_a} = i(c_a^\dagger - c_a) \tag{4.34}$$

They obey the anticommutation relations

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab} \tag{4.35}$$

$$\{\gamma^a, \gamma^{\frac{n}{2}+b}\} = 0 \tag{4.36}$$

$$\{\gamma^{\frac{n}{2}+a}, \gamma^{\frac{n}{2}+b}\} = 2\delta^{ab} \tag{4.37}$$

$$\tag{4.38}$$

as can be derived by simple application of the relations (4.26). This shows that such operators provide a concrete representation for the gamma matrices of $so(n)$. Using (4.32) we can write them explicitly as:

$$\gamma^a = \tau_3 \otimes \ldots \otimes \tau_3 \otimes \tau_1 \otimes 1 \otimes \ldots \otimes 1 \tag{4.39}$$

$$\gamma^{\frac{n}{2}+a} = \tau_3 \otimes \ldots \otimes \tau_3 \otimes \tau_2 \otimes 1 \otimes \ldots \otimes 1 \tag{4.40}$$

where

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.41}$$

are the Pauli matrices. For example, in the case of $so(8)$ that will interest us later, we have

$$\gamma^1 = \tau_1 \otimes 1 \otimes 1 \otimes 1 \tag{4.42}$$

$$\gamma^2 = \tau_3 \otimes \tau_1 \otimes 1 \otimes 1 \tag{4.43}$$

$$\gamma^3 = \tau_3 \otimes \tau_3 \otimes \tau_1 \otimes 1 \tag{4.44}$$

$$\gamma^4 = \tau_3 \otimes \tau_3 \otimes \tau_3 \otimes \tau_1 \tag{4.45}$$

$$\gamma^5 = \tau_2 \otimes 1 \otimes 1 \otimes 1 \tag{4.46}$$

$$\gamma^6 = \tau_3 \otimes \tau_2 \otimes 1 \otimes 1 \tag{4.47}$$

$$\gamma^7 = \tau_3 \otimes \tau_3 \otimes \tau_2 \otimes 1 \tag{4.48}$$

$$\gamma^8 = \tau_3 \otimes \tau_3 \otimes \tau_3 \otimes \tau_2 \tag{4.49}$$

knowing that $\tau_a\tau_b = i\epsilon_{abc}\tau_c + \delta_{ab}$ and multiplying the matrices at each site it is easy to verify that they obey the commutation relations (4.17).

An important property of the gamma matrices defined in this way is that they create or destroy one fermion, therefore if we have a state with an even total number of fermions then, applying the gamma matrices we get a state with an odd number of fermions and vice versa. The generators of rotations are given by the product of two gamma matrices:

$$\Sigma^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] \tag{4.50}$$

and therefore preserve the parity (odd of even) of the total number of fermions in the state. So, the $2^{\frac{n}{2}}$ states in the basis split in two sets which transform among themselves under rotations. We should remember that these fermions are an auxiliary concept. We are always dealing with one particle with $2^{\frac{n}{2}}$ polarizations. Going back to the example of $so(8)$ the states can be written as

$$\begin{aligned}|\psi\rangle = {} & \alpha_1|0000\rangle + \alpha_2|1100\rangle + \alpha_3|1010\rangle + \alpha_4|1001\rangle + \alpha_5|0110\rangle + \alpha_6|0101\rangle + \alpha_7|0011\rangle + \alpha_8|1111\rangle \\ & +\beta_1|1000\rangle + \beta_2|0100\rangle + \beta_3|0010\rangle + \beta_4|0001\rangle + \beta_5|1110\rangle + \beta_6|1011\rangle + \beta_7|1101\rangle + \beta_8|1110\rangle\end{aligned}$$

and the statement is that, under rotations, the upper row transform separate from the lower row. The reader can check this statement by finding explicitly the matrices Σ^{ij} using (4.49) and (4.50) although another example such as $so(6)$ might be simple to deal with.

The two representations into which the spinor representation splits are called left and right spinor representations. The corresponding $2^{\frac{n}{2}-1}$ dimensional spinors are called Weyl spinors. The operator that distinguishes the two representations is the total fermionic parity :

$$\gamma = (-1)^{\sum_a n_a} = \tau_3 \otimes \tau_3 \otimes \ldots \otimes \tau_3 \tag{4.51}$$

It is obvious that this operators squares to one and is easy to see that anticommutes with all gamma matrices since they increase or decrease the number of fermions by one. So we have

$$\gamma^2 = 1, \qquad \{\gamma, \gamma^j\} = 0, j = 1 \ldots n \tag{4.52}$$

meaning that γ is the extra matrix that we need if we want to represent the gamma matrices of $so(n+1)$. In that case, since γ does not change the number of fermions, infinitesimal rotations such as $\Sigma^{j(2n+1)}$ change the fermionic parity of the states and mix left and right spinors. For that reason, in odd dimension n, the spinor representation of dimension $2^{\frac{n-1}{2}}$ is irreducible and there are no left and right spinors.

After constructing the gamma matrices and, by commutation, the Σ^{ij} matrices, we can obtain a representation for a generic rotation by exponentiation:

$$R = e^{i\theta_{ij}\Sigma^{ij}} \tag{4.53}$$

Since the γ's are hermitian, so are the Σ^{ij} and therefore the representation is unitary, namely $RR^\dagger = 1$. This means that, out of the possible three new representations we can get, only one could be different: $R^* = (R^t)^{-1}$. This representation is actually the same as we can prove by finding a matrix C such that

$$C\gamma^i C^{-1} = [\gamma^i]^* = [\gamma^i]^t \tag{4.54}$$

If that is the case, then

$$C\Sigma^{ij}C^{-1} = -[\Sigma^{ij}]^* = -[\Sigma^{ij}]^t \qquad \Rightarrow \qquad CRC^{-1} = R^* = [R^t]^{-1} \tag{4.55}$$

We still need to find C. Looking at the expressions (4.34) or (4.40) we see that

$$[\gamma^a]^* = \gamma^a, \quad [\gamma^{\frac{n}{2}+a}]^* = -\gamma^{\frac{n}{2}+a} \tag{4.56}$$

So we need a matrix that produces that sign change. For example in the case of $so(8)$ one can see that a rotation in planes (56) and (78) will do the job. A perhaps simpler way to do this is to observe that, from (4.17) we deduce that

$$\gamma^i\gamma^j\gamma^i = -\gamma^j, \ \text{ if } \ i \neq j \tag{4.57}$$

$$\gamma^i\gamma^j\gamma^i = \gamma^j, \ \text{ if } \ i = j \tag{4.58}$$

Therefore if we multiply all gamma matrices which should change sign:

$$C = \gamma^{\frac{n}{2}+1}\gamma^{\frac{n}{2}+2}\dots\gamma^n \tag{4.59}$$

$$C^{-1} = \gamma^n\gamma^{n-1}\dots\gamma^{\frac{n}{2}+1} \tag{4.60}$$

we have

$$C\gamma^a C^{-1} = (-1)^{\frac{n}{2}}\gamma^a \tag{4.61}$$

$$C\gamma^{\frac{n}{2}+a}C^{-1} = (-1)^{\frac{n}{2}}\gamma^{\frac{n}{2}+a} \tag{4.62}$$

which has the desired effect up to the overall sign $(-1)^{\frac{n}{2}}$. This sign does not affect the matrices Σ^{ij} so we have just proved that the spinor representation is equal to its conjugate and found the matrix that relates the two. Since C is the product of $\frac{n}{2}$ matrices, it will convert a left spinor into a right spinor if $\frac{n}{2}$ is odd or a left into a left is

$\frac{n}{2}$ is even. So, if $n = 4k$, (k integer) the conjugate of a left(right) spinor is a left(right) spinor and if $n = 4k + 2$ the conjugate of a left spinor is a right spinor and vice versa. If $n = 2k+1$ then the spinor representation is irreducible and self-conjugate (since also $C\gamma C^{-1} = (-1)^{\frac{n}{2}}\gamma$).

Since a spinor and its conjugate transform in the same way we can ask ourselves if we could take the spinor to be real. This is not necessarily the case, for example, in $so(3)$, the spin $\frac{1}{2}$ representation is complex even if it is conjugate to itself. The second caveat is that we cannot impose $\zeta = \zeta^*$ since that is not Lorentz invariant but, in principle we can impose

$$\zeta^* = C\zeta \tag{4.63}$$

since both transform equally under rotations as we just found out. The problem is that we need

$$\zeta = (\zeta^*)^* = C^*C\zeta, \quad \text{namely,} \quad C^*C = 1 \tag{4.64}$$

which is not always true. In fact, from the definition of C and of the gamma matrices we get, if $n = 2k$:

$$C^* = (-1)^k C, \quad C^2 = (-1)^{\frac{k(k-1)}{2}} \tag{4.65}$$

The last one follows from the fact that

$$\gamma^1 \dots \gamma^p \gamma^1 \dots \gamma^p = (-1)^{\frac{p(p-1)}{2}} \tag{4.66}$$

as can be seen by commuting the gamma matrices so that we can use $(\gamma^i)^2 = 1$ and keeping track of all the minus signs. All in all we need

$$C^*C = (-1)^{\frac{k(k+1)}{2}} = 1, \quad \Rightarrow \quad \frac{k}{2} \text{ even or } \frac{k+1}{2} \text{ even} \tag{4.67}$$

In terms of n this means that $n = 8p$ or $n = 8p - 2$ for some integer p. If that is not the case however, we can still do something. If a representation is irreducible then the only matrix commuting with all rotations is the identity but the spinor representation is not, so there is another matrix, namely γ commuting with all rotations. We can then impose a reality condition

$$\zeta^* = \gamma C\zeta \tag{4.68}$$

In this case, since γ is real and $\gamma C\gamma = (-1)^k C$ we need $(-1)^{\frac{k(k-1)}{2}} = 1$ so we can still impose a reality condition if $n = 8p + 2$.

Now that we understand how to impose a reality condition, we would like to know if that means that the representation is real, namely if, for any rotation given by:

$$R = e^{i\theta_{ij}\Sigma^{ij}} \tag{4.69}$$

we can have, in some basis, that the matrix R has all real elements. That is equivalent to say that we have, in that basis, Σ^{ij} to be purely imaginary. A way to ensure that, is to have all gamma matrices real or all purely imaginary since $\Sigma^{ij} = \frac{i}{4}[\gamma^i, \gamma^j]$. Suppose that such a change of basis is given by a matrix S for which we propose, for reasons that become clearer later, the form

$$S = \frac{1 + \alpha C}{\sqrt{2}} \tag{4.70}$$

for some constant α (and in dimension $n = 8p + 2$ we replace $C \to \gamma C$). This matrix has to be invertible. The inverse is:

$$S^{-1} = \frac{1 - \alpha C}{\sqrt{2}} \quad \text{if } \alpha^2 = (-1)^{k+1} \text{ since } C^2 = (-1)^k \tag{4.71}$$

where $k = \frac{n}{2}$. Now we compute the gamma matrices in this new basis:

$$\tilde{\gamma}^i = S\gamma^i S^{-1} \tag{4.72}$$

and see that

$$\left(\tilde{\gamma}^i\right)^* = \left(S\gamma^i S^{-1}\right)^* = (-1)^k \tilde{\gamma}^i \tag{4.73}$$

after a short calculation where we use that $C^* = (-1)^k C$ and $\alpha^2 = (-1)^{k+1}$. That means that if k is even, the gamma matrices are real and if k is odd, they are purely imaginary. In both cases, the matrix R for any rotation is real so we have a real representation. It is easy to see also, that, if ζ satisfies the reality condition then we can find a real spinor η through:

$$\eta = \beta S\zeta, \tag{4.74}$$

Exercise: Find β in the previous equation such that η is real whenever $\zeta^* = C\zeta$.

Now we can study an example. In the case of $SO(8)$ we wrote the gamma matrices explicitly but they are not real. Let us compute C in that case:

$$C = \gamma^5\gamma^6\gamma^7\gamma^8 = -\tau_1 \otimes \tau_2 \otimes \tau_1 \otimes \tau_2 \tag{4.75}$$

which is real since $k = \frac{n}{2} = 4$ is even. If we want to transform the gamma matrices we have to do:

$$\tilde{\gamma}^i = S\gamma^i S^{-1} = \frac{1 + iC}{\sqrt{2}}\gamma^i \frac{1 - iC}{\sqrt{2}} = \frac{1}{2}\left(\gamma^i + (\gamma^i)^*\right) - \frac{i}{2}[\gamma^i, C] \tag{4.76}$$

Now, γ^i for $i = 1, 2, 3, 4$ are real and also:

$$C\gamma^i C = \gamma^i \quad \Rightarrow \quad [\gamma^i, C] = 0, \quad i = 1, 2, 3, 4 \tag{4.77}$$

So, $\gamma^{i=1,2,3,4}$ stay the same which is satisfying since they are already real. For $\gamma^{i=5,6,7,8}$, We have instead

$$\gamma^i + (\gamma^i)^* = 0, \quad \gamma^i C = -C\gamma^i, \quad \Rightarrow \quad \tilde{\gamma}^i = iC\gamma^i, \quad i = 5,6,7,8 \tag{4.78}$$

By explicit computation we find then:

$$\gamma^1 = \tau_1 \otimes 1 \otimes 1 \otimes 1 \tag{4.79}$$
$$\gamma^2 = \tau_3 \otimes \tau_1 \otimes 1 \otimes 1 \tag{4.80}$$
$$\gamma^3 = \tau_3 \otimes \tau_3 \otimes \tau_1 \otimes 1 \tag{4.81}$$
$$\gamma^4 = \tau_3 \otimes \tau_3 \otimes \tau_3 \otimes \tau_1 \tag{4.82}$$
$$\gamma^5 = -\tau_3 \otimes \epsilon \otimes \tau_1 \otimes \epsilon \tag{4.83}$$
$$\gamma^6 = \epsilon \otimes 1 \otimes \tau_1 \otimes \epsilon \tag{4.84}$$
$$\gamma^7 = -\epsilon \otimes \tau_1 \otimes \tau_3 \otimes \epsilon \tag{4.85}$$
$$\gamma^8 = \epsilon \otimes \tau_1 \otimes \epsilon \otimes 1 \tag{4.86}$$

where we replaced τ_2 by $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ using $\tau_2 = -i\epsilon$ to emphasize that all matrices are real.

Exercise: Show, by direct computation, that these new matrices obey the commutation relations of the gamma matrices (4.17).

We have seen before that, if the dimension is a multiple of 4, the left and right representations are conjugate to themselves. This means that, when the dimension is a multiple of 8, the spinors can be Weyl and Majorana at the same time. Namely the left and right spinors transform by themselves in real representations. In that case the number of independent real components is one fourth of the Dirac representation. Namely, spinors in $n = 8$ have 8 real components and can be left or right.

We can summarize our findings for the spinor representations of $SO(n)$ as follows:

Weyl spinors: In even dimension, $n = 2k$, the spinor representation is reducible and splits into left and right spinors.

Conjugate of Weyl spinors: In dimension $n = 4k$ the left and right representations are self-conjugate, namely the conjugate of a left spinor is a left spinor and the same with right spinors. In dimension $n = 4k + 2$ the conjugate of a left spinor is a right spinor and vice versa.

Majorana spinors: In dimension $n = 8p$, $n = 8p \pm 2$ we can do a change of basis so that the matrices of rotations are real. The gamma matrices are real if $n = 8p$ and imaginary if $n = 8p \pm 2$.

Majorana-Weyl spinors: If $n = 8p$ then, the left and right spinor representations are real. Spinors have one fourth the number of real components as the Dirac spinors of that dimension.

Now that we understand thoroughly the spinor representations we are interested in composing two of them. In the case of $SO(3)$ we know that two representations of spin $\frac{1}{2}$ compose to spin 1 or 0. Now, we should obtain the analogous result for $SO(n)$.

Suppose we have a rotation given by some angles θ_{ij}. We know that we can represent such rotation by an $n \times n$ matrix given by

$$A = e^{i\theta_{ij}J^{ij}} \tag{4.87}$$

or by an $2^{\frac{n}{2}} \times 2^{\frac{n}{2}}$ matrix given by

$$R = e^{i\theta_{ij}\Sigma^{ij}} \tag{4.88}$$

or by many others, given by tensor products, etc. But let us concentrate on these two. To emphasize the meaning of different representations, what we have is that

$$e^{i\theta_{ij}^{(1)}J^{ij}}e^{i\theta_{ij}^{(2)}J^{ij}} = e^{i\theta_{ij}^{(3)}J^{ij}} \quad \Rightarrow \quad e^{i\theta_{ij}^{(1)}\Sigma^{ij}}e^{i\theta_{ij}^{(2)}\Sigma^{ij}} = e^{i\theta_{ij}^{(3)}\Sigma^{ij}} \tag{4.89}$$

that is, the product of two rotations of angles $\theta_{ij}^{(1)}$ and $\theta_{ij}^{(2)}$ is another rotation of angles $\theta_{ij}^{(3)}$ and these last angles can be found using any of the representations.

Consider now the following matrix

$$\tilde{\gamma}^l = \left(e^{-it\theta_{ij}J^{ij}}\right)_{lk} e^{-it\theta_{ij}\Sigma^{ij}}\gamma^k e^{-it\theta_{ij}\Sigma^{ij}} \tag{4.90}$$

where t is a real parameter. Compute now

$$\partial_t\tilde{\gamma}^l = -it\theta_{ij}\left\{\left(e^{it\theta_{ij}J^{ij}}\right)_{lk}\left(J^{ij}\right)_{kp} e^{-it\theta_{ij}\Sigma^{ij}}\gamma^p e^{-it\theta_{ij}\Sigma^{ij}}\right. \tag{4.91}$$

$$\left. + \left(e^{it\theta_{ij}J^{ij}}\right)_{lk} e^{-it\theta_{ij}\Sigma^{ij}}[\Sigma^{ij},\gamma^k]e^{-it\theta_{ij}\Sigma^{ij}}\right\} \tag{4.92}$$

$$= 0 \tag{4.93}$$

where we used the identity (4.24). This means that $\tilde{\gamma}^l$ is independent of t. On the other hand we have $\tilde{\gamma}^l(t=0) = \gamma^l$ so $\tilde{\gamma}^l = \gamma^l$ for any t. In particular if $t = 1$ we get the rotation matrices A and R, so we just derived:

$$R^{-1}\gamma^k R = A_{kl}\gamma^l \tag{4.94}$$

which shows that, if we rotate the indices of the gamma matrix in the spinor representation, the index k rotates as a vector. Again, we see that the gamma matrix is a Clebsch-Gordan coefficient that composes the two spinor representations into the vector representation.

We are ready now to compose two spinor representations. If we have two spinors of $2^{\frac{n}{2}}$ components, η, ζ that rotate as

$$\eta \to R\eta, \quad \zeta \to R\zeta \tag{4.95}$$

when we multiply them we get $2^{\frac{n}{2}} \times 2^{\frac{n}{2}} = 2^n$ components out of which we want to extract linear combinations transforming by themselves under rotations. For example we can form a scalar by doing:

$$s = \eta^t C \zeta \tag{4.96}$$

To see it is a scalar we have to use (see eq.(4.55)) that:

$$CRC^{-1} = (R^t)^{-1}, \;\Rightarrow\; C^{-1}R^tC = R^{-1} \tag{4.97}$$

Indeed, the scalar s transforms as:

$$s \to \tilde{s} = \eta^t R^t C R \zeta = \eta^t C C^{-1} R^t C R \zeta = \eta^t C R^{-1} R \zeta = s \tag{4.98}$$

namely is invariant. We can also form a vector:

$$v^k = \eta^t C \gamma^k \zeta \tag{4.99}$$

Again, we check, using eq.(4.94) :

$$v^k \to \eta^t R^t C \gamma^k R \zeta = \eta^t C R^{-1} \gamma^k R \zeta = A_{kl} \eta^t C \gamma^l \zeta = A_{kl} v^l \tag{4.100}$$

which is the usual rotations for vectors. In fact now we can form several antisymmetric tensors by doing:

$$\begin{aligned}
v^{k_1 k_2} &= \eta^t C \gamma^{[k_1} \gamma^{k_2]} \zeta && (4.101)\\
v^{k_1 k_2 k_3} &= \eta^t C \gamma^{[k_1} \gamma^{k_2} \gamma^{k_3]} \zeta && (4.102)\\
v^{k_1 k_2 \ldots k_j} &= \eta^t C \gamma^{[k_1} \gamma^{k_2} \ldots \gamma^{k_j]} \zeta && (4.103)\\
v^{k_1 k_2 \ldots k_n} &= \eta^t C \gamma^{[k_1} \gamma^{k_2} \ldots \gamma^{k_n]} \zeta && (4.104)\\
& && (4.105)
\end{aligned}$$

The square brackets mean that we antisymmetrize the corresponding indices. We need to do so because the symmetric part can be extracted using the anticommutation rules of the gamma matrices. An antisymmetric tensor of j indices has $\begin{pmatrix} n \\ j \end{pmatrix}$ components, therefore we found

$$1 + n + \begin{pmatrix} n \\ 2 \end{pmatrix} + \begin{pmatrix} n \\ 3 \end{pmatrix} + \ldots \begin{pmatrix} n \\ 2 \end{pmatrix} = 2^n \tag{4.106}$$

independent components (as follow from using Newton's binomial formula to expand $(1+1)^n$). Therefore we exhausted all independent products. This gives the product of two spinor representations in any dimension. Now there are some particulars in even dimension. We can ask what happens if we compose two left representations or a left and a right for example. The answer follows simply from the fact that a gamma matrix times a left spinor gives a right spinor and vice versa. The only caveat is the matrix C that converts a left representation into a right one if $n = 4k + 2$ or the left into a left (and right into a right) if $n = 4k$. For example the vector representation is obtained out of a left and a right representation if $n = 4k$ and from two lefts or two rights if $n = 4k + 2$. In general we have:

$\mathbf{n = 4k}$**:**

LL and RR give: scalar, two index tensor ,etc. Namely, antisymmetric tensors with an even number of indices.

LR or RL give: antisymmetric tensors with odd number of indices (including vector rep.)

$\mathbf{n = 4k + 2}$**:**

LL and RR give: antisymmetric tensors with odd number of indices (including vector rep.)

LR or RL give: scalar, two index tensor ,etc. Namely, antisymmetric tensor with even number of indices.

4.2 More on the case of $SO(8)$

We have found that, for $SO(8)$ we can write the gamma matrices in a real representation:

$$\begin{array}{ll}
\gamma^1 = \ \ \tau_1 \otimes 1 \otimes 1 \otimes 1 & \gamma^5 = -\tau_3 \otimes \epsilon \otimes \tau_1 \otimes \epsilon \\
\gamma^2 = \ \ \tau_3 \otimes \tau_1 \otimes 1 \otimes 1 & \gamma^6 = \ \ \epsilon \otimes 1 \otimes \tau_1 \otimes \epsilon \\
\gamma^3 = \ \ \tau_3 \otimes \tau_3 \otimes \tau_1 \otimes 1 & \gamma^7 = -\epsilon \otimes \tau_1 \otimes \tau_3 \otimes \epsilon \\
\gamma^4 = \tau_3 \otimes \tau_3 \otimes \tau_3 \otimes \tau_1 & \gamma^8 = \ \ \epsilon \otimes \tau_1 \otimes \epsilon \otimes 1
\end{array} \tag{4.107}$$

Since the representation is real, the charge conjugation matrix is the identity, namely we do not need any matrix to map the representation to its conjugate. We also know that the 16 dimensional space of states on which these matrices act can be divided into two eight dimensional spaces corresponding to the left and right spinors. This amounts to a reordering of the states of the basis and therefore the gamma matrices in that basis are still real and of the form:

$$\gamma^i = \begin{pmatrix} 0 & \rho^i \\ \hat{\rho}^i & 0 \end{pmatrix} \tag{4.108}$$

where ρ^i and $\hat{\rho}^i$ are eight dimensional matrices. The fact that the diagonal blocks are zero is a reflection of the fact that a gamma matrix maps a left spinor into a right spinor and vice versa as we saw before. Moreover, the matrices are symmetric which implies:

$$\hat{\rho}^i = \left(\rho^i\right)^t \tag{4.109}$$

namely we only need to know the matrices ρ^i to reconstruct all the gamma matrices. At this point it is useful to divide the spinor index $\alpha = 1\ldots 16$ in two: $\alpha = (a, \dot{a})$, where $a, \dot{a} = 1\ldots 8$ and they correspond to left and right spinors respectively. With that convention the indices of the matrices ρ^i are:

$$\rho^{i\, a\dot{b}}, \quad \hat{\rho}^{i\, \dot{a}b}, \qquad \text{with} \qquad \hat{\rho}^{i\, \dot{a}b} = \rho^{i\, b\dot{a}}. \tag{4.110}$$

From the properties of the gamma matrices we derive:

$$\rho^i \left(\rho^j\right)^t + \rho^j \left(\rho^i\right)^t = 2\delta^{ij} \tag{4.111}$$

or

$$\rho^{i\, a\dot{c}}\rho^{j\, b\dot{c}} + \rho^{j\, a\dot{c}}\rho^{i\, b\dot{c}} = 2\delta^{ij}\delta^{ab} \tag{4.112}$$

From the matrices (4.107) we can extract the rho matrices as:

$$\begin{aligned}
\rho^1 &= \tau_1 \otimes 1 \otimes 1 & \qquad \rho^5 &= -1 \otimes \tau_1 \otimes \epsilon \\
\rho^2 &= \tau_3 \otimes \tau_1 \otimes 1 & \qquad \rho^6 &= \tau_1 \otimes \tau_3 \otimes \epsilon \\
\rho^3 &= \tau_3 \otimes \tau_3 \otimes \tau_1 & \qquad \rho^7 &= -\tau_1 \otimes \epsilon \otimes 1 \\
\rho^4 &= \tau_3 \otimes \tau_3 \otimes \tau_3 & \qquad \rho^8 &= \epsilon \otimes \tau_1 \otimes \epsilon
\end{aligned} \tag{4.113}$$

which can be seen to satisfy (4.111) by direct computation.

Exercise Check the last statement, namely, by direct computation verify that the rho matrices defined in (4.113) satisfy the properties (4.111).

In eq.(4.112), there is an intriguing symmetry between the indices (i, j) and (a, b) since both run from 1 to 8 and enter equivalently in the relation. In fact we can define a new set of 8×8 matrices labeled by a as:

$$\tilde{\rho}^a, \quad \text{with} \quad \tilde{\rho}^{a\, i\dot{b}} = \rho^{i\, a\dot{b}} \tag{4.114}$$

We can then rewrite the relation (4.112) as:

$$\tilde{\rho}^{a\, i\dot{c}}\tilde{\rho}^{b\, j\dot{c}} + \tilde{\rho}^{a\, j\dot{c}}\tilde{\rho}^{b\, i\dot{c}} = 2\delta^{ij}\delta^{ab} \tag{4.115}$$

or, equivalently:

$$\tilde{\rho}^a \left(\tilde{\rho}^b\right)^t + \tilde{\rho}^b \left(\tilde{\rho}^a\right)^t = 2\delta^{ab} \tag{4.116}$$

Thus, we can define also new gamma matrices labeled by a:

$$\tilde{\gamma}^a = \begin{pmatrix} 0 & \tilde{\rho}^a \\ (\tilde{\rho}^a)^t & 0 \end{pmatrix} \tag{4.117}$$

which satisfy

$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 2\delta^{ab} \tag{4.118}$$

Exercise Verify this last statement.

The $\tilde{\gamma}^a$ are 16×16 matrices labeled by an index in the left spinor representation and which act on a space of states sum of the vector representation and the right spinor representation. Essentially, the gamma matrices compose the vector representation together with left and right spinor representations to the identity. In the case of $SO(8)$ the three representations are real and eight-dimensional and one can think of gamma matrices labeled by i or a or $\dot{a}$. Each set rotating according to the corresponding representation, vector, left or right spinor, with their corresponding indices rotating according to the representation they belong to:

$$\gamma^i_{a\dot{b}}, \qquad \tilde{\gamma}^a_{i\dot{b}}, \qquad \tilde{\gamma}^{\dot{b}}_{ia} \tag{4.119}$$

At this stage this seems to be just a curiosity but will become very important in the next subsection.

4.3 Green-Schwarz superstring: spectrum

The superstring lives in ten dimensions and therefore, in light cone gauge, the relevant group of transverse rotations is $SO(8)$, namely, all expressions are manifestly invariant under $SO(8)$. There are eight bosonic variables corresponding to the transverse coordinates $X^{i=1\dots8}$. They obey the wave equation whose general solution is of the form:

$$X^i(\sigma,\tau) = X^i_L(\sigma+\tau) + X^i_R(\sigma-\tau) \tag{4.120}$$

Therefore, the bosonic sector works as in the bosonic string. To that, in the light-cone Green-Schwarz formulation of the superstring one adds, on the world-sheet, a set of right moving and left moving fermionic variables.

The fermionic variables are taken to transform in one of the eight-dimensional spinor real representations[4]. If both, the left and right moving variables transform in the same spinor representation, the theory is called type IIB superstring, whereas if they transform in opposite representations the theory is called type IIA. For that

[4]One should not confuse the idea of right and left moving variables on the world-sheet with the idea of left and right spinors which are unrelated.

reason, IIA strings are non-chiral, namely the theory is invariant under interchange of left and right spinors. Type IIB on the other hand is a chiral theory. Massive states transform in representations of $SO(9)$ and therefore they have no chirality. In fact the massive spectrum of type IIA and type IIB are exactly the same, only the massless states are different as we are going to see. To be concrete, introduce the world sheet variable

$$S^a(\sigma,\tau) = S^a(\sigma+\tau), \qquad S^a(\sigma+2\pi,\tau) = S^a(\sigma,\tau) \tag{4.121}$$

The index $a = 1\ldots 8$ transforms in the left spinor representation. Spinors transforming in the right spinor representation we denote with a dotted index $\dot{a}$. These variables are real which is expressed as:

$$(S^a)^\dagger = S^a \tag{4.122}$$

We have to impose equal time anticommutation relations similar to the ones we imposed for bosons. We have to respect rotational invariance so we naturally impose:

$$\{S^a(\sigma,\tau), S^b(\sigma',\tau)\} = \delta^{ab}\delta(\sigma-\sigma') \tag{4.123}$$

For rotational invariance the matrix on the right hand side of the equation should be the charge conjugation matrix, but for $SO(8)$ in the basis where the gamma matrices are real, the charge conjugation matrix is the identity. We now expand in modes as

$$S^a(\sigma,\tau) = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} e^{-in(\sigma+\tau)} S^a_n \tag{4.124}$$

where we took into account that S^a is a function of $\sigma+\tau$ only. The fact that $S^a(\sigma,\tau)$ is hermitian translates into

$$(S^a_n)^\dagger = S^a_{-n} \tag{4.125}$$

The Fourier modes can be obtained through:

$$S^a_n = \frac{1}{\sqrt{2\pi}}\, e^{in\tau} \int_0^{2\pi} e^{in\sigma} S^a(\sigma,\tau)\, d\sigma \tag{4.126}$$

Using the anti-commutation relations (4.123) we obtain

$$\{S^a_n, S^b_m\} = \delta^{ab}\delta_{m+n} \tag{4.127}$$

In the mode expansion (4.124) we notice that modes with positive n multiply a wave with negative frequency $e^{-in\tau}$. This means that those modes carry negative energy and therefore S^a_n should be an annihilation operator, namely it decreases the energy of the system. For the same reason S^a_n with negative n is a creation operator. The vacuum is therefore defined as:

$$S^a_{n>0}|0\rangle = 0 \tag{4.128}$$

From (4.125) and (4.127), the operators S^a_n obey:

$$\{(S^a_n)^\dagger\,, S^b_n\} = \delta^{ab} \tag{4.129}$$

which means that they are standard fermionic creation and annihilation operators. The space of states is labeled by the occupation number of each mode which can only be zero or one since we cannot put more than one fermion in each state. We should also introduce right moving fermions given by

$$\tilde{S}^a(\sigma,\tau) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{in(\sigma-\tau)} \tilde{S}^a_n \tag{4.130}$$

for type IIB or

$$\tilde{S}^{\dot{a}}(\sigma,\tau) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} e^{in(\sigma-\tau)} \tilde{S}^{\dot{a}}_n \tag{4.131}$$

for type IIA. Notice that we put the sign in the exponent so that $\tilde{S}^a_{n>0}$ is still an annihilation operator. In fact everything is the same for the right moving modes so the total oscillator space of states is given by:

$$|\psi\rangle = |N^i_n, \tilde{N}^i_n, N^a_n, \tilde{N}^a_n\rangle, \quad N^i_n, \tilde{N}^i_n \geq 0, \quad N^a_n, \tilde{N}^a_n = 0, 1 \tag{4.132}$$

where we added the bosonic sector. We labeled the right moving fermions with a as corresponds to type IIB. In type IIA we should label them with $\dot{a}$. The fermionic occupation numbers N^a_n are the operators

$$N^a_n = (S^a_n)^\dagger\ S^a_n, \qquad (n > 0). \tag{4.133}$$

with no sum over a or n. As we know, increasing the occupation number in the bosonic sector increases the mass of the string. Including the contribution from the fermionic oscillators, the mass of the string is given by

$$M^2 = \frac{2}{\alpha'}\left(N + \tilde{N} + N_f + \tilde{N}_f\right) = \frac{2}{\alpha'}\left[\sum_{i=1}^{8}\sum_{m=1}^{\infty} m\left(N^i_m + \tilde{N}^i_m\right) + \sum_{a=1}^{8}\sum_{m=1}^{\infty} m\left(N^a_m + \tilde{N}^a_m\right)\right] \tag{4.134}$$

In terms of the variables $S^a(\sigma,\tau)$ we can write the fermionic contribution also as

$$M^2_f = \frac{i}{\alpha'} \int_0^{2\pi} d\sigma \left(S^a \partial_\sigma S^a - \tilde{S}^a \partial_\sigma \tilde{S}^a\right) \tag{4.135}$$

This is rather straight-forward but there is a crucial point we ignored. The fermions have zero modes S_0^a which, from (4.127), obey

$$\{S_0^a, S_0^b\} = \delta^{ab} \tag{4.136}$$

and which do not appear in the expression for the mass. We also have, from (4.125):

$$(S_0^a)^\dagger = S_0^b \tag{4.137}$$

Therefore, we have to represent the operators S_0^a as hermitian matrices acting on a space of states. The anticommutation relations are actually exactly the same as those of the gamma matrices $\tilde{\gamma}^a$ that we found in (4.118). Therefore we can represent these operators as symmetric and real matrices acting on a sixteen dimensional space of states sum of the vector and right spinor representation:

$$S^a = \frac{1}{\sqrt{2}}\tilde{\gamma}^a_{ib} \tag{4.138}$$

These sixteen states all have the same energy. Therefore the vacuum is degenerate. This is a crucial difference with the bosonic string where the vacuum is unique and corresponds to a scalar, the tachyon. We have to consider also the right moving fermions which have another sixteen states in the vector times right spinor representation for type IIB or vector times left spinor representation for IIA. In total we have 256 states. To know how they transform under rotations we have to remember the rules of composition derived in the previous subsection. What we obtain is:

IIA

$$\Big[|i\rangle \oplus |\dot{a}\rangle\Big] \otimes \Big[|j\rangle \oplus |b\rangle\Big] = (\mathbf{1}+\mathbf{28}+\mathbf{35}+\mathbf{8}+\mathbf{56})_B + (\mathbf{8}+\mathbf{56}+\mathbf{8}+\mathbf{56})_F$$

IIB

$$\Big[|i\rangle \oplus |\dot{a}\rangle\Big] \otimes \Big[|j\rangle \oplus |\dot{b}\rangle\Big] = (\mathbf{1}+\mathbf{28}+\mathbf{35}+\mathbf{1}+\mathbf{28}+\mathbf{35})_B + (\mathbf{8}+\mathbf{56}+\mathbf{8}+\mathbf{56})_F \tag{4.139}$$

Let us analyze this result. From composing the two vector representations in the left and right sectors we get a scalar, *i.e.* the trace, a two-index antisymmetric tensor which has 28 components and a two-index traceless symmetric tensor which has 35 components. This is the same in type IIA and IIB and gives the dilaton, B-field and graviton. In type IIB, composing two left spinors gives a scalar, a two-index antisymmetric tensor and a four-index self dual antisymmetric tensor which has 35 independent components. In type IIA, composing a left and a right spinor gives a vector, eight components, and a three index antisymmetric tensor, 56 components. Composing the vector representation with the right spinors give a left spinor, eight components and a spin $\frac{3}{2}$ representation that has 56 components. Similarly with a vector and a left spinor. We can therefore rewrite the result as:

IIA

$$\Big[|i\rangle \oplus |\dot{a}\rangle\Big] \otimes \Big[|j\rangle \oplus |b\rangle\Big] = (\phi, B_{ij}, G_{ij}, A^{[1]}_i, A^{[3]}_{ijk})_B + (\psi_a + \Psi_{i\dot{a}} + \psi_{\dot{a}} + \Psi_{ia})_F$$

IIB

$$\Big[|i\rangle \oplus |\dot{a}\rangle\Big] \otimes \Big[|j\rangle \oplus |\dot{b}\rangle\Big] = (\phi, B_{ij}, G_{ij}, \chi, \tilde{B}_{ij}, A^{[4]}_{ijkl})_B + (\psi_a + \Psi_{i\dot{a}} + \psi_a + \Psi_{i\dot{a}})_F \tag{4.140}$$

where we have

$$A^{[4]}_{ijkl} = \frac{1}{4!}\epsilon_{ijklmnop}A^{[4]}_{mnop} \tag{4.141}$$

and

$$\rho^{ia\dot{b}}\Psi_{ia} = 0, \qquad \rho^{ib\dot{a}}\Psi_{i\dot{a}} = 0 \tag{4.142}$$

We then have 256 ground states, half of which are bosonic and the other half fermionic. The 64 bosons that come from multiplying the two vector representations are called Neveau-Schwarz-Neveau-Schwarz (NS-NS) bosons and the 64 that come from multiplying the two spinor representations are called Ramond-Ramond (RR) bosons. These names come from the Neveau-Schwarz-Ramond representation of the superstring which we do not describe in this notes. The NS-NS sector consists of the dilaton, graviton and B-field. The fields in the RR sector are p-forms $A^{[p]}_{i_1\ldots i_p}$. In fact one can also construct higher order forms, for example in type IIA we can construct a 5-form and a 7-form by using the antisymmetric products of the gamma matrices as we saw in the previous subsection. They cannot be independent however since there are only 64 states in the product of the left and right representations. By studying the gamma matrices one in fact finds that:

$$A^{[5]}_{i_1\ldots i_5} = \frac{1}{3!}\epsilon_{i_1\ldots i_5 j_1\ldots j_3}A^{[3]}_{j_1\ldots j_3} \tag{4.143}$$

$$A^{[7]}_{i_1\ldots i_7} = \epsilon_{i_1\ldots i_7 j_1}A^{[1]}_{j_1} \tag{4.144}$$

In type IIB we have a 6-form and an 8-form:

$$A^{[6]}_{i_1\ldots i_6} = \epsilon_{i_1\ldots i_8 j_1 j_2}\tilde{B}_{j_1 j_2} \tag{4.145}$$

$$A^{[8]}_{i_1\ldots i_8} = \epsilon_{i_1\ldots i_8}\chi \tag{4.146}$$

We know that a string is a source for the B-field since it couples to it through the world-sheet action:

$$S_{int} = \int d\sigma d\tau \partial_\sigma X^\mu \partial_\tau X^{nu} B_{\mu\nu} \tag{4.147}$$

Higher order forms can couple to p-branes which are p dimensional branes that propagate in time. Their trajectory is given by the coordinates X^μ as a function of $p+1$ parameters: $X^\mu(\sigma_1, \ldots, \sigma_p, \tau)$. The world-volume action contains a term:

$$S_{int} \int d\tau d\sigma_1 \ldots d\sigma_p \; \partial_{\sigma_1} X^{\mu_1} \ldots \partial_{\sigma_p} X^{\mu_p} \partial_\tau X^{\mu_{p+1}} \; A^{[p+1]}_{\mu_1 \cdots \mu_{p+1}} \tag{4.148}$$

We can then anticipate that type IIA contains 0-branes (which are particles), 2-branes, 4-branes, 6-branes and 8-branes. On the other hand type IIB contains (-1)-branes (instantons), 1-branes (string objects different from the fundamental string), 3-branes, 5-branes, and 7-branes. The fact that the corresponding forms are not independent but dual to each other will imply that the branes are also dual to each other. The A-fields are dual when their number of indices add up to eight so the branes are dual when their spacial dimensions add up to 6. For example 0 and 6-branes are dual, and so are 2 and 4-branes, (-1) and 7-branes and 1 and 5-branes. The 3-brane is self-dual because the 4-form is self-dual. We are going to study these objects in the next section. Here we go back to the superstring spectrum.

The particles that we saw, namely those corresponding to the world-sheet vacuum, are massless and there is no tachyon which is a significant improvement over the bosonic string. The massive spectrum is constructed as we saw before, by applying the creation operators of the different modes. The only point is that we have to take into account that the vacuum is degenerate. Since in the vacuum sector we have that half the states are bosonic and half fermionic, we are going to get always half and half at each level. For example at level one the states are

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle, \quad \alpha^i_{-1}\tilde{S}^a_{-1}|0\rangle, \quad S^a_{-1}\tilde{\alpha}^i_{-1}|0\rangle, \quad S^a_{-1}\tilde{S}^b_{-1}|0\rangle \tag{4.149}$$

which are $256 \times 256 = 2^{16}$ states, half bosonic and half fermionic. When doing this calculation we should take into account that there are 256 vacua. We also imposed the level matching condition which is the same as in the bosonic string, the total energy of left and right movers should be the same. The fact that, at each level, the number of bosons and fermions is always the same is a manifestation of a symmetry generated by the zero modes of S^a and $\tilde{S}^a$. When applied to a state they change it from bosonic to fermionic and vice versa without changing the mass. We analyze this symmetry called supersymmetry in the next subsection.

4.4 Space-time supersymmetry of the superstring

A symmetry is an operator that commutes with the Hamiltonian or, in this case, the mass squared operator. When applied to a state it gives another state with the same mass. We found already sixteen such operators:

$$Q^a = \sqrt{2p^+}S_0^a, \quad \tilde{Q}^a = \sqrt{2p^+}\tilde{S}_0^a \tag{4.150}$$

where the factors $\sqrt{2p^+}$ are introduced for later convenience. It is interesting that one can construct another sixteen hermitian operators:

$$Q^{\dot{a}} = \frac{1}{2\alpha'}\frac{1}{\sqrt{\pi p^+}}\int d\sigma \rho^{i\,b\dot{a}} S^b \left(2\pi\alpha'\Pi^i + \partial_\sigma X^i\right) \tag{4.151}$$

$$\tilde{Q}^{\dot{a}} = \frac{1}{2\alpha'}\frac{1}{\sqrt{\pi p^+}}\int d\sigma \rho^{i\,b\dot{a}} \tilde{S}^b \left(2\pi\alpha'\Pi^i - \partial_\sigma X^i\right) \tag{4.152}$$

which are constructed so that they contain only left and right moving modes respectively. This is clearly seen in their mode expansion:

$$Q^{\dot{a}} = \frac{1}{\sqrt{\alpha' p^+}}\rho^{i\,\dot{a}b}\sum_{m=-\infty}^{\infty} S_m^b \alpha_{-m}^i \tag{4.153}$$

$$\tilde{Q}^{\dot{a}} = \frac{1}{\sqrt{\alpha' p^+}}\rho^{i\,\dot{a}b}\sum_{m=-\infty}^{\infty} \tilde{S}_m^b \tilde{\alpha}_{-m}^i \tag{4.154}$$

where we used the convention $\alpha_0^i = \tilde{\alpha}_0^i = \sqrt{\frac{\alpha'}{2}}p^i$. Using the known commutation relations between the modes we can readily compute the commutation relations of the Q's:

$$\{Q^a, Q^b\} = 2p^+\delta^{ab} \tag{4.155}$$

$$\{Q^a, Q^{\dot{b}}\} = \rho^{i\,a\dot{b}}p^i \tag{4.156}$$

$$\{Q^{\dot{a}}, Q^{\dot{b}}\} = \delta^{\dot{a}\dot{b}}\left[\frac{1}{p^+}p_\perp^2 + \frac{4}{\alpha' p^+}\left(\sum_{ni} nN_n^i + \sum_{na} nN_n^a\right)\right] = p^- \tag{4.157}$$

where, in the last line, we get the contribution to the energy from the right moving oscillators alone. However, we should remember that, from the level matching condition, the total contribution to p^- from the left and right movers is the same. A similar result is valid for the charges $\tilde{Q}$. In total we have 32 real supercharges, as the Q's are called. They all commute with the mass squared operator. Among themselves they anticommute to a translation and for that reason sometimes they are described as the square root of the translations.

5. Open strings and D-branes

In the previous section we saw that, if the RR fields have sources, those should be extended objects. These objects are called D-branes and appear in the study of open strings. In this section we start by considering bosonic open strings and then open superstrings. After that we study D-branes.

5.1 Bosonic open strings

As its name suggests, open strings are strings with two end points. The world-sheet is now a strip instead of a cylinder but the action is still the area of the world-sheet. If we parameterize the world-sheet by σ with $0 \leq \sigma \leq \pi$ and τ with $\tau_i \leq \tau \leq \tau_f$, the action is

$$S = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma \int_{\tau_i}^{\tau_f} d\tau \, \sqrt{(\partial_\sigma X.\partial_\tau X)^2 - (\partial_\sigma X)^2(\partial_\tau X)^2} \tag{5.1}$$

as before. The difference is that now we do not impose periodicity in σ. If we try to minimize the action, the first order variation of S is:

$$\delta S = \int d\sigma d\tau \left[\frac{\partial \mathcal{L}}{\partial X^\mu} - \partial_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\sigma X^\mu)} - \partial_\tau \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)}\right] \delta X^\mu \tag{5.2}$$

$$+ \int_{\tau_i}^{\tau_f} d\tau \, \frac{\partial \mathcal{L}}{\partial(\partial_\sigma X^\mu)} \, \delta X^\mu \bigg|_{\sigma=0}^{\sigma=\pi} + \int_0^\pi d\sigma \, \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)} \, \delta X^\mu \bigg|_{\tau=\tau_i}^{\tau=\tau_f} \tag{5.3}$$

The variation should be zero for arbitrary values of δX^μ. This imposes the equation of motion

$$\left[\frac{\partial \mathcal{L}}{\partial X^\mu} - \partial_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\sigma X^\mu)} - \partial_\tau \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)}\right] = 0 \tag{5.4}$$

However we still have to cancel the boundary terms. If we fix the initial and final shape of the string we should have

$$\delta X^\mu(\sigma, \tau = \tau_i) = 0, \qquad \delta X^\mu(\sigma, \tau = \tau_f) = 0 \tag{5.5}$$

which cancels the last term. For closed strings the other boundary term was zero because of the periodicity of the string. In fact there was no boundary in sigma. Now we have to impose some boundary condition at $\sigma = 0, \pi$ such that the boundary term vanishes. We see two possibilities:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\sigma X^\mu)} = 0, \qquad \text{or} \qquad \delta X^\mu = 0, \qquad \text{at} \;\; \sigma = 0, \pi \tag{5.6}$$

More insight into the boundary conditions follows if we remember how we derived the conservation of energy-momentum in the closed string. In flat space the Lagrangian in eq.(5.1) is actually independent of X^μ, it depends only on its derivatives. Therefore the equation of motion reduces to:

$$\partial_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\sigma X^\mu)} + \partial_\tau \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)} = 0 \tag{5.7}$$

We can now integrate in σ and obtain:

$$\partial_\tau P^\mu = \partial_\tau \int d\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)} = \left. \frac{\partial \mathcal{L}}{\partial(\partial_\sigma X^\mu)} \right|_{\sigma=0}^{\sigma=\pi} \tag{5.8}$$

In the closed string the right hand side is zero because of periodicity and we conclude that the energy-momentum defined as $P^\mu = \int d\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)}$ is conserved. For the open string it is only conserved if we impose the boundary condition

$$\frac{\partial \mathcal{L}}{\partial(\partial_\sigma X^\mu)} = 0 \tag{5.9}$$

at both ends of the string. Since conservation of energy momentum seems like a good thing, the other boundary condition $\delta X^\mu = 0$ was ignored for a long time. The first to completely understand what it meant was Polchinski. What he argued is that such boundary condition fixes the end point of the string to live in some subspace. Such subspace however should be thought as a dynamical object. The presence of this object breaks translational invariance in the same way as, for example, an ordinary wall does. The momentum perpendicular to the wall is not conserved whereas the parallel one does. These objects are precisely the D-branes and their study is an extremely interesting subject within string theory. However we are anticipating things a bit, we continue now with the study of open strings but considering both types of boundary conditions.

As in the case of the closed string we can take conformal gauge. In order to do so we redefine (σ, τ) in such a way that the conditions

$$\eta_{\mu\nu} \partial_\sigma X^\mu \partial_\tau X^\nu = (\partial_\sigma X \partial_\tau X) = 0 \tag{5.10}$$

$$\eta_{\mu\nu} \left(\partial_\sigma X^\mu \partial_\sigma X^\nu + \partial_\tau X^\mu \partial_\tau X^\nu\right) = (\partial_\sigma X)^2 + (\partial_\tau X)^2 = 0 \tag{5.11}$$

are satisfied. As we know, this largely simplifies the equations of motion, in fact, they reduce to the wave equation:

$$\left(\partial_\sigma^2 - \partial_\tau^2\right) X^\mu = 0 \tag{5.12}$$

It also simplifies the boundary conditions. They can be written as:

$$\text{Neumann:}\;\; \partial_\sigma X^\mu = 0 \tag{5.13}$$

$$\text{Dirichlet:}\;\; \partial_\tau X^\mu = 0 \tag{5.14}$$

where we used the standard names that they are given. Notice that we wrote $\partial_\tau X^\mu = 0$ which implies $\delta X^\mu = 0$, namely the μ coordinate of the end point of the string is fixed and independent of τ. We should remember that we can impose different boundary conditions for each coordinate and at each end point. Notice also that now we have a standard problem in analysis, we have to solve the wave equation with Neumann or Dirichlet boundary conditions, namely with the normal or parallel derivatives to the boundary equal to zero. A simple way to solve the problem is by Fourier analysis. Suppose first that we have Neumann boundary conditions at both ends, namely $\partial_\sigma X^\mu = 0$ for $\sigma = 0, \pi$. A solution of the wave equation is a linear combination of left and right moving waves. We can take

$$X^\mu = e^{-i\alpha\tau} \cos \alpha\tau \tag{5.15}$$

which satisfies $\partial X^\mu = 0$ at $\sigma = 0$. We now want $\partial_\sigma X^\mu = 0$ also at $\sigma = \pi$. This requires $\alpha =$ integer. With that in mind we write the most generic solution as:

$$X^\mu = x^\mu + p^\mu \tau + \sum_{n \neq 0} x_n e^{-in\tau} \cos(n\sigma) \tag{5.16}$$

where we x_n are the amplitudes of each oscillator and we also included a linear and a constant term that satisfy the wave equation and the boundary conditions. Notice that a term $X^\mu = L\sigma$ satisfies the wave equation but not the Neumann boundary condition. A similar result we can obtain by considering the expansion we had for the closed string

$$X^\mu = x^\mu + p^\mu \tau \alpha' + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\frac{1}{n} \alpha_n^\mu e^{-in(\sigma+\tau)} + \frac{1}{n} \tilde{\alpha}_n^\mu e^{in(\sigma-\tau)} \right) \tag{5.17}$$

and imposing $\partial_\sigma X^\mu = 0$, $\sigma = 0, \pi$. This gives $\alpha_n^\mu = \tilde{\alpha}_n^\mu$, namely we have only one set of oscillators. The expansion is:

$$\begin{aligned} X^\mu &= x^\mu + p^\mu \tau \alpha' + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu \left(e^{-in(\sigma+\tau)} + e^{in(\sigma-\tau)} \right) & (5.18) \\ &= x^\mu + p^\mu \tau \alpha' + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma & (5.19) \end{aligned}$$

Doing the same for the case of Dirichlet boundary conditions we find that the condition is $\alpha_n^i = -\tilde{\alpha}_n^i$ we basically have to replace $\cos n\sigma \to \sin n\sigma$. Also, now we admit a term $L^\mu \sigma$ but not $p^\mu \tau$. The case in which it is Neumann at $\sigma = 0$ and Dirichlet at $\sigma = \pi$ requires a similar analysis. We propose a solution

$$X^\mu = e^{-i\alpha\tau} \cos \alpha\tau \tag{5.20}$$

but now we need $\cos\alpha\pi = 0$ namely $\alpha = (n+\frac{1}{2})\pi$. So we have a generic expansion:

$$X^\mu = x^\mu + \sum_n x_n e^{-i(n+\frac{1}{2})\tau}\cos(n+\frac{1}{2})\sigma \tag{5.21}$$

Summarizing we have the expansions:

$$\begin{aligned}
\mathbf{NN}:\ X^\mu &= x_0^\mu + p^\mu\tau\alpha' + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma \\
\mathbf{DD}:\ X^\mu &= x_0^\mu + \frac{L^\mu}{\pi}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\sin n\sigma \\
\mathbf{ND}:\ X^\mu &= x_0^\mu + i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma \\
\mathbf{DN}:\ X^\mu &= x_0^\mu + \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\sin n\sigma
\end{aligned} \tag{5.22}$$

Notice that for the cases **ND** and **DN** there are no zero modes since $n\in\mathbb{Z}+\frac{1}{2}$, *i.e.* $n = \ldots -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$. The functions $X^\mu(\sigma,\tau)$ satisfy:

$$\begin{aligned}
&\mathbf{NN}:\ \partial_\sigma X^\mu(0,\tau) = 0, && \partial_\sigma X^\mu(\pi,\tau) = 0 \\
&\mathbf{DD}:\ \ X^\mu(0,\tau) = x_0^\mu && X^\mu(\pi,\tau) = x_0^\mu + L^\mu \\
&\mathbf{ND}:\ \partial_\sigma X^\mu(0,\tau) = 0, && X^\mu(\pi,\tau) = x_0^\mu \\
&\mathbf{DN}:\ \ X^\mu(0,\tau) = x_0^\mu && \partial_\sigma X^\mu(0,\tau) = 0,
\end{aligned} \tag{5.23}$$

where x_0 is a constant which, for Dirichlet boundary conditions is interpreted as the position of the D-brane. Now we should also impose the constraints. As we know, a simple way to impose the constraints and identify the physical degrees of freedom is to go to light-cone gauge as we did in the closed string. This means that we redefine (σ,τ) keeping the conformal gauge conditions in such a way that:

$$X^+ = x^+ + p^+\tau \tag{5.24}$$

Note that we have $\partial_\sigma X^+ = 0$ but not $\partial_\tau X^+ = 0$ so we can take light-cone gauge only if there is at least one spacial direction with Neumann boundary conditions. Time always is Neumann otherwise the string cannot move in time. In this gauge, the physical modes are the transverse ones which can have any boundary condition and can be expanded as in (5.22). Thus, the physical variables are $\alpha_n^{i=1\ldots D-2}$ with n integer or half-integer for each i according to the boundary conditions in that direction.

Before discussing the quantization we are going to consider an example of a solution and also a symmetry called T-duality. The example is, in fact, the same rotating string we saw in the closed string case and given by:

$$t = \kappa\tau \tag{5.25}$$

$$x = \kappa \sin\sigma \cos\tau \tag{5.26}$$

$$y = \kappa \sin\sigma \sin\tau \tag{5.27}$$

We can easily check that the solution satisfies the conformal constraints and the wave equation. The Neumann boundary condition $\partial_\sigma X^\mu = 0$ is satisfied for t since t is independent of σ, and for x, y is satisfied if we take $-\frac{\pi}{2} \le \sigma \le \frac{\pi}{2}$. As shown in fig.4 the solution looks like the closed string rotating string solution but, if we recall that the closed string was folded on itself, we see that it is only half of it. For that reason the energy and angular momentum are also half, thus the Regge relation $E \sim \sqrt{J}$ is still valid but with a different coefficient.

5.2 T-duality

At the classical level, T-duality is a symmetry that, given a solution to the equations of motion of the string, allows us to construct new solutions. Consider conformal gauge and the world-sheet coordinates $\sigma_+ = \sigma + \tau$ and $\sigma_- = \sigma - \tau$. The equation of motion and conformal constraints read:

$$\partial_{\sigma_+}\partial_{\sigma_-} X^\mu = 0 \tag{5.28}$$

$$(\partial_{\sigma_+} X)^2 = 0 \tag{5.29}$$

$$(\partial_{\sigma_-} X)^2 = 0 \tag{5.30}$$

Excercise Verify that indeed these are the same as (5.12) and (5.11).

The general solution to the equations of motion is:

$$X^\mu(\sigma,\tau) = X_L^\mu(\sigma_+) + X_R^\mu(\sigma_-) \tag{5.31}$$

It is simple to verify that, if we do the sign change $X_R^\mu(\sigma_-) \to -X_R^\mu(\sigma_-)$ for one or more of the X^μ, then the new solution satisfies all constraints and equations of motion. This procedure is call T-duality in the direction in which we flipped the sign of X_R^μ.

For the case of the open string solution we describe in (5.27) we can do T-duality in x to obtain:

$$x = \kappa \sin\sigma \cos\tau = \frac{\kappa}{2}\sin\sigma_+ + \frac{\kappa}{2}\sin\sigma_- \to \frac{\kappa}{2}\sin\sigma_+ - \frac{\kappa}{2}\sin\sigma_- = \kappa\cos\sigma\sin\tau \tag{5.32}$$

On the other hand, t and y remain the same so the new solution reads:

$$t = \kappa\tau \tag{5.33}$$

$$x = \kappa\cos\sigma\sin\tau \tag{5.34}$$

$$y = \kappa\sin\sigma\sin\tau \tag{5.35}$$

We notice that x now satisfies Dirichlet boundary conditions at $\sigma = \pm\frac{\pi}{2}$. The solution is shown in fig.4. If we do a further T-duality in direction y we get the solution

$$t = \kappa\tau \tag{5.36}$$

$$x = \kappa\cos\sigma\sin\tau \tag{5.37}$$

$$y = -\kappa\cos\sigma\cos\tau \tag{5.38}$$

Now both, x and y satisfy Dirichlet boundary conditions and the solution looks like half the closed string solution as shown also in fig.4.

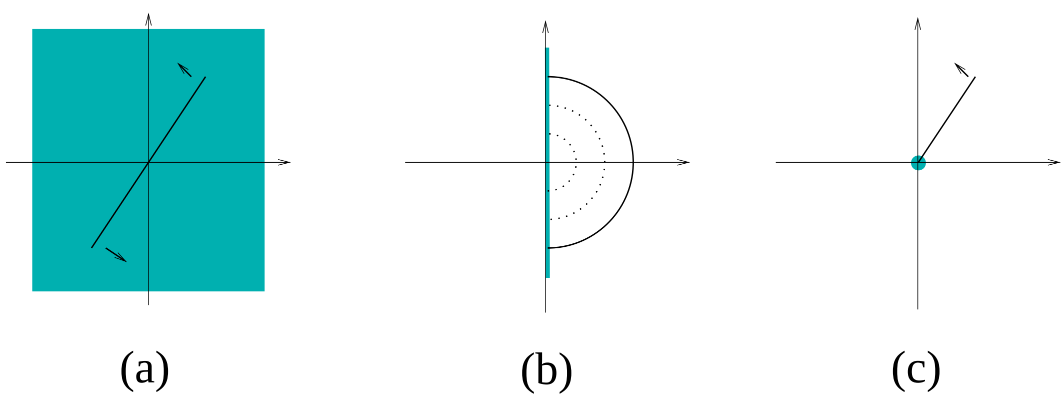

Figure 4: Rotating open string and its T-duals. The cases (b) and (c) correspond to a string attached to a 1-brane and a 0-brane respectively. In case (a) we can say that there is a 2-brane filling the space (x, y). Note that case (b) is a semi-circular string that pulsates whereas (a) and (c) are rotating strings.

We see that in the case where we have Dirichlet boundary conditions, the momentum is not conserved in the correspoding direction, because in that direction the motion of the center of mass of the string is oscillatory. It seems as if the string is attached to an object of infinite mass which absorbs the momentum. In case (b) the object is a line along the y axis, in case (c) it is a point at the origin. These objects are precisely the D-branes. What we see is that if we take into account T-duality as an important symmetry, this gives another motivation to consider both boundary conditions, Dirichlet and Neumann.

5.3 Open string spectrum

To quantize the open strings we proceed as for closed stirngs. We take light cone gauge and quantize the amplitudes α_n^i replacing them by a quantum operator. The difference is that we have now only one set. The commutation relations are

$$[\alpha_n^i, \tilde{\alpha}_m^j] = n\delta^{ij}\delta_{m+n} \tag{5.39}$$

and the mass of the string is given by

$$M^2 = \frac{2}{\alpha'}\left(\sum_{i=1\ldots D-2}\sum_{n\geq 1} nN_n^i - a\right) \tag{5.40}$$

where $N_n \geq 0$ is the occupation number of the corresponding oscillator. If we have a direction with half integer modes then we should sum over half-integer n for that direction. In the case of all Neumann boundary conditions we have that the spectrum is given by:

$$\begin{array}{ll} \text{level 0:} & |0\rangle \\ \text{level 1:} & \alpha_{-1}^i|0\rangle \\ \text{level 2:} & \alpha_{-2}^i|0\rangle\ ,\quad \alpha_{-1}^i\alpha_{-1}^j|0\rangle \end{array} \tag{5.41}$$

At level 0 we have one state which should correspond to a scalar particle. At level 1 we have a vector of $SO(D-2)$ which can only be a massless vector particle of $SO(D-1,1)$. At level 2 we have states that fill the two-index traceless symmetric represetation of $SO(D-1)$ and therefore it should be a massive two-index traceless symmetric tensor. To obtain this result we should take the normal order constant $a = 1$ and therefore the mass is

$$M^2 = \frac{2}{\alpha'}\left(\sum_{i=1\ldots D-2}\sum_{n\geq 1} nN_n^i - 1\right) \tag{5.42}$$

Therefore, the bosonic open string has a tachyon, same as the bosonic closed string. For that reason we consider next the open superstring.

5.4 Open superstrings

When we include fermionic modes we have to find the correct boundary condition for them. In the bosonic sector we found that the number of modes is reduced by half since we do not have independent left and right moving modes. The same is the case with the fermions. The key point in finding the fermionic boundary conditions is that we want to obtain a supersymmetric theory, namely some supersymmetry should be preserved. For that to be the case we should have that the boundary conditions commute with some of the supercharges. Each type of boundary condition should be treated separately so we start by considering the case were the boundary conditions are all Neumann.

5.4.1 NN boundary conditions in all directions

The supercharges are

$$Q^{\dot{a}} = \frac{1}{2\alpha'}\frac{1}{\sqrt{\pi p^+}}\int d\sigma \rho^{i\,\dot{a}b} S^b \left(2\pi\alpha'\Pi^i + \partial_\sigma X^i\right) = \frac{1}{\sqrt{\alpha' p^+}}\,\rho^{i\,\dot{a}b}\sum_{m=-\infty}^{\infty} S^b_m \alpha^i_{-m} \quad (5.43)$$

$$\tilde{Q}^{\dot{a}} = \frac{1}{2\alpha'}\frac{1}{\sqrt{\pi p^+}}\int d\sigma \rho^{i\,\dot{a}b} \tilde{S}^b \left(2\pi\alpha'\Pi^i - \partial_\sigma X^i\right) = \frac{1}{\sqrt{\alpha' p^+}}\,\rho^{i\,\dot{a}b}\sum_{m=-\infty}^{\infty} \tilde{S}^b_m \tilde{\alpha}^i_{-m} \quad (5.44)$$

$$Q^a = \sqrt{2p^+}S^a_0 \quad (5.45)$$

$$\tilde{Q}^a = \sqrt{2p^+}\tilde{S}^a_0 \quad (5.46)$$

We compute first:

$$[Q^{\dot{a}}, \partial_\sigma X^i] = \sqrt{\frac{\pi}{p^+}}\rho^{j\,\dot{a}b}\partial_\sigma\int d\sigma' S^b(\sigma')\left[\Pi^j(\sigma'), X^i(\sigma)\right] = -i\sqrt{\frac{\pi}{p^+}}\rho^{i\,\dot{a}b}\partial_\sigma S^b \quad (5.47)$$

$$[\tilde{Q}^{\dot{a}}, \partial_\sigma X^i] = \sqrt{\frac{\pi}{p^+}}\rho^{j\,\dot{a}b}\partial_\sigma\int d\sigma' \tilde{S}^b(\sigma')\left[\Pi^j(\sigma'), X^i(\sigma)\right] = -i\sqrt{\frac{\pi}{p^+}}\rho^{i\,\dot{a}b}\partial_\sigma \tilde{S}^b \quad (5.48)$$

We see that none of these supersymmetries commute with the boundary condition $\partial_\sigma X^i = 0$. However we have that

$$[Q^{\dot{a}} - \tilde{Q}^{\dot{a}}, \partial_\sigma X^i] = -i\sqrt{\frac{\pi}{p^+}}\rho^{i\,\dot{a}b}\partial_\sigma\left(S^b - \tilde{S}^b\right) \quad (5.49)$$

$$[Q^{\dot{a}} + \tilde{Q}^{\dot{a}}, \partial_\sigma X^i] = -i\sqrt{\frac{\pi}{p^+}}\rho^{i\,\dot{a}b}\partial_\sigma\left(S^b + \tilde{S}^b\right) \quad (5.50)$$

$$(5.51)$$

So if we impose

$$\partial_\sigma\left(S^b - \tilde{S}^b\right) = 0 \qquad \text{at} \qquad \sigma = 0, \pi \quad (5.52)$$

half of the supersymmetry will be preserved. And if we, instead, impose $\partial_\sigma\left(S^b+\tilde{S}^b\right)=0$ at $\sigma=0,\pi$ then the other half is preserved. So, we have a choice, but in both cases only eight of the 16 supersymmetries $Q^{\dot{a}}$, $\tilde{Q}^{\dot{a}}$ are preserved. If we look at the mode expansion

$$S^a(\sigma,\tau) = \frac{1}{\sqrt{2\pi}}\sum_n e^{-in(\sigma+\tau)}S^a_n \tag{5.53}$$

$$\tilde{S}^a(\sigma,\tau) = \frac{1}{\sqrt{2\pi}}\sum_n e^{-in(\sigma+\tau)}\tilde{S}^a_n \tag{5.54}$$

The condition

$$\partial_\sigma\left(S^b-\tilde{S}^b\right)=0 \;\text{ at }\; \sigma=0,\pi, \quad \text{implies} \quad S^a_n=-\tilde{S}^a_n \tag{5.55}$$

namely an identification between left and right modes similar to $\alpha^i_n=\tilde{\alpha}^i_n$ for the bosonic sector. It is natural to identify also the zero modes:

$$S^a_0+\tilde{S}^a_0=0 \tag{5.56}$$

In fact this is necessary since

$$[Q^{\dot{a}}-\tilde{Q}^{\dot{a}},S^b_0+\tilde{S}^b_0]=\frac{1}{\sqrt{\alpha' p^+}}\rho^{i\,\dot{a}b}\left(\alpha^i_0-\tilde{\alpha}^i_0\right)=0 \tag{5.57}$$

because $\alpha^i_0=\tilde{\alpha}^i_0=\sqrt{\frac{\alpha'}{2}}p^i$ for Neumann directions. So (5.56) is an invariant condition. If we also notice that

$$[Q^a-\tilde{Q}^a,S^b_0+\tilde{S}^b_0]=\sqrt{2p^+}[S^a_0-\tilde{S}^a_0,S^b_0+\tilde{S}^b_0]=0 \tag{5.58}$$

but

$$[Q^a+\tilde{Q}^a,S^b_0+\tilde{S}^b_0]=\sqrt{2p^+}[S^a_0+\tilde{S}^a_0,S^b_0+\tilde{S}^b_0]=2\sqrt{2p^+}\delta^{ab}\neq 0 \tag{5.59}$$

then we see that there are 16 preserved supersymmetries in total:

$$\mathbb{Q}^a=Q^a-\tilde{Q}^a, \quad \mathbb{Q}^{\dot{a}}=Q^{\dot{a}}-\tilde{Q}^{\dot{a}} \tag{5.60}$$

Before continuing we can repeat the calculation in terms of modes to perhaps better understand the result. We saw that the Neumann boundary condition is

$$\alpha^i_n-\tilde{\alpha}^i_n=0 \tag{5.61}$$

Using the mode expansion (5.46) we find

$$[Q^{\dot{a}}-\tilde{Q}^{\dot{a}},\alpha^i_n-\tilde{\alpha}^i_n]=\frac{1}{\sqrt{\alpha' p^+}}\rho^{i\,\dot{a}b}\left(S^b_n+\tilde{S}^b_n\right) \tag{5.62}$$

so the correct boundary condition is $S^b_n + \tilde{S}^b_n = 0$ as we already found. It is easier to work with modes but that can obscure the fact that we are imposing the conditions at the boundary. In fact we can see that $S^b_n + \tilde{S}^b_n = 0$ implies

$$\boxed{\begin{aligned} S(\sigma = 0, \tau) &= -\tilde{S}(\sigma = 0, \tau) \ , \\ S(\sigma = \pi, \tau) &= -\tilde{S}(\sigma = \pi, \tau) \end{aligned}} \tag{5.63}$$

Now we are ready to describe the string spectrum. The zero modes are S^a so again they can be represented by the matrices $\tilde{\gamma}^a_{k\dot{b}}$ of eq.(4.118) which act on a space of states sum of the vector representation and right spinor representation. Therefore the open superstring has 16 vacua, eight bosonic in a vector representation and eight fermionic in a right spinor representation. The rest of the spectrum is constructed by applying creation operators to the vacua:

$$\begin{array}{lrl} \text{level 0:} & 16 \text{ states:} & |0\rangle \rightarrow |k\rangle \oplus |\dot{b}\rangle \\ \text{level 1:} & 2^8 \text{ states:} & \alpha^i_{-1}|0\rangle, \qquad S^a_{-1}|0\rangle \\ \text{level 2:} & 9 \times 2^8 \text{ states:} & \alpha^i_{-2}|0\rangle, \quad \alpha^i_{-1}\alpha^j_{-1}|0\rangle \ , \quad S^a_{-2}|0\rangle \ , \quad S^a_{-1}S^b_{-1}|0\rangle \ , \quad \alpha^i_{-1}S^a_{-1}|0\rangle \end{array} \tag{5.64}$$

where in computing the number of states we took into account the degeneracy of the ground state. Since we have a massless vector particle, the theory is a gauge theory instead of a gravitational theory. In fact, the massless content is that of $\mathcal{N} = 1$, $d = 10$ super Yang-Mills theory. Such theory is anomalous meaning that it has a problem at the quantum level. However we can use it to determine the world-volume theories of lower dimensional branes as we see later. The theory that we are considering should be understood as a theory on a D9-brane, since the ends of the string can move in any direction. We see, at the same time that we can only have Neumann boundary conditions in all directions if we are in IIB string theory because, in type IIA we have S^a and $\tilde{S}^{\dot{a}}$ and we can never identify them. We can now consider lower dimensional branes. For that we have to understand Dirichlet boundary conditions.

5.4.2 NN + DD boundary conditions

Suppose we have a D5 brane that spans the directions $0 \rightarrow 5$. It is usually convenient to write it as

	0	1	2	3	4	5	6	7	8	9
$D5$	×	×	×	×	×	×				

(5.65)

The coordinates 0 and 1 we use to define $X^{\pm}$ for light-cone gauge. From the transverse coordinates we have to impose Neumann boundary conditions in directions 2,3,4,5 and Dirichlet for directions 6,7,8,9. This means that the end points of the string are free to move along the brane, namely directions 0 to 5 but are fixed at some value in the other directions, *e.g.* $X^{6,7,8,9} = 0$ (at $\sigma = 0, \pi$). We can now compute:

$$[Q^{\dot a}, \partial_\tau X^i] = \sqrt{\frac{\pi}{p^+}} \rho^{j\,\dot a b} \partial_\sigma \int d\sigma' S^b(\sigma') \left[\partial_\sigma X^j(\sigma'), \Pi^i(\sigma)\right] = -i \sqrt{\frac{\pi}{p^+}} \rho^{i\,\dot a b} \partial_\sigma S^b \tag{5.66}$$

$$[\tilde Q^{\dot a}, \partial_\tau X^i] = \sqrt{\frac{\pi}{p^+}} \rho^{j\,b\dot a} \partial_\sigma \int d\sigma' \tilde S^b(\sigma') \left[\partial_\sigma X^j(\sigma'), \Pi^i(\sigma)\right] = i \sqrt{\frac{\pi}{p^+}} \rho^{i\,\dot a b} \partial_\sigma \tilde S^b \tag{5.67}$$

where we used $\partial_\tau X^i = 2\pi\alpha'\Pi^i$. There is a crucial minus sign in $[\tilde Q^{\dot a}, \partial_\tau X^i]$ with respect to the Neumann case, so, at first sight we should impose $S^a_n = \tilde S^a_n$ instead of $S^a_n = -\tilde S^a_n$. However, that is not possible since we have to preserve $\partial_\sigma X^i = 0$ for $i = 2,3,4,5$ *and* $\partial_\tau X^i = 0$ for $i = 6,7,8,9$. We have to find then another boundary condition. For that we can try to preserve the following supersymmetries:

$$\mathbb{Q}^{\dot a} = Q^{\dot a} - \Gamma^{\dot a \dot b} \tilde Q^{\dot b} \tag{5.68}$$

for some matrix Γ that we still have to find. At this stage it is convenient to consider Dirac spinors which allows us to treat IIA and IIB together. Given the spinor S^a we construct a Dirac spinor that we denote S^α by putting the left part $S^{\alpha=a} = S^a$ and the right part zero: $S^{\alpha=\dot a} = 0$. The same with the Q's and all other Weyl spinors. We then write

$$\mathbb{Q} = Q - \Gamma \tilde Q \tag{5.69}$$

where $\tilde Q$ is a left spinor for type IIB and a right spinor for IIA whereas Q is a left spinor for both. The commutation relations can now be written:

$$[\mathbb{Q}, \partial_\sigma X^i] = -i \sqrt{\frac{\pi}{p^+}} \partial_\sigma \left(\gamma^i S - \Gamma \gamma^i \tilde S\right) \tag{5.70}$$

$$[\mathbb{Q}, \partial_\tau X^i] = -i \sqrt{\frac{\pi}{p^+}} \partial_\sigma \left(\gamma^i S + \Gamma \gamma^i \tilde S\right) \tag{5.71}$$

Therefore, we need then to impose

$$\partial_\sigma \left(\gamma^i S - \Gamma \gamma^i \tilde S\right) = 0 \quad \text{for} \quad i = 2,3,4,5 \quad (NN) \tag{5.72}$$

$$\partial_\sigma \left(\gamma^i S + \Gamma \gamma^i \tilde S\right) = 0 \quad \text{for} \quad i = 6,7,8,9 \quad (DD) \tag{5.73}$$

Mutliplying both sides by γ^i and using $(\gamma^i)^2 = 1$, we obtain:

$$\partial_\sigma S = \gamma^i \Gamma \gamma^i \partial_\sigma \tilde S \quad \text{for} \quad i = 2,3,4,5 \quad (NN) \tag{5.74}$$

$$\partial_\sigma S = -\gamma^i \Gamma \gamma^i \partial_\sigma \tilde S \quad \text{for} \quad i = 6,7,8,9 \quad (DD) \tag{5.75}$$

Since the left hand side is the same in both cases we should have

$$\gamma^i \Gamma \gamma^i = \pm \Gamma \tag{5.76}$$
$$\gamma^i \Gamma \gamma^i = \mp \Gamma \tag{5.77}$$

where we allow for a sign ambiguity. The important point is that the signs have to be opposite to each other in the two lines. We solved a similar problem when we needed to find the charge conjugation matrix. The solution is

$$\Gamma = \prod_{i=2,3,4,5} \gamma^i \tag{5.78}$$

If we consider a p-brane along directions $0, 1, \ldots, p$ then we have (taking into account one direction for the light-cone gauge):

$$\Gamma = \prod_{i=2,\ldots p} \gamma^i \tag{5.79}$$

which, using the anticommutation relations $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ results in

$$\gamma^i \Gamma \gamma^i = (-)^p \, \Gamma, \qquad i = 2 \ldots p \quad (NN) \tag{5.80}$$
$$\gamma^i \Gamma \gamma^i = -(-)^p \, \Gamma, \qquad i = p+1 \ldots 9 \quad (DD) \tag{5.81}$$

Therefore we should impose the boundary conditions

$$\partial_\sigma S = (-)^p \, \Gamma \partial_\sigma \tilde{S} \tag{5.82}$$

which implies

$$S_n = -(-)^p \, \Gamma \, \tilde{S}_n \tag{5.83}$$

If we impose the same for the zero modes, namely

$$S_0 = -(-)^p \, \Gamma \, \tilde{S}_0 \tag{5.84}$$

we obtain

$$\boxed{\begin{aligned} S(\sigma = 0, \tau) &= -(-)^p \, \Gamma \, \tilde{S}(\sigma = 0, \tau) \,, \\ S(\sigma = \pi, \tau) &= -(-)^p \, \Gamma \, \tilde{S}(\sigma = \pi, \tau) \end{aligned}} \tag{5.85}$$

and the preserved supersymmetry is

$$\mathbb{Q}^{\dot{a}} = Q^{\dot{a}} - \Gamma^{\dot{a}\dot{b}} \, \tilde{Q}^{\dot{b}} \tag{5.86}$$
$$\mathbb{Q}^{a} = Q^{a} - (-)^p \, \Gamma^{ab} \, \tilde{Q}^{b} \tag{5.87}$$

for type IIB or

$$\mathbb{Q}^{\dot a} = Q^{\dot a} - \Gamma^{\dot a b}\tilde Q^{b} \tag{5.88}$$
$$\mathbb{Q}^{a} = Q^{a} - \Gamma^{a\dot b}\tilde Q^{\dot b} \tag{5.89}$$

for type IIA. At this point however we notice a crucial point. In type IIB both Q and $\tilde Q$ have the same chirality, so we need Γ to have indices Γ^{ab} and $\Gamma^{\dot a\dot b}$. In type IIA we need the components $\Gamma^{a\dot b}$ and $\Gamma^{\dot a b}$. But we already have $\Gamma = \prod_{i=2,\ldots p}\gamma^i$. Since γ^i has indices $\gamma^{i\,a\dot b}$ and $\gamma^{i\,\dot a b}$ we need p =odd for type IIB and p =even for type IIA. So we can have the following supersymmetric branes:

$$\begin{array}{lllllll} \text{Type IIA:} & D0 & & D2 & D4 & D6 & D8 \\ \text{Type IIB:} & D(-1) & D1 & D3 & D5 & D7 & D9 \end{array} \tag{5.90}$$

With the light-cone gauge formalism we are using here we cannot describe $D0$ and $D(-1)$ branes because we do not have one Neumann spacial direction to combine with time. Nevertheless they can be described in conformal gauge and so we included them here. The nice thing is that this precisely fits with the ideas of the previous section where we needed exactly these same branes to source the RR fields. It is natural to identify both but we still have to show that these D-branes actually couple to the RR fields. Let us now ellaborate more on the condition (5.84). As before we can see that this condition is preserved by the supersymmetries in eq. (5.87) or (5.89). To check that we should remmeber the mode expansion for the Dirichlet modes (5.23) which determines the zero mode part of the supercharges:

$$Q^{\dot a} = \frac{1}{\sqrt{2p^+}}\sum_{i=2}^{p}\rho^{i\,\dot a b}p^i + \frac{1}{\pi\sqrt{2p^+}}\sum_{i=p+1}^{9}L^i\rho^{i\,\dot a b}S_0^b + \frac{1}{\sqrt{\alpha' p^+}}\,\rho^{i\,\dot a b}\sum_{m\neq 0}S_m^b\alpha_{-m}^i \tag{5.91}$$
$$\tilde Q^{\dot a} = \frac{1}{\sqrt{2p^+}}\sum_{i=2}^{p}\rho^{i\,\dot a b}p^i - \frac{1}{\pi\sqrt{2p^+}}\sum_{i=p+1}^{9}L^i\rho^{i\,\dot a b}S_0^b + \frac{1}{\sqrt{\alpha' p^+}}\,\rho^{i\,\dot a b}\sum_{m\neq 0}\tilde S_m^b\alpha_{-m}^i \tag{5.92}$$

where we work on type IIB and we used that $\alpha_0^i = \tilde\alpha_0^i = \sqrt{\frac{\alpha'}{2}}p^i$. If we now commute $Q - \Gamma\tilde Q$ with the condition (5.84) we find that it is preserved precisely because, for the DD directions, the term $L^i\sigma$ contributes with opposite sign to the left and right moving supercharges.

We are interested now in computing the spectrum. The zero modes are still the S_0^a so the vacuum is degenerate with 16 states transforming as the vector plus the right spinor representation. The momenta p^i that label the states can only have components parallel to the brane. If the two D-branes between which the string stretches are separated, namely $L^i \neq 0$ we expect the mass of the string to be non-zero and proportional to $\frac{L}{\alpha'}$ even in the ground state. To check that we recall that the mass is given by

$$M^2 = \frac{1}{2\pi(\alpha')^2} \int_0^{2\pi} d\sigma \, \left[\partial_\sigma X^i \partial_\sigma X^i + \partial_\tau X^i \partial_\tau X^i\right] - P^i P^i + \frac{i}{\alpha'} \int_0^{2\pi} d\sigma \left(S^a \partial_\sigma S^a - \tilde{S}^a \partial_\sigma \tilde{S}^a \right) \tag{5.93}$$

which gives:

$$M^2 = \frac{L^i L^i}{(\pi\alpha')^2} + \frac{2}{\alpha'} \left(N + \tilde{N} + N_f + \tilde{N}_f \right) \tag{5.94}$$

Besides the usual contribution from the oscillators, the mass has a contribution $\frac{L^2}{(\pi\alpha')^2}$ from the fact that the string is stretched. Therefore, we have massless particles only if $L^2 = L^i L^i = 0$, namely the D-branes are on top of each other. In that case, we have that the massless modes are still 16 but they propagate only in a subspace. The field theory describing the massless modes is the dimensional reduction of $\mathcal{N} = 1$ in $d = 10$ to the corresponding dimension of the brane. For example for a $D3$-brane the world-volume theory is the dimensionl reduction of $\mathcal{N} = 1$, $d = 10$ to $3 + 1$ dimensions which is $\mathcal{N} = 4$ Super Yang-Mills theory. The vector field A^μ has four componenets parallel to hte brane that behave as a vector in the world-volume theory and six perpendicular components that behave as scalar under rotations parallel to the brane. In fact, from the Lorentz group $SO(9, 1)$ only $SO(3, 1) \times SO(6)$ survives under the presence of the $D3$-brane because we split the coordinates into NN and DD boundary conditions. From the transverse group $SO(8)$ only $SO(2) \times SO(6)$ survives. The fact that there is a gauge field, namely a massless vector field A^μ, implies that there is a gauge symmetry which eliminates the extra component of A^μ. For what we said we seem to have an abelian theory with group $U(1)$ since there is only one gauge field. However when we have N branes on top of each other, the vector field A^μ carries two indices: A^μ_{pq} where $p, q = 1 \ldots N$ label the D-branes between which the string is stretching. These are new quantum numbers that do not exist for closed strings. A^μ becomes an $N \times N$ matrix and the theory is non Abelian, we need a larger symmetry to get rid of all the extra components. In fact the gauge group is $U(N)$ with N^2 parameters, as needed to get rid of the extra components of N^2 massless fields. If the D-branes are not on top of each other then there are only N massless vectors from the open strings on each D-brane and the gauge group is $U(1)^N$.

A. Lorentz group

Special relativity is based on the fact that the laws of physics are the same in all inertial frames. The transformation between inertial frames moving with respect to each other at velocity v are the Lorentz transformations or boosts given by:

$$\tilde{t} = \frac{t - vx}{\sqrt{1-v^2}} \tag{1.1}$$

$$\tilde{x} = \frac{x - vt}{\sqrt{1-v^2}} \tag{1.2}$$

where we set $c = 1$ by an adequate choice of unit to measure time. They can also be written as:

$$\tilde{t} = \cosh\beta\ t - \sinh\beta\ x \tag{1.3}$$

$$\tilde{x} = \cosh\beta\ x - \sinh\beta\ t \tag{1.4}$$

where $v = \tanh\beta$. In this way they look similar to rotations on a plane:

$$\tilde{x} = \cos\theta\ x + \sin\theta\ y \tag{1.5}$$

$$\tilde{y} = \cos\theta\ y - \sin\theta\ x \tag{1.6}$$

where the rotation is by an angle θ. Since space and time are related, it is convenient to introduce a notation of cuadrivectors, where the position is given in terms of a vector of four components:

$$x = (t, x, y, z) \tag{1.7}$$

The components of x we denote with a greek index x^μ, $\mu = 0, 1, 2, 3$. The spacial components are denoted with a latin index: x^i, $i = 1, 2, 3$.

This notation is convenient to describe the group that includes all rotations and boosts which is called the Lorentz group. It can be represented as the group of 4×4 matrices $\Lambda^\mu{}_\nu$ that define a transformation: [5]

$$\tilde{x}^\nu = \Lambda^\nu{}_\mu x^\mu \tag{1.8}$$

such that they leave the interval invariant:

$$(\Delta s)^2 = -(\Delta t)^2 + \Delta x^i \Delta x^i = \Delta x^\mu \Delta x^\nu \eta_{\mu\nu} \tag{1.9}$$

[5] We use the convention that repeated indices are summed (*e.g.* in this case there is an implicit sum over μ).

where the matrix $\eta_{\mu\nu}$ is given by

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.10}$$

To leave the interval invariant, the matrix $\Lambda^{mu}{}_{\nu}$ has to satisfy:

$$\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu} \tag{1.11}$$

or in matrix notation:

$$\eta = \Lambda^t \eta \Lambda \tag{1.12}$$

For a boost in direction x and a rotation in the plane (x, y) the matrix Λ is:

$$\Lambda^{\text{boost}} = \begin{pmatrix} \cosh\beta & -\sinh\beta & 0 & 0 \\ -\sinh\beta & \cosh\beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.13}$$

$$\Lambda^{\text{rot.}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.14}$$

It is important to consider also infinitesimal Lorentz transformations. In that case we can take:

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \epsilon\, \eta^{\mu\alpha}\, \omega_{\alpha\nu} + \dots \tag{1.15}$$

where ϵ is an infinitesimal parameter and $\omega_{\mu\nu}$ is a four by four matrix. If Λ satisfies (1.12), then we have, in matrix notation

$$\eta = (1 + \epsilon\omega^t\eta)\eta(1 + \epsilon\eta\omega) \simeq \eta + \epsilon(\omega + \omega^t) + \dots \tag{1.16}$$

Therefore, we need to have:

$$\omega \text{ is an antisymmetric matrix: } \omega_{\mu\nu} = -\omega_{\nu\mu} \tag{1.17}$$

B. Problems

B.1 Classical strings

Consider a string moving in flat space. The metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2.1}$$

Suppose we have a solution of the form

$$t(\sigma,\tau) = \kappa\tau \tag{2.2}$$
$$x(\sigma,\tau) = A\cos\left[(n-1)(\tau+\sigma)\right] + B\cos(\tau-\sigma) \tag{2.3}$$
$$y(\sigma,\tau) = A\sin\left[(n-1)(\tau+\sigma)\right] + B\sin(\tau-\sigma) \tag{2.4}$$
$$z(\sigma,\tau) = 0 \tag{2.5}$$

where A, B, κ are constants and n is an integer number.

1) Find a condition on the constants A, B and κ for the conformal constraints (2.72) to be satisfied.

2) Assuming the conditions on **1)** show that the equations of motion are satisfied.

3) Compute the energy E and angular momentum J of the string. Eliminate the constants to obtain a relation $E(J)$ similar to the Regge trajectory (*i.e.* $E \sim \sqrt{J}$).

4) Use a computer porgram to plot the shape of the string for different values of n and understand its motion.

As a guide for the computation notice that the case $n = 2$ is the rotating string described in the text.

B.2 Quantum string and string spectrum

In the text we analyze the equation of motion for a massless vector particle and for a graviton. We show that the number of physical components is given by $D-2$ and $\frac{(D-1)(D-2)}{2} - 1$ respectively as appropriate to representations of $SO(D-2)$. Repeat the analysis for the B-field. That is, take an antisymmetric tensor $B_{\mu\nu}$ and impose a gauge invariance. Find the equation of motion compatible with such gauge invariance and by adequate gauge choices find the number of physical components of the B-field. The calculations should be similar to the ones for the graviton in the main text.

References