Strings in AdS pp-waves

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Based on:

arXiv:0802.2039 **A. Tseytlin, M.K.**

arXiv:0804.3438 **R. Ishizeki, A. Tirziu, M.K.**

arXiv:0812.2431 **R. Ishizeki, A. Tirziu, A. Tseytlin , M.K.**

Summary

- **Introduction**

  String / gauge theory duality (AdS/CFT)

  AdS pp-waves

  - Properties (boundary metric)
  - Application to AdS/CFT

  Twist two operators in gauge theories (**QCD**)

  AdS/CFT and twist two operators (GKP rot. string)

  Higher twist operators: spiky strings

  Other applications / results.

- **Infinite spin limit of the spiky string**

  Limiting shape and near boundary string

- **Gauge theories in pp-waves and the pp-wave anomaly**

- **PP-wave anomaly**

  Gauge theory in a pp-wave → pp-wave anomaly
  → cusp anomaly / anomalous dim. of twist two ops.

  - **Strong coupling** String calculation:
    Wilson loop in pp-wave w/ AdS/CFT
  - **Small coupling** Field theory calculation:
    Wilson loop in pp-wave (Class. source).

- **Wilson loops in a pp-wave**

- **Periodic spikes in a pp-wave and a thermodynamic limit of the SL(2,R) spin chain**

$$O_{[n]} = Tr\left(\nabla_{+}^{S/n}\Phi\,\nabla_{+}^{S/n}\Phi\,\nabla_{+}^{S/n}\Phi\ldots\nabla_{+}^{S/n}\Phi\right)$$

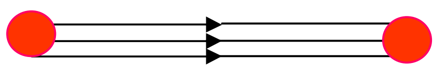
- **Conclusions**

String/gauge theory duality: Large N limit ('t Hooft)

QCD [SU(3)] → Large N-limit [SU(N)]

↓

q $\bar{q}$



Effective strings

Strong coupling

More precisely: $$N \to \infty, \lambda = g_{YM}^2 N \ \ fixed$$ ('t Hooft coupl.)

Lowest order: sum of planar diagrams (infinite number)

AdS/CFT correspondence (Maldacena)

Gives a precise example of the relation between strings and gauge theory.

Gauge theory	String theory
$\mathcal{N} = 4$ SYM SU(N) on R^4	IIB on $AdS_5 x S^5$
A_μ , Φ^i, Ψ^a	radius R
Operators w/ conf. dim. Δ	String states w/ $E = \frac{\Delta}{R}$

$$g_s = g_{YM}^2 ; \qquad R / l_s = (g_{YM}^2 N)^{1/4}$$

$$N \to \infty , \lambda = g_{YM}^2 N$$ fixed ➡

λ large → string th.
λ small → field th.

AdS pp-waves

Flat space

$$ds^2 = 2dx_+dx_- + dx_i^2, \qquad {\scriptstyle i=1,2,} \qquad x_\pm = \frac{x \pm t}{\sqrt{2}}$$

AdS space (Poincare coordinates)

$$ds^2 = \frac{1}{z^2}\left(2dx_+dx_- + dx_i^2 + dz^2\right)$$

Pp-wave

$$ds^2 = 2dx_+dx_- - \mu^2 x_i^2 dx_+^2 + dx_i dx_i$$

AdS pp-wave

$$ds^2 = \frac{1}{z^2}\left[2dx_+dx_- - \mu^2\left(z^2 + x_i^2\right)dx_+^2 + dx_i dx_i + dz^2\right]$$

Properties

Conformal mapping

The metric:

$$ds^2 = 2dx_+dx_- - \mu^2 x_i^2 dx_+^2 + dx_i dx_i$$

is conformally flat! But this is a local equivalence. Equivalently the AdS pp-wave is (locally) AdS space (written in different coordinates). (Brecher, Chamblin, Reall).

Indeed, the mapping:

$$\tilde{x}_+ = \mu^{-1} \tan \mu x_+ \ , \qquad \tilde{x}_- = x_- - \frac{1}{2}\mu x_i^2 \tan \mu x_+ \ , \qquad \tilde{x}_i = \frac{1}{\cos \mu x_+} x_i$$

gives:

$$2d\tilde{x}_+ d\tilde{x}_- + d\tilde{x}_i^2 = \frac{1}{\cos^2 \mu x_+} \left(2dx_+ dx_- - \mu^2 x_i^2 dx_+^2 + dx_i^2\right)$$

and

$$\tilde{x}_+ = \mu^{-1} \tan \mu x_+ \ , \qquad \tilde{x}_- = x_- - \frac{1}{2}\mu(x_i^2 + z^2) \tan \mu x_+$$

$$\tilde{x}_i = \frac{1}{\cos \mu x_+} x_i \ , \qquad \tilde{z} = \frac{1}{\cos \mu x_+} z \ ,$$

gives:

$$\frac{1}{\tilde{z}^2} \left(2d\tilde{x}_+ d\tilde{x}_- + d\tilde{x}_i^2 + d\tilde{z}^2\right) = \frac{1}{z^2} \left[2dx_+ dx_- - \mu^2 (x_i^2 + z^2) dx_+^2 + dx_i^2 + dz^2\right]$$

Application

By taking the limit z→0 in the AdS pp-wave

$$ds^2 = \frac{1}{z^2}\left[2dx_+dx_- - \mu^2\left(z^2 + x_i^2\right)dx_+^2 + dx_i dx_i + dz^2\right]$$

We get the corresponding boundary metric which is a pp-wave in usual flat space

$$ds^2 = 2dx_+dx_- - \mu^2 x_i^2 dx_+^2 + dx_i dx_i$$

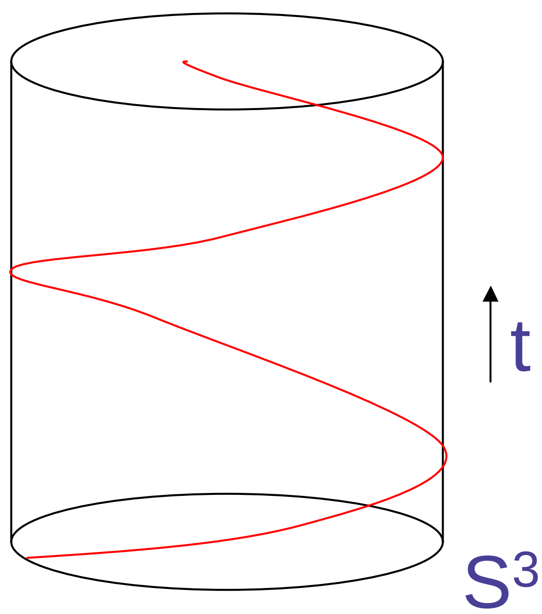
$\mathcal{N} = 4$ SYM SU(N) on R^4 pp-wave dual to

IIB on AdS_5 pp-wave x S^5

According to AdS/CFT

Motivation

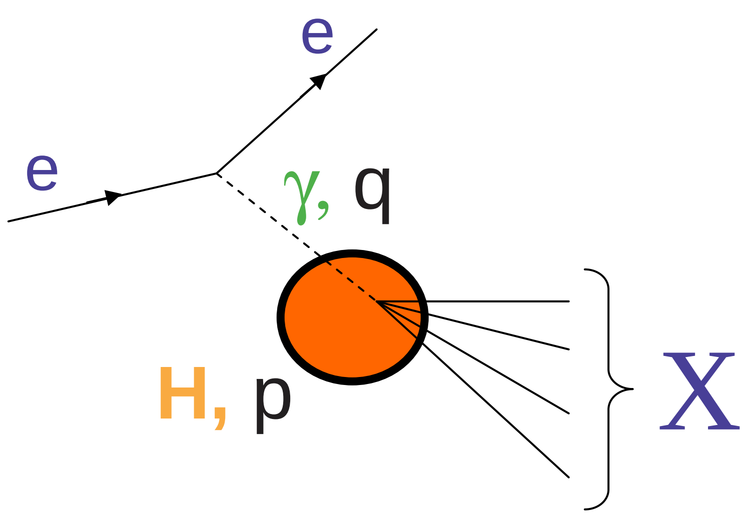
Consider the usual AdS/CFT in global coordinates and, in the boundary (RxS^3) a particle moving at the speed of light (along a light-like geodesic). Such particle experiences, close to itself, the **Penrose limit** of the boundary metric. This is precisely a flat space pp-wave.



$$ds^2 = 2dx_+dx_- - \mu^2 x_i^2 dx_+^2 + dx_i dx_i$$

Such states arise when studying operators with angular momentum. In AdS/CFT they are dual to rotating strings (GKP, spiky). The spikes see the AdS-pp-wave metric.

Twist two operators in gauge theories (QCD)



$q^2 \to \infty$,
$\omega = -2\, p.q / q^2$ fixed

or $q_+ \to \infty$, q_- fixed
Near l.c. expansion

OPE: ($z^2 \to 0$, light-like, twist), ($z^2 \to 0$, euclidean, conf. dim.)

$$\hat{T} J(z) J(0) = \sum_{\mathcal{O}} \mathcal{O}^{\Delta}_{\mu_1 \mu_2 \ldots \mu_S} z^{\mu_1} z^{\mu_2} \ldots z^{\mu_S} |z|^{\Delta - 6 - S}$$

This (after including indices correctly) is plugged into:

$$\int d^4 z e^{-iqz} \langle N | \hat{T} J^{\nu}(z) J^{\mu}(0) | N \rangle = \left(\frac{q^{\mu} q^{\nu}}{q^2} - \eta^{\mu\nu} \right) T_1(q^2, \omega) + \frac{1}{p^2} \left(p^{\mu} - \frac{pq}{q^2} q^{\mu} \right) \left(p^{\nu} - \frac{pq}{q^2} q^{\nu} \right) T_2(\omega, q^2)$$

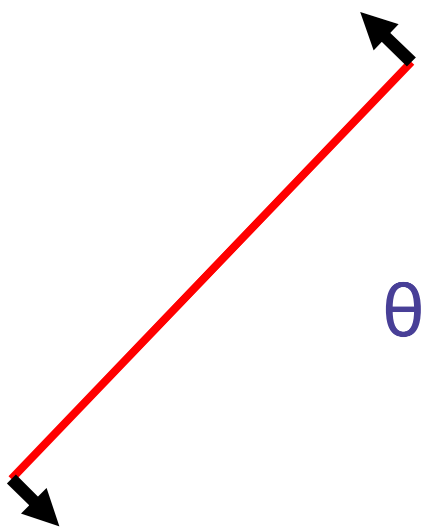
Twist two operators from rotation in AdS$_5$

(Gubser, Klebanov, Polyakov)

$$Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 - Y_5^2 - Y_6^2 = -R^2$$

$\sinh^2 \rho; \Omega_{[3]}$ $\qquad$ $\cosh^2 \rho; t$

$$ds^2 = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega^2_{[3]}$$



$\theta = \omega t$

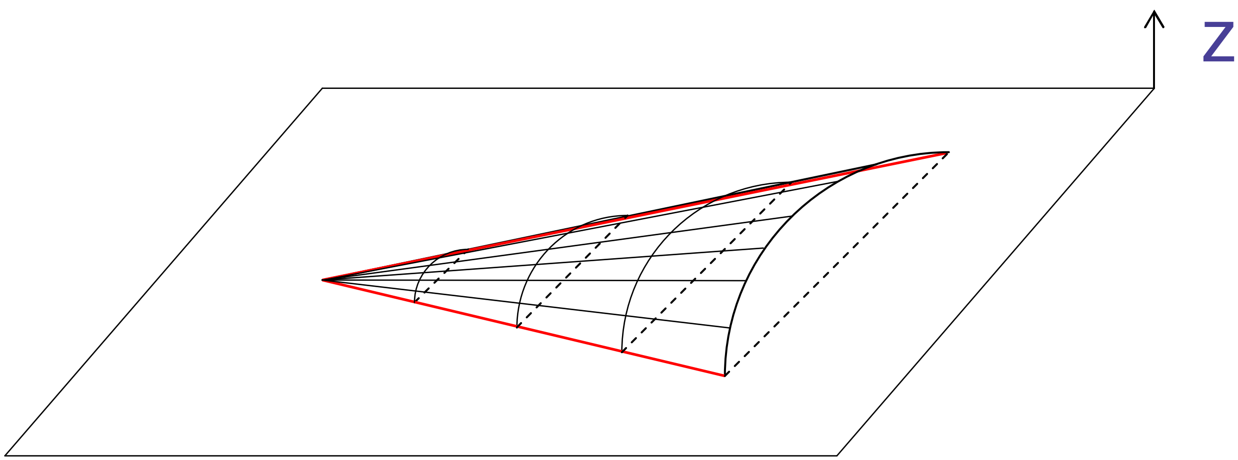
$$E \cong S + \frac{\sqrt{\lambda}}{\pi} \ln S, \quad (S \to \infty)$$

$$O = Tr\left(\Phi \nabla_+^S \Phi\right), \quad x_+ = z + t$$

Twist two ops. from cusp anomaly (MK, Makeenko)

The anomalous dimensions of twist two operators can also be computed by using the **cusp anomaly** of light-like Wilson loops (**Korchemsky and Marchesini**).

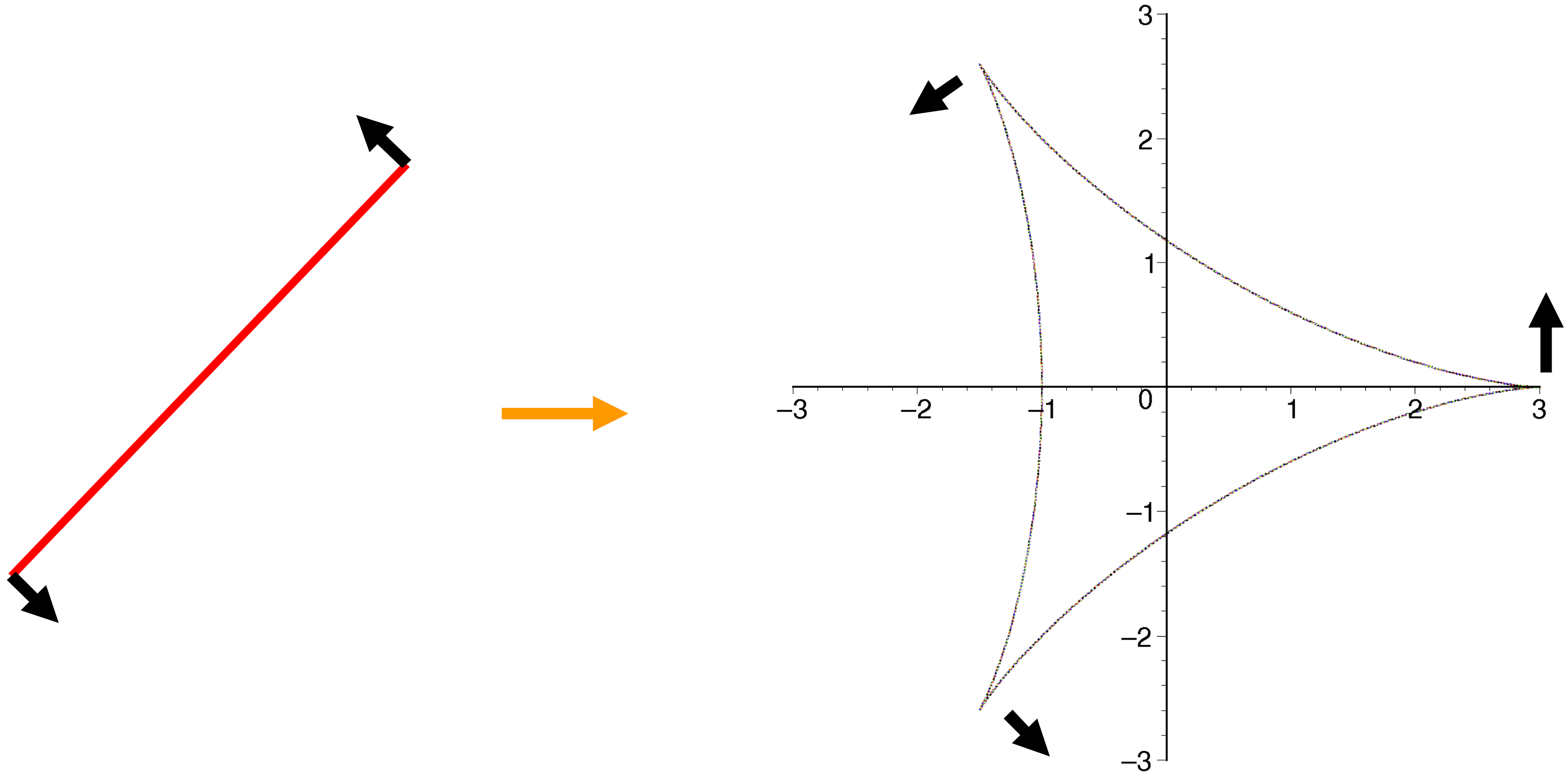
In **AdS/CFT** Wilson loops can be computed using surfaces of minimal area in AdS_5 (**Maldacena, Rey, Yee**)



The result **agrees** with the rotating string calculation.

Generalization to higher twist operators (MK)

$$O_{[2]} = Tr\left(\Phi\, \nabla_+^S \Phi\right) \quad \longrightarrow \quad O_{[n]} = Tr\left(\nabla_+^{S/n}\Phi\, \nabla_+^{S/n}\Phi\, \nabla_+^{S/n}\Phi \ldots \nabla_+^{S/n}\Phi\right)$$



In flat space such solutions are easily found:

$$x = A\cos[(n-1)\sigma_+] + A(n-1)\cos[\sigma_-]$$

$$y = A\sin[(n-1)\sigma_+] + A(n-1)\sin[\sigma_-]$$

Spiky strings in AdS:

$$ds^2 = -\cosh^2\rho \; dt^2 + d\rho^2 + \sinh^2\rho \; d\theta^2$$

Ansatz: $t = \tau, \qquad \theta = \omega\tau + \sigma, \qquad \rho = \rho(\sigma)$

Action and momenta:

$$I = T \int d\theta d\sigma \, \sqrt{\rho'^2(\cosh^2\rho - \omega^2 \sinh^2\rho) + \sinh^2\rho \cosh^2\rho}$$

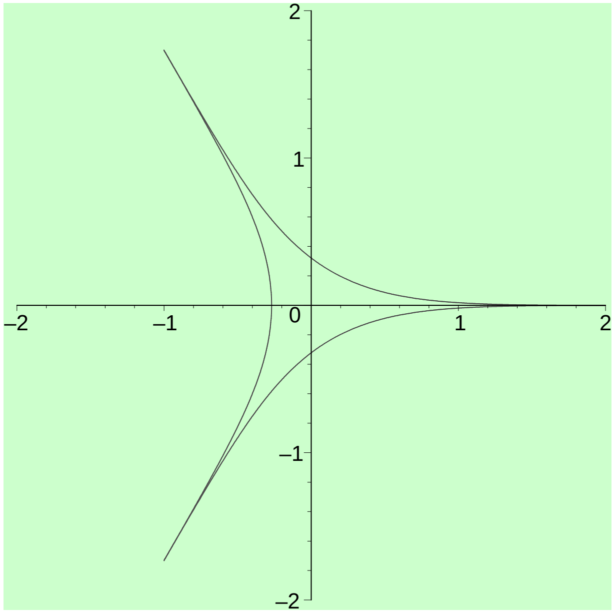
$$T = \frac{\sqrt{\lambda}}{2\pi}$$

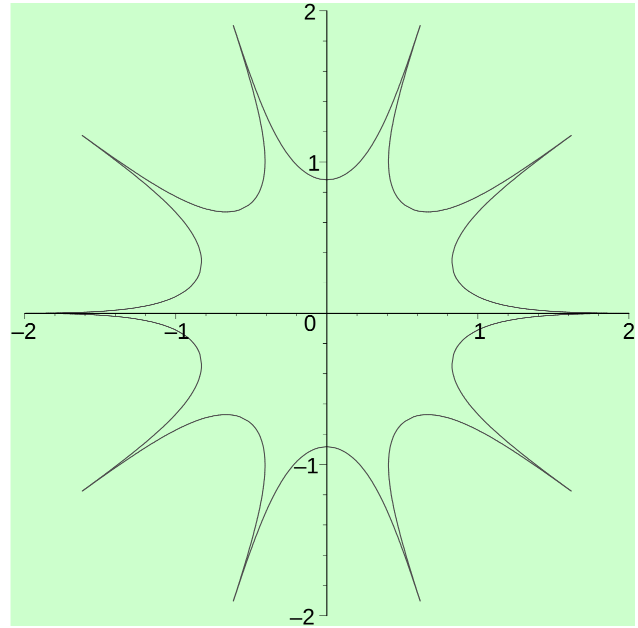
$$P_t = E = T \int d\sigma \, \frac{\cosh^2\rho \, (\rho'^2 + \sinh^2\rho)}{\sqrt{\rho'^2(\cosh^2\rho - \omega^2 \sinh^2\rho) + \sinh^2\rho \cosh^2\rho}}$$

$$P_\theta = -S = -\omega T \int d\sigma \, \frac{\rho'^2 \sinh^2\rho}{\sqrt{\rho'^2(\cosh^2\rho - \omega^2 \sinh^2\rho) + \sinh^2\rho \cosh^2\rho}}$$

Eq. of motion:

$$\rho'^2 = \frac{\sinh^2\rho \cosh^2\rho}{\sinh^2\rho_0 \cosh^2\rho_0} \, \frac{\sinh^2\rho \cosh^2\rho - \sinh^2\rho_0 \cosh^2\rho_0}{\cosh^2\rho - \omega^2 \sinh^2\rho}$$





$$E \cong S + \left(\frac{n}{2}\right)\frac{\sqrt{\lambda}}{\pi}\ln S, \quad (S \to \infty)$$

$$O = Tr\left(\nabla_+^{S/n}\Phi\,\nabla_+^{S/n}\Phi\,\nabla_+^{S/n}\Phi \ldots \nabla_+^{S/n}\Phi\right)$$

More recently, match between spectral curves in string and spin chain sides. (Dorey, Losi).

For all couplings we are lead to define $f(\lambda)$ through:

$E = S + (n/2)\, f(\lambda) \ln S$ (large S)

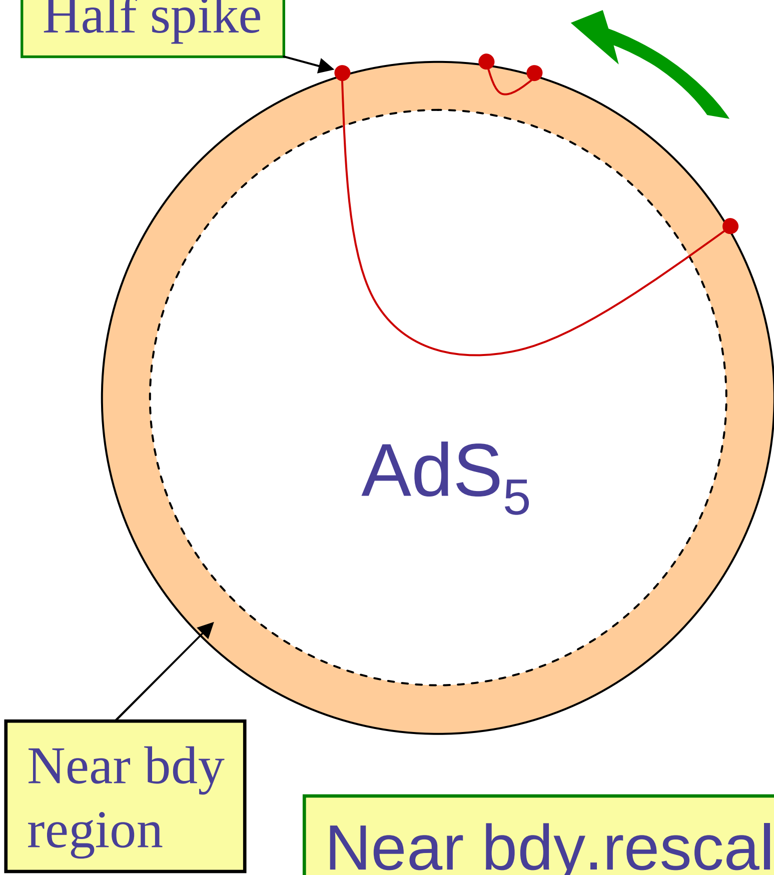
$$\begin{cases} f(\lambda) = \dfrac{\lambda}{2\pi^2} + O(\lambda^2) \\ f(\lambda) = \dfrac{\sqrt{\lambda}}{\pi} + \text{cst.} + O(\dfrac{1}{\sqrt{\lambda}}) \end{cases}$$

Other applications / results for $f(\lambda)$

- Gluon scattering amplitudes
  (Bern, Dixon, Smirnov …)
- Scattering amplitudes in AdS/CFT
  (Alday, Maldacena,…)
- Anomalous dimension $f(\lambda)$ at all loops
  (Beisert, Eden, Staudacher ,…)

Near boundary limit

We can take $\rho_0 \to \infty$ and get a solution close to the bdy.



$$ds^2 = -\cosh^2\rho \; dt^2 + d\rho^2 + \sinh\rho^2 \; d\Omega_{[3]}^2$$
$$= -\cosh^2\rho \; dt^2 + d\rho^2 + \sinh\rho^2 \, \frac{(1-\frac{x_i^2}{4})^2 \; d\theta^2 + dx_i dx_i}{(1+\frac{x_i^2}{4})^2}$$

Define light-like coords.

$$\rho = -\ln(2z) \, , \qquad t = x_+ - x_- \, , \qquad \theta = x_+ + x_- \, ,$$

Near bdy.rescaling.
Zooms in the shaded region

$$x_+ \longrightarrow \mu^{-1} \, x_+$$
$$x_- \longrightarrow 8\mu \, \varepsilon^2 \, x_-$$
$$x_i \longrightarrow 4\varepsilon \, x_i$$
$$z \longrightarrow \varepsilon \, z$$

AdS!

In the limit $\epsilon \to 0$ we get the metric:

$$ds^2 = \frac{1}{z^2}\left[2dx_+dx_- - \mu^2\left(z^2 + x_i^2\right)dx_+^2 + dx_i dx_i + dz^2\right]$$

The tip of the spike sees an AdS pp-wave in Poincare coordinates. When $z \to 0$, the boundary metric becomes a pp-wave in usual flat space:

$$ds^2 = 2dx_+dx_- - \mu^2 x_i^2 dx_+^2 + dx_i dx_i$$

$\mathcal{N} = 4$ SYM SU(N) on R^4 pp-wave dual to

IIB on AdS_5 pp-wave x S^5

According to AdS/CFT

This duality should contain all the information about $f(\lambda)$

We just argued that to compute $f(\lambda)$ we need to study $\mathcal{N} = 4$ in a pp-wave or strings in an AdS pp-wave.

What should we compute?
Before it was $E-S = f(\lambda) \ln S$ for the folded string.

Following the change of coordinates we find that

$$P_+ = P_t + P_\theta = E - S, \qquad P_- = -P_t + P_\theta = -E - S$$

Since $E \sim S$ we can say that for a single charge moving along x_+ we should have

$$P_+ = \gamma(\lambda) \ln |P_-| \ , \quad \text{with} \quad \gamma(\lambda) = \frac{1}{4} f(\lambda)$$

This leads us to define a PP wave anomaly

According to the previous calculations we need to compute $P_{\pm}$ in the presence of a charge (WL):

$$P_{\pm} \equiv \int dx_{-} d^2 x_i \sqrt{-g} \, \langle \, T^{+}_{\pm} \; W \, \rangle_{\text{pp-wave}} \; ,$$

$$W = \frac{1}{N} \mathrm{tr} \, \mathcal{P} e^{-i g_{\mathrm{YM}} \int A^a_+ t^a dx^+} \; .$$

$$P_+ = \gamma(\lambda) \ln |P_-|$$

PP-wave anomaly

Equivalently: $\gamma(\lambda) = -\frac{1}{2} \lim_{\varepsilon \to 0} \varepsilon \frac{\partial}{\partial \varepsilon} \, P_+$

$\mathcal{E}$ UV cut-off

PP wave anomaly Strong coupling

We put a particle in a pp-wave background and compute $P_{\pm}$ which amounts to computing a Wilson loop. We can use AdS/CFT since we know That the dual of the pp-wave is the AdS pp-wave.

The Wilson loop is $x_+=\tau$ and the string solution is simply:

$$x_+ = \tau, \qquad z = \sigma, \qquad x_- = x_i = 0$$

Giving:

$$P_+ = T\int_\epsilon^L dz \frac{\sqrt{\mu^2 z^2}}{z^2} \approx \frac{\mu T}{2} \ln\frac{L^2}{\epsilon^2}$$

$$P_- = T\int_\epsilon^\infty dz \frac{1}{z^2\sqrt{\mu^2 z^2}} \approx -\frac{T}{2\mu\epsilon^2}$$

or

$$P_+ \approx \frac{T}{2}\ln|P_-| = \frac{\sqrt{\lambda}}{4\pi}\ln|P_-|$$

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi}$$

OK

PP wave anomaly Small coupling

Again we need to compute a Wilson loop in the pp-wave background. At lowest order it is a classical source in the linearized approximation.

We need to solve Maxwell eqns. in the pp-wave with a source moving according to $x_+=\tau$.

$$\partial_\mu F^{\mu +} = g_{\rm YM}\ \delta(x_-)\ \delta^{(2)}(x_i)\ , \qquad \partial_\mu F^{\mu -} = \partial_\mu F^{\mu i} = 0$$

$$\begin{aligned} F^{+-} &= F_{-+} = -\frac{2g_{\rm YM}}{\pi^2}\frac{\mu x_-}{\mu^2 r^4 + 4x_-^2}\ , \\ F_{+i} &= \frac{g_{\rm YM}}{\pi^2}\frac{\mu^3 r^2 x_i}{\mu^2 r^4 + 4x_-^2}\ , \qquad F^{-i} = F_{+i} + \mu^2 r^2 F_{-i} = 0 \\ F^{+i} &= F_{-i} = -\frac{g_{\rm YM}}{\pi^2}\frac{\mu x_i}{\mu^2 r^4 + 4x_-^2}\ . \end{aligned}$$

$r^2=x_1^2+x_2^2$

We can now compute the energy momentum tensor

$$T^{\mu}{}_{\nu} = F^{\mu\alpha} F_{\alpha\nu} - \frac{1}{4} \delta^{\mu}{}_{\nu} F^{\alpha\beta} F_{\beta\alpha}$$

Obtaining

$$T^{+}{}_{+} = \frac{g_{\rm YM}^2}{2\pi^4} \frac{\mu^2}{\mu^2 r^4 + 4x_-^2} \,, \qquad T^{+}{}_{-} = -\frac{g_{\rm YM}^2}{\pi^4} \frac{\mu^2 r^2}{(\mu^2 r^4 + 4x_-^2)^2}$$

Using

$$P_+ = \frac{N}{2} \int_{-\infty}^{\infty} dx_- d^2x \; T^{+}{}_{+} \,, \qquad P_- = \frac{N}{2} \int_{-\infty}^{\infty} dx_- d^2x \; T^{+}{}_{-}$$

we get

$$P_+ = \frac{\lambda}{4\pi^4} 2\pi \int_{-\infty}^{\infty} dx_- \int_0^{\infty} dr \; r \; \frac{\mu^2}{\mu^2 r^4 + 4x_-^2 + \mu^2 \varepsilon^4} \approx \frac{\lambda}{8\pi^2} \; \mu \; \ln \frac{L^2}{\varepsilon^2} \,,$$

$$P_- = -\frac{\lambda}{2\pi^4} 2\pi \int_{-\infty}^{\infty} dx_- \int_0^{\infty} dr \; r \; \frac{\mu^2 \; r^2}{(\mu^2 r^4 + 4x_-^2 + \mu^2 \varepsilon^4)^2} \approx -\frac{\lambda}{8\pi^2 \mu \varepsilon^2} \,,$$

Or

$$P_+ \approx \frac{\lambda}{8\pi^2}\ \ln|P_-|$$

Giving $$f(\lambda) = \frac{\lambda}{2\pi^2}$$ OK

The symmetry of the pp-wave under

$$\tilde{x}_+ = x_+ \ , \qquad \tilde{x}_- = \xi^2 x_- \ , \qquad \tilde{x}_i = \xi x_i$$

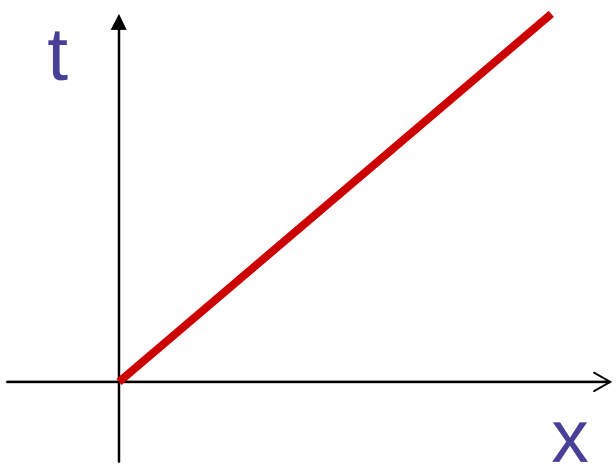
implies that $P_+ \sim \ln P_-$ and therefore that the anomalous dimension grows logarithmically with the spin.

Other Wilson loops in the pp-wave

$$ds^2 = 2\, dx_+\, dx_- - \mu^2\, x_\perp^{\,2}\, dx_+^{\,2} + dx_\perp^{\,2}\,,$$

$$x_\pm = \frac{x \pm t}{\sqrt{2}}$$

- Light-like line: $x_\perp = 0,\ x_- = 0,\ x_+ = \tau$

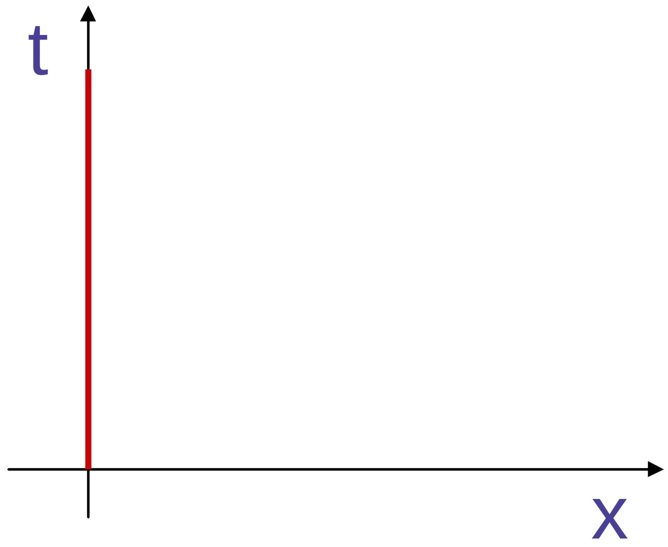


$$x_+ = \tau,$$

$$z = \sigma,$$

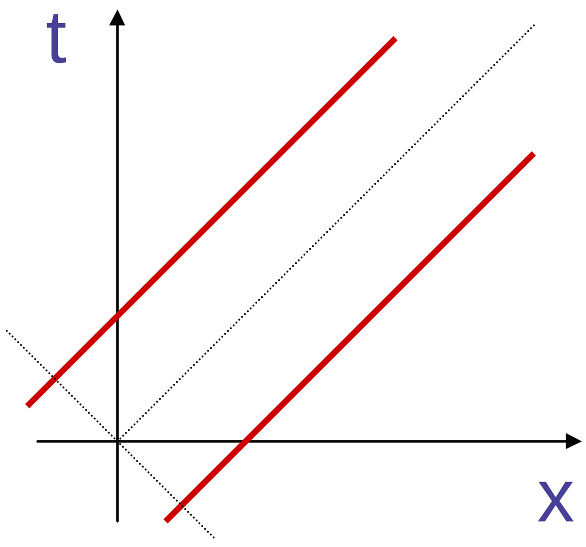
$$x_- = x_i = 0$$

- Time-like line: $x = x_\perp = 0,\ t = \tau$



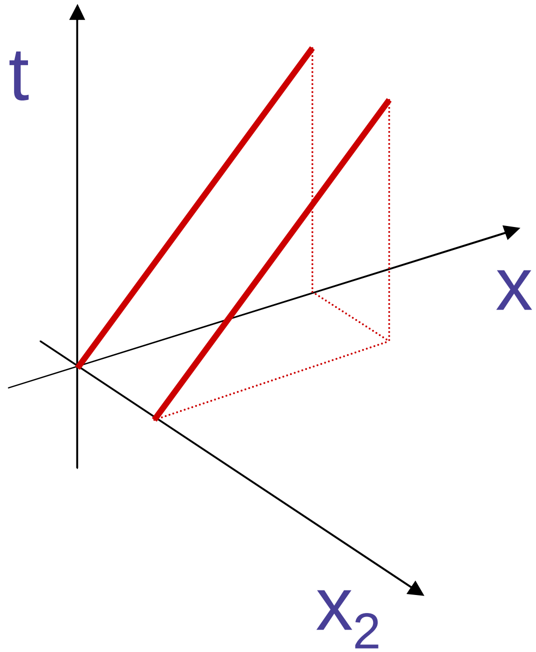
$$t = \tau$$
$$z = \sigma$$

- Parallel lines in x_+ direction: $x_\perp = 0$, $x_- = \pm a$, $x_+ = \tau$



$$x_+ = \tau$$
$$x_- = \sigma$$
$$z = \sqrt{2}(\sigma_0^2 - \sigma^2)^{\frac{1}{4}}$$
$$x_i = 0$$

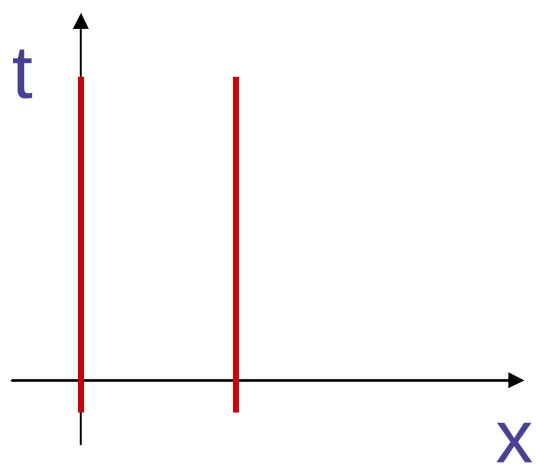
- Parallel lines in x_+ direction: x_2=a,b , x_- =0, $x_+=\tau$



a=0 light-like

$$x_+ = \tau$$
$$x_- = 0$$
$$x_2 = \sigma$$
$$z = z(\sigma)$$

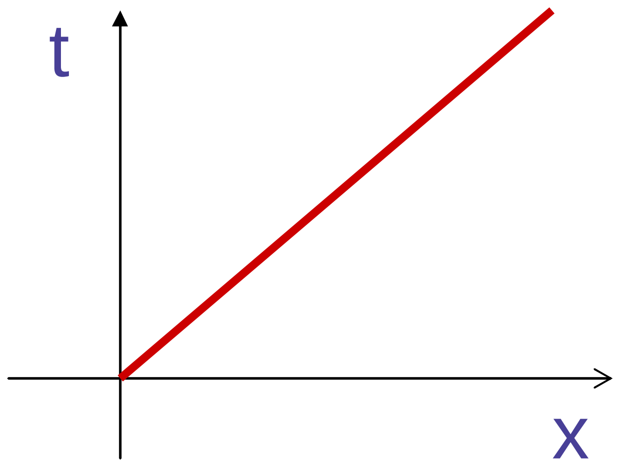
- Parallel lines in time-like direction: x=±a, t=τ

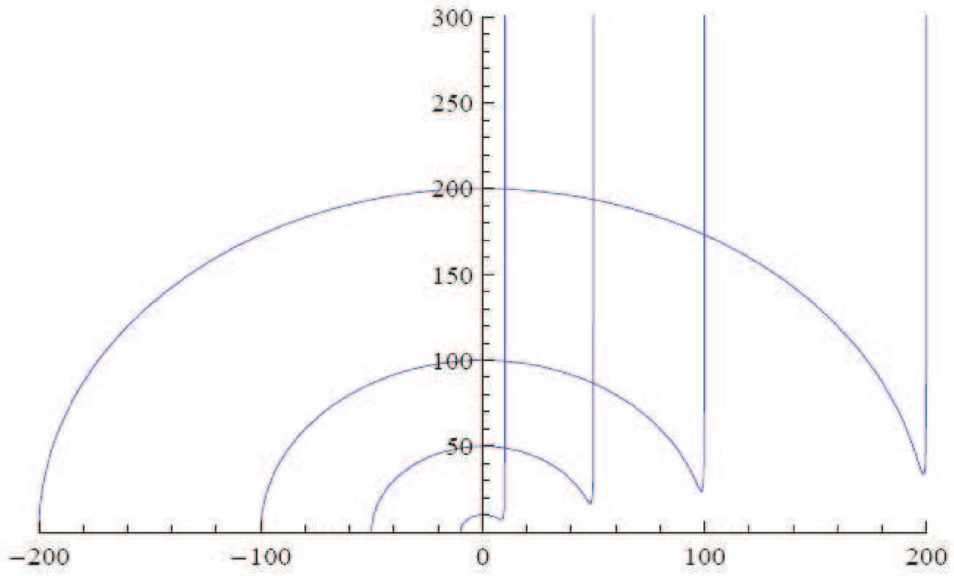


$$t = \tau$$
$$x = \sigma$$
$$z = z(\sigma)$$

This allows us to obtain new Wilson loop solutions in usual AdS in Poincare coordinates.

- Light-like line: $x_\perp = 0$, $x_- = 0$, $x_+ = \tau$



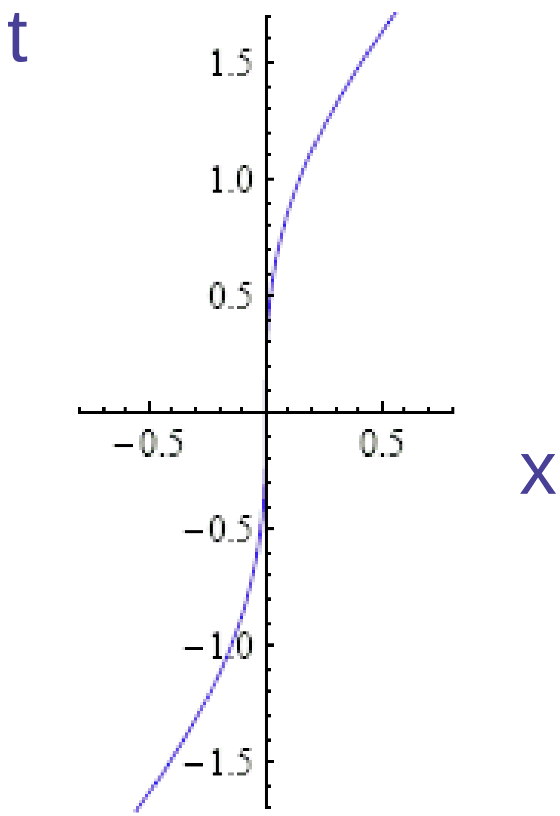


Plot of $\tilde{z}$ versus $\tilde{x}$ for $\mu = 1$, and $\tilde{t} = -200, -100, -50, -10$.

$$\tilde{z} = \sqrt{-\frac{2\tilde{x}_-}{\mu^2 \tilde{x}_+}(1 + \mu^2 \tilde{x}_+^2)}$$

- Time-like line: $x = x_\perp = 0,\ t = \tau$

The Wilson loop now changes shape.
Particle decelerating and then accelerating.



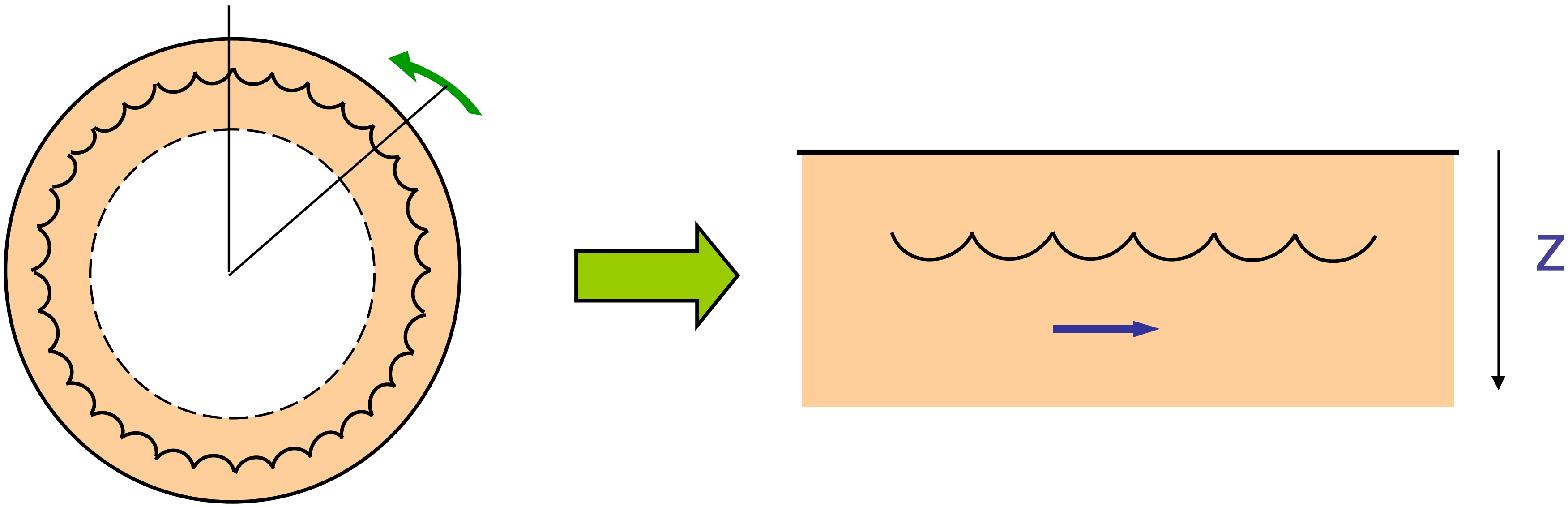
$$\tilde{z} = \sqrt{-\frac{2}{\mu^2 \tilde{x}_+}(\tilde{x}_- + \frac{\arctan \mu \tilde{x}_+}{\mu})(1 + \mu^2 \tilde{x}_+^2)}$$

We can do the same with the other solutions. Generically we obtain surfaces in which z is a function of two variables and therefore very difficult to find directly.

Physically these solution seem to represent Wilson loops in the presence of propagating gluons given by spikes coming out from the horizon.

Also, in general the energy is not conserved so it requires certain power to move the quarks in these particular way.

Periodic spike solution in AdS pp-wave



Spiky string
in global AdS

Periodic spike
in AdS pp-wave

If we do not take number of spikes to infinity we get a single spike:

Such solutions actually exist!

One can use the ansatz
(in AdS pp-wave background):

$t = \tau, \quad x = v\,\tau + \sigma, \quad z = z(\sigma)$

for a string moving at constant velocity v.
Somewhat better, the symmetry suggest using

$$x_+ = \tau, \quad x_- = \sigma, \quad z = z(x - vt) = z\left(\tau - \frac{1}{\eta_0^2}\sigma\right),$$

with

$$\eta_0 = e^{-\beta}, \quad v = \coth\beta$$

We get for the (x_-) separation between spikes

$$\Delta x_- = 2\mu \int_{z_0}^{z_1} z^2\, dz \sqrt{\frac{z^2 - z_1^2}{z_0^4 - z^4}} \qquad z_1 = \sqrt{2}\,\frac{\eta_0}{\mu}$$

and for the conserved momenta

$$P_- = \frac{2}{\mu z_0^2} \int_{z_1}^{z_0} \frac{dz}{z^2} \sqrt{\frac{z_0^4 - z^4}{z^2 - z_1^2}}$$

$$P_+ = \frac{2\mu}{z_0^2} \int_{z_1}^{z_0} dz \left[z^2 \sqrt{\frac{z^2 - z_1^2}{z_0^4 - z^4}} + \left(1 - \frac{z_1^2}{2z^2}\right) \sqrt{\frac{z_0^4 - z^4}{z^2 - z_1^2}} \right]$$

Where z_0 and z_1 are the position of the valleys and spikes

Equivalenty

$$\Delta x_- = \frac{\mu z_0^2}{\sqrt{1+b}}\Big[(1+b)\mathrm{E}(p) - b\mathrm{K}(p) - b^2\Pi(1-b,p)\Big]$$

$$P_- = \frac{2}{\mu z_0^2}\frac{1}{b\sqrt{1+b}}\left[-bK(p) + (1+b)E(p) - b^2\Pi(1-b,p)\right]$$

$$P_+ = \frac{\mu}{\sqrt{1+b}}\left[(2+b)K(p) - (1+b)E(p) - b^2\Pi(1-b,p)\right]$$

with

$$p = \sqrt{\frac{1-b}{1+b}} \qquad b = \frac{z_1^2}{z_0^2}$$

The metric $ds^2 = \frac{1}{z^2}\left[2dx_+dx_- - \mu^2\left(z^2 + x_i^2\right)dx_+^2 + dx_idx_i + dz^2\right]$
has the symmetry

$$z \longrightarrow \xi z, \quad x_i \longrightarrow \xi x_i, \quad x_+ \longrightarrow x_+ \quad x_- \longrightarrow \xi^2 x_-$$

implying the existence of many equivalent solutions.
We can use invariant quantities to characterize each class of solutions. We take:

$$P_-\Delta x_- = \frac{2T}{b(1+b)}\Big[(1+b)\mathrm{E}(p) - b\mathrm{K}(p) - b^2\Pi(1-b,p)\Big]^2$$

$$P_+ = \frac{\mu T}{\sqrt{1+b}}\Big[(1+b)\mathrm{E}(p) - (2+b)\mathrm{K}(p) + b^2\Pi(1-b,p)\Big]$$

Field theory interpretation?

The same can be obtained from taking the limit in the spiky string. The metric is

$$ds^2 = -\cosh^2\rho\; dt^2 + d\rho^2 + \sinh^2\rho\; d\theta^2$$

The angle between spikes, spin and energy are:

$$\Delta\theta = \frac{2\pi}{n} = 2\sqrt{\frac{1-v_3^2}{1-v_2}}\int_{v_2}^{v_3}\frac{dv}{1-v^2}\sqrt{\frac{v(v-v_2)}{v_3^2-v^2}}$$

$$\frac{\mathcal{S}}{n} = \frac{\sqrt{1+v_2}}{4\pi v_3}\int_{v_2}^{v_3}\frac{dv}{v(1+v)}\sqrt{\frac{v_3^2-v^2}{v(v-v_2)}}$$

$$\frac{\mathcal{E}}{n} = \frac{1-v_3^2}{2\pi v_3\sqrt{1-v_2}}\int_{v_2}^{v_3}\frac{dv}{1-v^2}\sqrt{\frac{v(v-v_2)}{v_3^2-v^2}} + \frac{\sqrt{1-v_2}}{4\pi v_3}\int_{v_2}^{v_3}\frac{dv}{v(1-v)}\sqrt{\frac{v_3^2-v^2}{v(v-v_2)}}$$

$$v \equiv \frac{1}{\cosh 2\rho}$$

Take $v_3 \longrightarrow \epsilon^2 v_3, \qquad v_2 \longrightarrow \epsilon^2 v_2\ , \qquad \epsilon \longrightarrow 0$

In this limit, $n \sim \frac{1}{\epsilon^2} \to \infty$. However, the following quantities remain fixed:

$$\frac{\mathcal{E}+\mathcal{S}}{n^2} \to \frac{1}{2\pi^2}\int_{v_2}^{v_3}\frac{dv}{v}\sqrt{\frac{v_3^2-v^2}{v(v-v_2)}}\int_{v_2}^{v_3}\frac{dv'}{v_3}\sqrt{\frac{v'(v'-v_2)}{v_3^2-v'^2}},$$

$$\frac{\mathcal{E}-\mathcal{S}}{n} \to \int_{v_2}^{v_3}\frac{dv}{2\pi v_3}\sqrt{\frac{v(v-v_2)}{v_3^2-v^2}}+\int_{v_2}^{v_3}\frac{dv}{2\pi v}\left(1-\frac{v_2}{2v}\right)\sqrt{\frac{v_3^2-v^2}{v(v-v_2)}}$$

or, defining $b \equiv \frac{v_2}{v_3}$

$$\bar{P}_- \equiv \frac{E+S}{n^2} = \frac{2T}{\pi b(1+b)}\left[(1+b)\mathrm{E}(p)-(2+b)\mathrm{K}(p)-b^2\Pi(1-b,p)\right]^2$$

$$\bar{\gamma} \equiv \frac{E-S}{n} = \frac{T}{\sqrt{1+b}}\left[(1+b)\mathrm{E}(p)-(2+b)\mathrm{K}(p)+b^2\Pi(1-b,p)\right]$$

We expect these results to correspond to a thermodynamic limit of the **SL(2,R)** chain, simply because we take $n \longrightarrow \infty$. In fact it should correspond to a particular translationally invariant state:

$$O_{[n]} = Tr\left(\nabla_+^{S/n}\Phi\,\nabla_+^{S/n}\Phi\,\nabla_+^{S/n}\Phi\ldots\nabla_+^{S/n}\Phi\right)$$

with $n \longrightarrow \infty$, keeping $\bar{P}_- \equiv \frac{E+S}{n^2}$, $\bar{\gamma} \equiv \frac{E-S}{n}$ fixed.

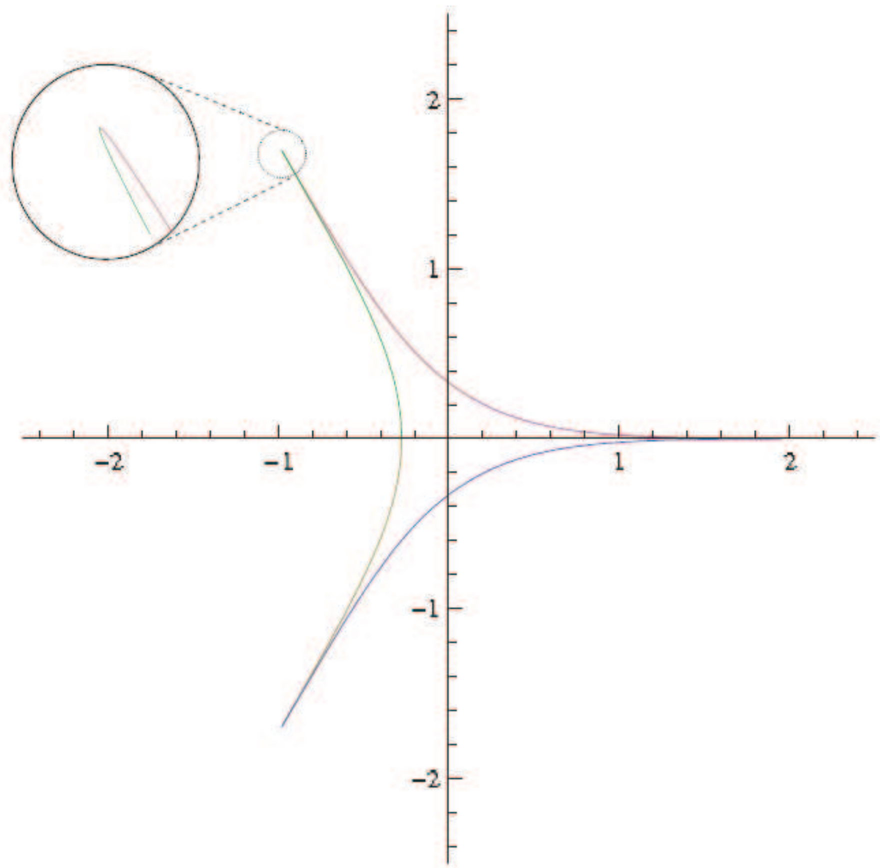
The **SL(2,R)** chain can be studied similarly to the **SU(2)** chain (Minahan-Zarembo) and was considered in the field theory by Stefanski, Tseytlin and by Belitsky Gorsky Korchemski.

Generalization to rotation on S^5

If we consider a maximal circle on S^5 we have the metric

$$ds^2 = -\cosh^2\rho\; dt^2 + d\rho^2 + \sinh^2\rho\; d\theta^2 + d\phi^2$$

And we want to add angular momentum J on ϕ. Result:



The limit now reads $n \longrightarrow \infty$, with

$$\frac{\mathcal{E}+\mathcal{S}}{n^2} \sim \frac{\mathcal{E}-\mathcal{S}}{n} \sim \frac{\mathcal{J}}{n} \sim \frac{m}{n} \quad = \quad \texttt{finite}$$

Here we included a possible winding number m.
Again one can reproduce the result by a string in an AdS-pp-wave with an extra circle:

$$ds^2 = \frac{1}{z^2}\left(dz^2 + 2dx_+dx_- - \mu^2 z^2 dx_+^2\right) + d\phi^2$$

and gives a prediction for the **SL(2,R)** chain when J and E-S are comparable.

Conclusions:

- We studied strings moving in an AdS pp-wave dual to states of a gauge theory in a pp-wave.

- In particular we showed that there is an anomaly (the pp-wave anomaly) that can be computed in this manner and turns out to be equal to the cusp anomaly. We verify this at weak coupling on the field theory side.

- We consider Wilson loops in the pp-wave.

- We found a new periodic spike solution which we argued should be dual to the thermodynamic limit of the SL(2,R) chain.