

Spin chains and strings
in $Y^{p,q}$ and $L^{p,q|r}$ manifolds

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Summary

Introduction:

N=4 AdS/CFT: Strings \leftrightarrow Long ops.

- Point-like strings \leftrightarrow BPS ops.
- String excitations \leftrightarrow insertions
- Long strings \leftrightarrow semiclassical states

N=1 AdS/CFT

• $\text{AdS}_5 \times X_5$; X_5 is Sasaki-Einstein

- X_5 :

$T^{1,1}$	$SU(2) \times SU(2) \times U(1)$
$Y^{p,q}$	$SU(2) \times U(1) \times U(1)$
$L^{p,q} r$	$U(1) \times U(1) \times U(1)$

$Y_{p,q}$

- Massless geodesics and long operators:
matching of R and flavor charges.
- Extended strings (Qualitative)
Eff. action appears in f.t. and is similar but not equal to
string side.

$L_{p,q|r}$

- Derivation of dual field theory
- Massless geodesics and long operators:
matching of R and flavor charges.

N=4 SYM \Leftrightarrow II B on AdS₅ x S⁵

Strings? Long operators \leftrightarrow Strings
(BMN, GKP) an. dimension \leftrightarrow Energy

Frolov-Tseytlin: Long strings

Minahan-Zarembo: Spin chains

Bethe Ansatz (BFST)

$X = \phi_1 + i \phi_2$; $Y = \phi_3 + i \phi_4$; $O = \text{Tr}(XY \dots Y)$

Long chains ($J \rightarrow \infty$) are classical

$$S_{\text{eff.}} = J \left[\int \cos \theta \dot{\phi} - \frac{\lambda}{J^2} \int \left(\partial_{\sigma} \vec{n} \right)^2 \right]$$

$$\vec{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

$S_{\text{eff.}}$: Action of spin chain and fast strings.

(M.K.; Ryzhov, Tseytlin, M.K.; ...)

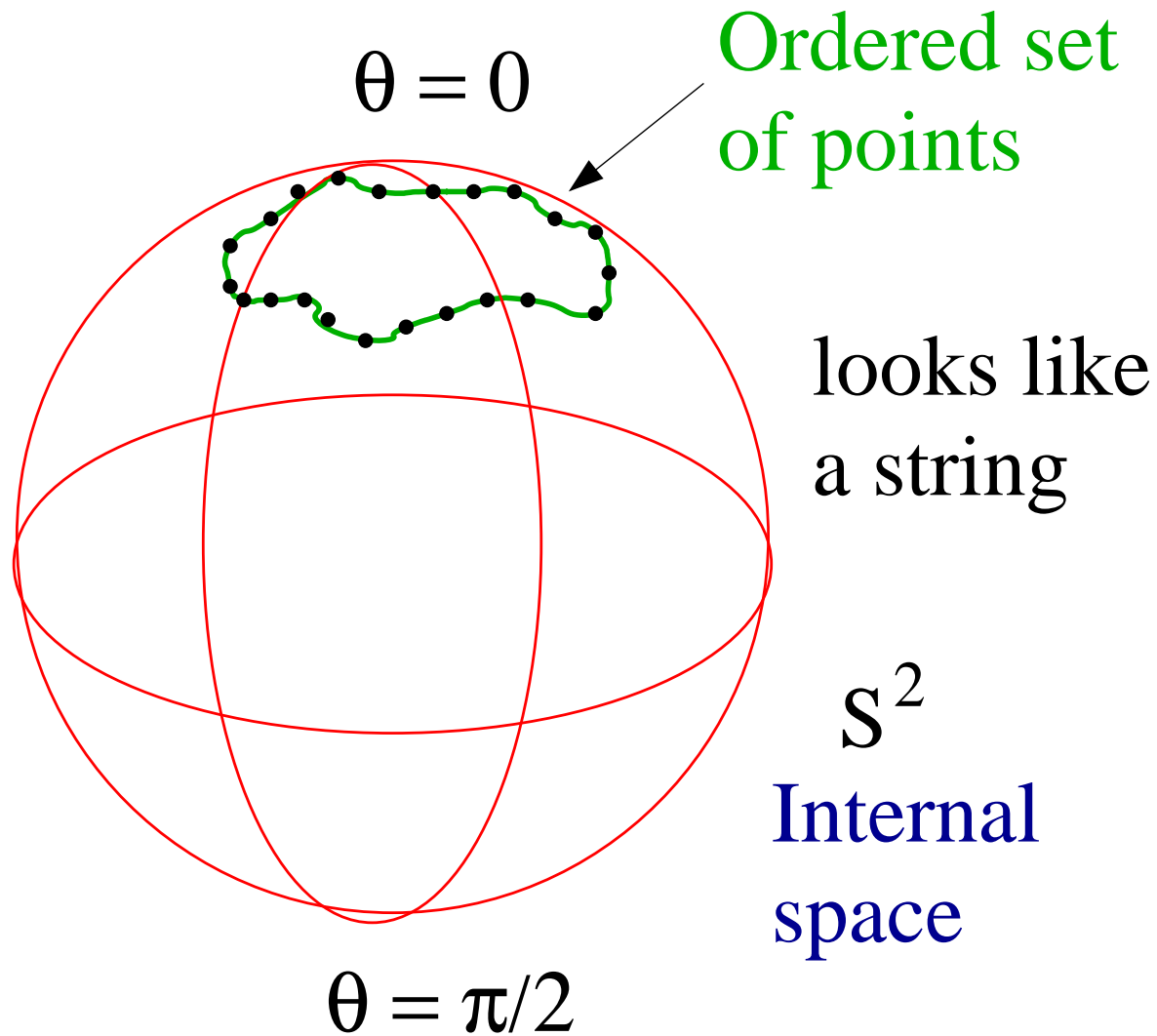
Strings as states of $N=4$ SYM on $R \times S^3$

The operator $O = \sum c_i [\text{Tr}(XY \dots Y)]_i$
maps to a state of the field th. on S^3

State: delocalized and has a large number
of particles (X and Y).

$$\text{Q.M. : } |\psi\rangle = \cos(\theta/2) \exp(i\phi/2) |X\rangle \\ + \sin(\theta/2) \exp(-i\phi/2) |Y\rangle$$

We can use $v_i = |\psi(\theta_i, \phi_i)\rangle$ to
construct: $O = \text{Tr}(v_1 v_2 v_3 \dots v_n)$
(Coherent state)



Strings are useful to describe states of a large number of particles (in the large N limit)

N=1 AdS₅ x X₅; X₅: Sasaki-Einstein

$$ds^2 = dx^2_{[4]} + \frac{dr^2 + r^2 dX_5^2}{r^2}$$



CY cone

Put D3 branes at $r = 0$ and take near horizon limit:

$$ds^2 = r^2 dx^2_{[4]} + \frac{dr^2}{r^2} + dX_5^2, \text{ AdS}_5 \times X_5$$

T^{1,1} (conifold) Klebanov-Witten

Y^{p,q} Gauntlett, Martelli, Sparks, Waldram

Benvenuti, Franco, Hanany, Martelli, Sparks

L^{p,q|r} Cvetič, Lu, Page, Pope; Benvenuti, M.K.;

Franco, Hanany, Martelli, Sparks, Vegh, Wecht;

Butti, Forcella, Zaffaroni

T^{1,1}

$$ds^2 = \frac{1}{9} [d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2]^2 + \frac{1}{6} \sum_{i=1}^2 [d\theta_i^2 + \sin \theta_i d\phi_i^2]$$

Y^{p,q} (Gauntlett, Martelli, Sparks, Waldram)

$$ds^2 = w(y) [d\alpha + f(y)(d\psi - \cos \theta)]^2 + \frac{1-y}{6} (d\theta^2 + \sin^2 \theta d\phi^2) \\ + \frac{dy^2}{6p(y)} + \frac{q(y)}{9} (d\psi - \cos \theta d\phi)^2$$

If $\alpha \rightarrow \beta = 6\alpha + \psi$; only $p(y)$ appears.

$$p(y) = \frac{a - 3y^2 + 2y^3}{3(1-y)}$$

; $y_1 < y < y_2$

$$y_{1,2} = \left(2p \mp 3q - \sqrt{4p^2 - 3q^2} \right) / (4p)$$

Point-like strings (massless geodesics)

$$S = \frac{\sqrt{\lambda}}{2} \int d\tau \left(-\partial_\tau t \partial_\tau t + g_{ab} \partial_\tau x^a \partial_\tau x^b \right)$$

$$\text{Const.: } -\partial_\tau t \partial_\tau t + g_{ab} \partial_\tau x^a \partial_\tau x^b = 0$$

$$t = \kappa \tau; \quad P_t = \Delta = \sqrt{\lambda} \quad \kappa$$

$$P_a = \frac{\delta L}{\delta(\partial_\tau x^a)}; \quad H = g^{ab} P_a P_b$$

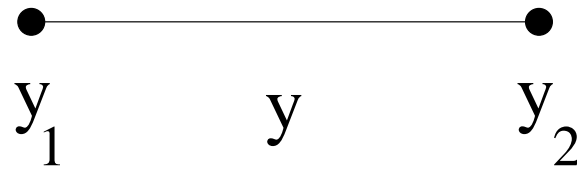
$$\Delta^2 = \left(\frac{3}{2} Q_R \right)^2 + \frac{1}{6p(y)} (P_\alpha + 3y Q_R)^2 + 6p(y) P_y^2 + \frac{6}{1-y} (J^2 - P_\psi^2)$$

$$\Delta \geq (3/2) Q_R; \quad Q_R = 2P_\psi - (1/3) P_\alpha$$

BPS geodesics

$$\Delta = (3/2) Q_R \quad \Rightarrow \quad P_y = 0, \quad J = P_\psi, \quad y_0 = -\frac{P_\alpha}{3 Q_R}$$

Therefore, y is the ratio between U(1) and R-charges.

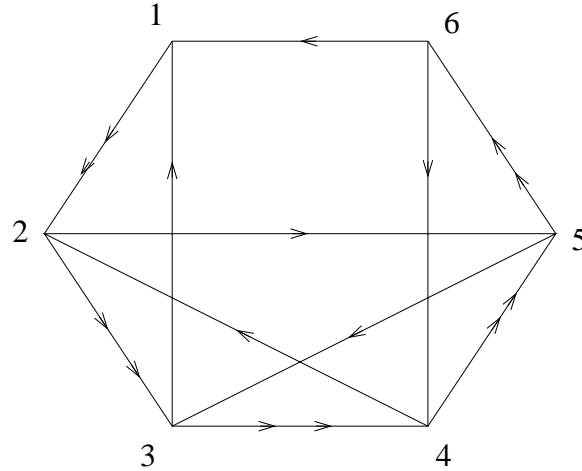


Also: $J = \frac{1}{2} (1 - y_0) Q_R$ (from $J = P_\psi$)

Dual gauge theories

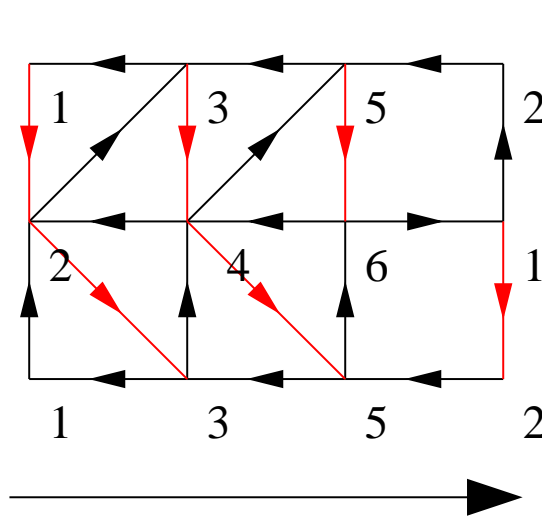
(Benvenuti, Franco, Hanany, Martelli, Sparks)

Example: $Y^{3,2}$



Periodic rep.:

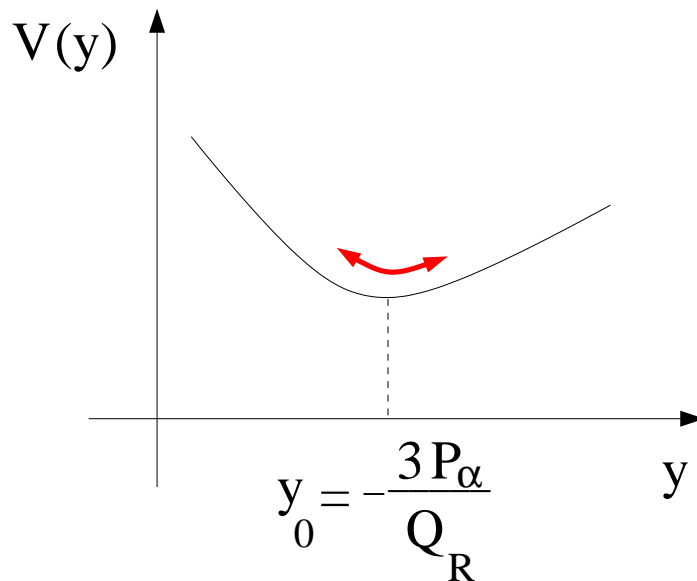
(Hanany,
Kennaway,
Franco,
Vegh,
Wecht)



$SU(2)$
 $U(1)$

$U(1)$

Small Fluctuations (BMN, GKP, Gomis-Ooguri, Klebanov et al.)



Quantize: $\delta\Delta=2n$, $\delta Q_R=0$, $\delta P_\alpha=0$, $\delta J=0$

Also ($J > P_\psi$): $\delta\Delta=2n$, $\delta Q_R=0$, $\delta P_\alpha=0$, $\delta J=n$

Agree with quantum numbers of $U(1)$ and $SU(2)$ currents
 \Rightarrow we identify these non-BPS geodesics with insertions of the currents. (Also Sonnenschein et al.)

Extended Strings

We consider closed extended strings such that each point moves approximately along a BPS geodesic.

Effective action for such strings:

$$ds^2 = -dt^2 + \frac{1}{6} g_{ij} dx^i dx^j + \frac{1}{9} (d\psi + A_j dx^j)^2$$

We introduce a coordinate $\psi_1 = \psi - 3t$

$$ds^2 = \frac{2}{3} dt (d\psi_1 + A_j dx^j) + \frac{1}{6} g_{ij} dx^i dx^j + \frac{1}{9} (d\psi_1 + A_j dx^j)^2$$

Now we use ansatz: $t = \kappa \tau$

and take the limit: $\partial_\tau X \rightarrow 0$, $\kappa \rightarrow \infty$, $\kappa \partial_\tau X$ fixed.

We get a reduced action:

$$S = \frac{1}{3} \int \kappa (\partial_\tau \psi_1 + A_i \partial_\tau x^i) - \frac{1}{12} g_{ij} \partial_\sigma x^i \partial_\sigma x^j$$

In our case:

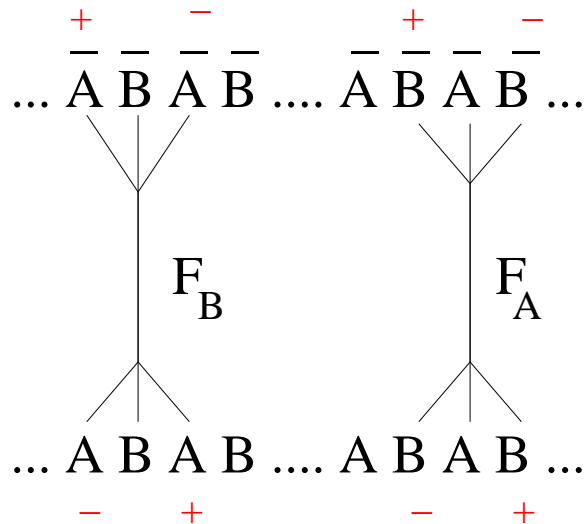
$$S = \frac{Q_R}{4\pi} \left\{ \int \left(\partial_t \psi_1 - y \partial_t \beta - (1-y) \cos \theta \partial_t \phi \right) \right. \\ \left. - \frac{4\pi^2}{9} \frac{\lambda}{Q_R^2} \left[(1-y) (\partial_\sigma n)^2 + \frac{(\partial_\sigma y)^2}{p(y)} + p(y) \left(\partial_\sigma \beta - \cos \theta \partial_\sigma \phi \right)^2 \right] \right\}$$

Example of $T^{1,1}$ (Angelova, Pando-Zayas, M.K. ; Strassler et al.)

$$S_{\text{eff.}} = J \left\{ \int \cos \theta_1 \dot{\phi}_1 + \cos \theta_2 \dot{\phi}_2 - \frac{\lambda}{J^2} \int \left[\left(\partial_\sigma \vec{n}_1 \right) + \left(\partial_\sigma \vec{n}_2 \right) \right] \right\}$$

Field theory:

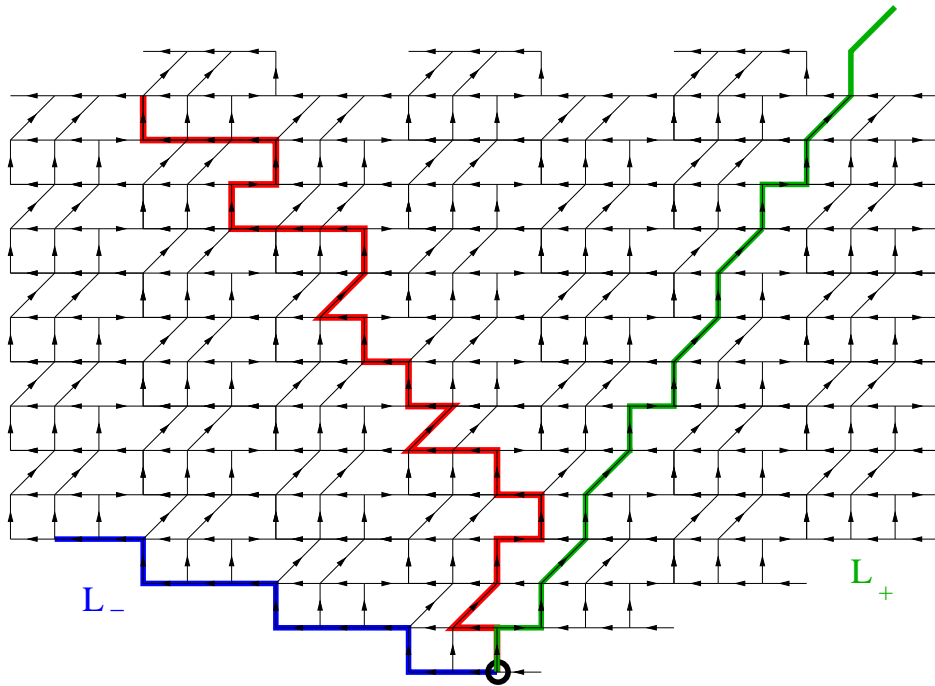
$\text{Tr}(\text{ABABAB} \dots \text{ABAB})$; $\text{SU}(2) \times \text{SU}(2)$



$$W = \epsilon_{ab} \epsilon_{cd} \text{Tr} (A^a B^c A^b B^d)$$

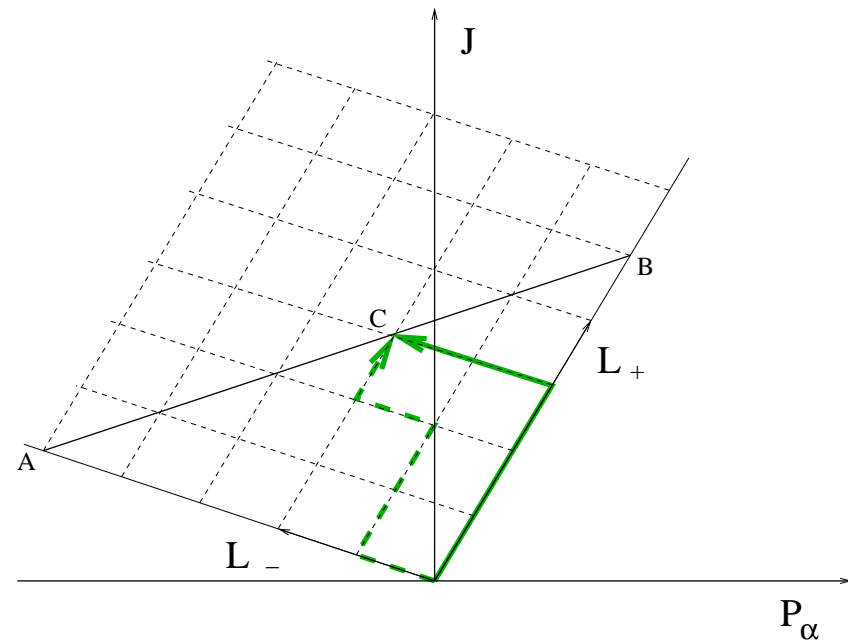
$+, -$: doublet of $\text{SU}(2)$

For $Y^{p,q}$



$y(\sigma) \rightarrow \text{slope}(\sigma);$
 $\beta: \text{conjugate of } y$
 $+ \text{spin} \rightarrow (\theta, \phi)$

Effective model:
(max and min slopes)



Hamiltonian

$$H = h_{\text{eff.}} \sum (1 - P_{i i+1})$$

Coherent States:

$$O = \rho_{1i} \exp(iP_{\alpha}^{(1)} \alpha_i) U(\theta_i, \phi_i, \psi_i) L_1 \\ + \rho_{2i} \exp(iP_{\alpha}^{(2)} \alpha_i) U(\theta_i, \phi_i, \psi_i) L_2$$

$$\rho_1^2 + \rho_2^2 = 1 \quad ; \quad \rho_1^2 y_1 + \rho_2^2 y_2 = y$$

The coherent state action is

$$S = \int d\tau \text{Im} \langle O | \partial_{\tau} | O \rangle - \int d\tau \langle O | H | O \rangle$$

which gives:

$$S = \frac{Q_R}{4\pi} \left\{ \int \left(\partial_t \psi_1 - y \partial_t \beta - (1-y) \cos \theta \partial_t \phi \right) \right. \\ \left. - 2\pi^2 \frac{h_{eff}}{Q_R^2} \left[(1-y)(\partial_\sigma n)^2 + \frac{(\partial_\sigma y)^2}{p(y)} + p(y) \left(\partial_\sigma \beta - \cos \theta \partial_\sigma \phi \right)^2 \right] \right\}$$

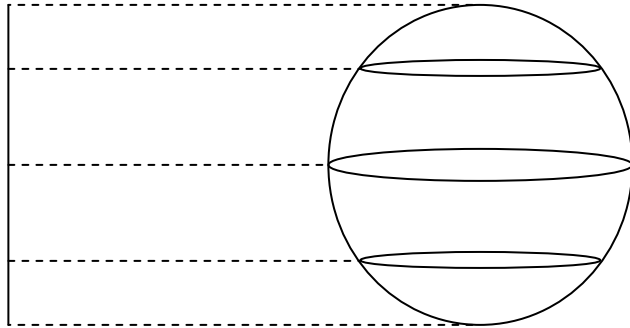
where $p(y) = (y_2 - y)(y - y_1)$

instead of $p(y) = \frac{a - 3y^2 + 2y^3}{3(1-y)}$

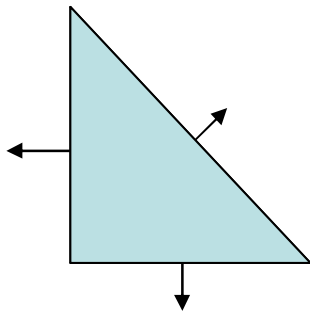
It is interesting that a string picture (and action) for the operators emerges from the analysis.

Toric Varieties

S^2



$$(z_1, z_2) \equiv w(z_1, z_2)$$

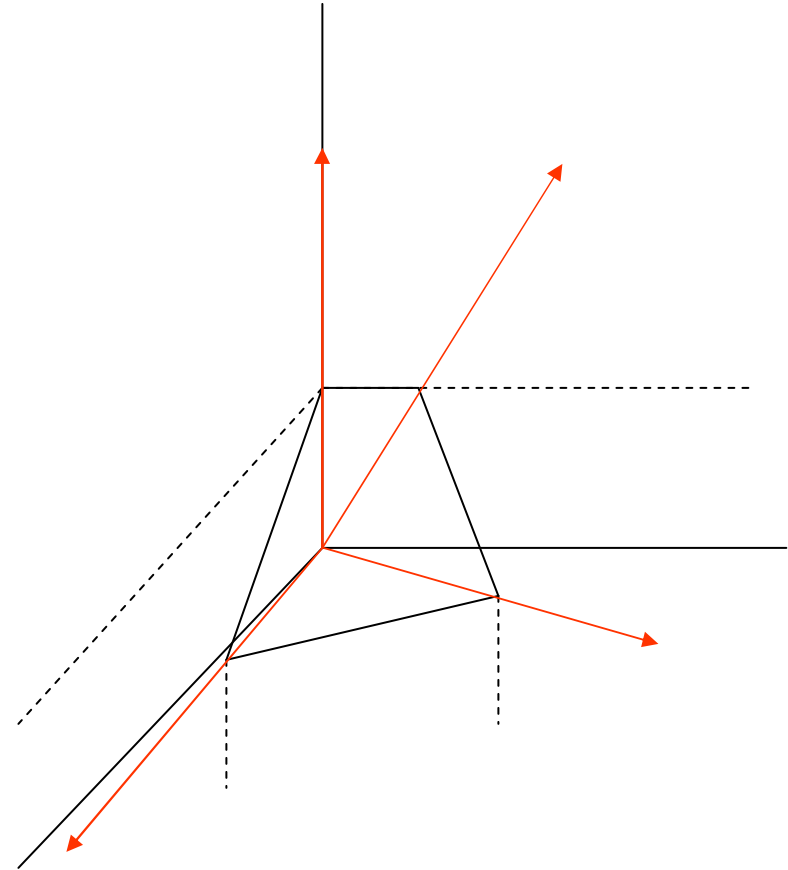
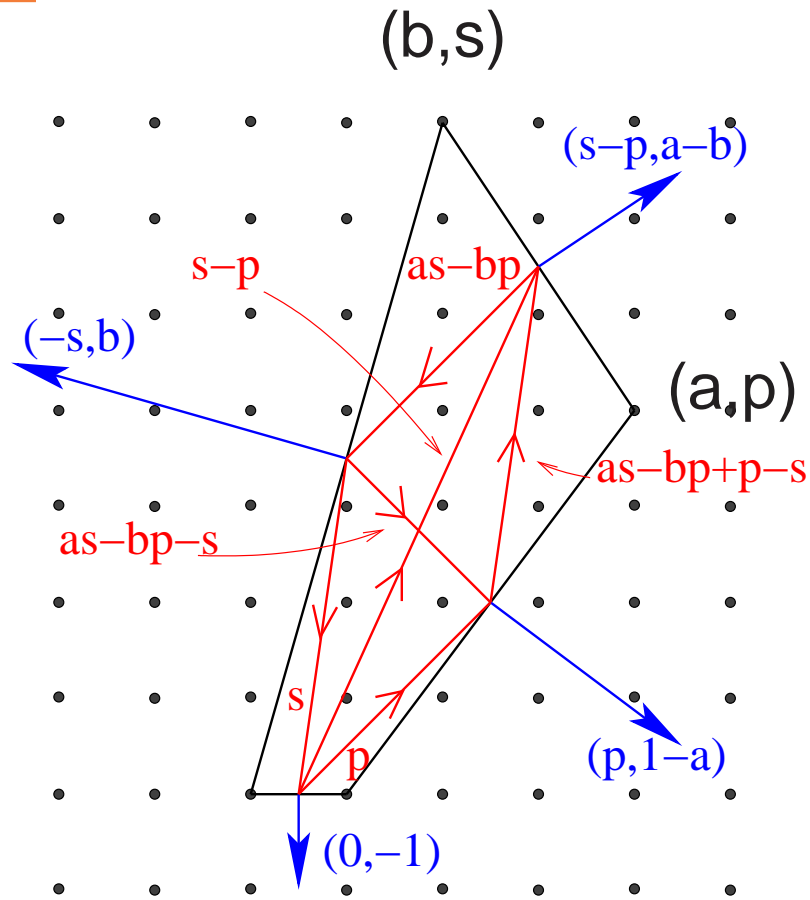


$$(\mathbb{C}^3)^* / \mathbb{C}$$

$$(z_1, z_2, z_3) \equiv w(z_1, z_2, z_3)$$

$$(z_1 e^{i\alpha}, z_2 e^{i\beta}, z_3)$$

$\lfloor pq|r$



$$r = p + q - s \quad q = a s - p b$$

(Also **Butti, Zaffaroni**, generic polygon)

L_{p,q|r}

Metric: (Cvetic, Lu, Page, Pope)

$$ds^2 = \left(\frac{1}{3} d\psi_R + A \right)^2 + \frac{1}{4} ds_{[4]}^2$$

$$A = \frac{\alpha - 3\alpha y - 3x + 3xy}{6\alpha} d\phi + \frac{\beta + 3\beta y - 3x - 3xy}{6\alpha} d\psi$$

$$ds_{[4]}^2 = \frac{\rho^2}{f(x)} dx^2 + \frac{f(x)}{\rho^2} \left(\frac{1-y}{\alpha} d\phi + \frac{1+y}{\beta} d\psi \right)^2 \\ + \frac{\rho^2}{g(y)} dy^2 + \frac{g(y)}{\rho^2} \left(\frac{\alpha-x}{\alpha} d\phi - \frac{\beta-x}{\beta} d\psi \right)^2$$

$$\rho^2 = \alpha + \beta - y(\alpha - \beta) - 2x$$

$$\psi_R = 3\xi + \phi + \psi$$

$$x_1 < x < x_2$$

$$-1 < y < 1$$

To write the metric in this way we needed to identify the angle conjugated to the R-charge.

This can be done by finding the covariantly constant holomorphic three form in the Calabi-Yau.

$$\Omega_{[3]} = \sqrt{f(x)g(y)} e^{i\psi_R} r^3 \eta_1 \wedge \eta_2 \wedge \eta_3$$

$$\eta_1 = \frac{x - \beta}{f(x)} dx - \frac{1 + y}{g(y)} dy + \frac{2i}{\alpha} d\phi \quad \eta_3 = \frac{dr}{r} + i(d\xi + \sigma)$$

$$\eta_2 = \frac{x - \alpha}{f(x)} dx + \frac{1 - y}{g(y)} dy + \frac{2i}{\beta} d\psi \quad \sigma = \frac{(\alpha - x)(1 - y)}{2\alpha} d\phi + \frac{(\beta - x)(1 + y)}{2\beta} d\psi$$

Massless geodesics

$$H = \frac{9}{2}P_{\psi_R}^2 + \frac{1}{2b_2}P_y^2 + \frac{1}{2b_1}P_x^2 + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)$$

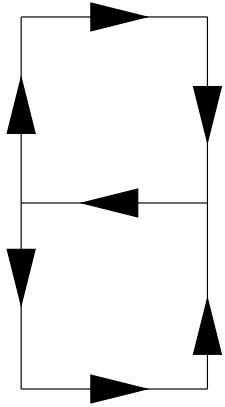
$$\sigma_1 = -\frac{-d_2P_\phi - d_1P_\psi + 3(a_1d_2 + a_2d_1)P_{\psi_R}}{c_1d_2 + c_2d_1}$$

$$\sigma_2 = -\frac{-c_2P_\phi + c_1P_\psi + 3(a_1c_2 - a_2c_1)P_{\psi_R}}{c_1d_2 + c_2d_1}$$

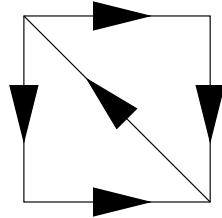
$$a_1 = \frac{(\alpha - x)(1 - y)}{2\alpha} - \frac{1}{3}, \quad a_2 = \frac{(\beta - x)(1 + y)}{2\beta} - \frac{1}{3}, \quad b_1 = \frac{\rho^2}{4f(x)}, \quad b_2 = \frac{\rho^2}{4g(y)}$$

$$c_1 = \frac{\sqrt{f(x)}(1 - y)}{2\rho} \frac{1}{\alpha}, \quad c_2 = \frac{\sqrt{f(x)}(1 + y)}{2\rho} \frac{1}{\beta}, \quad d_1 = \frac{\sqrt{g(y)}(\alpha - x)}{2\rho} \frac{1}{\alpha}, \quad d_2 = \frac{\sqrt{g(y)}(\beta - x)}{2\rho} \frac{1}{\beta}$$

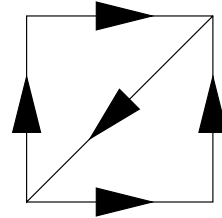
In general



p



r-p



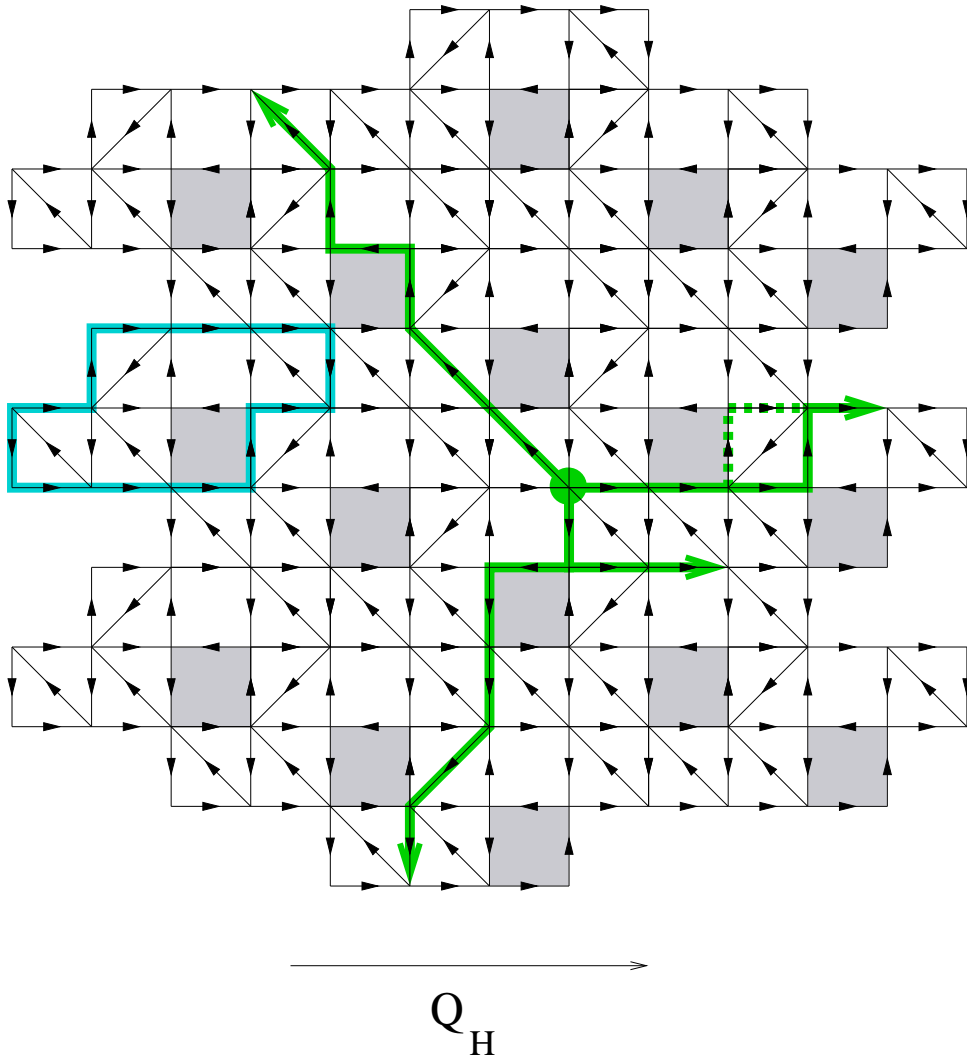
q-r

gives $L_{p,q|r}$.

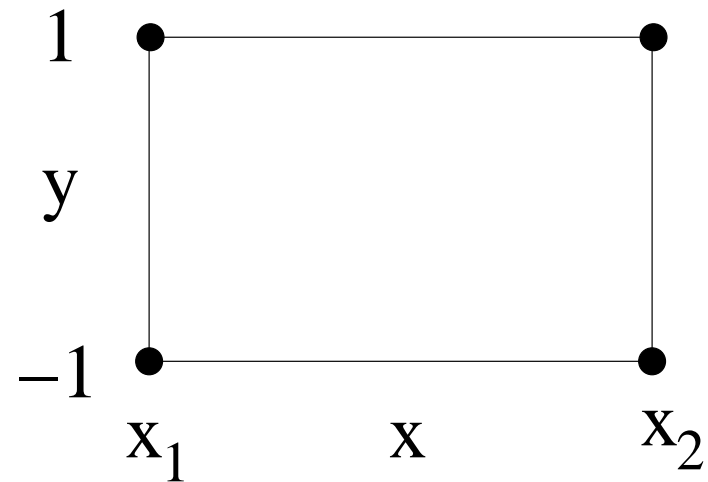
Using a-max one can compute the R-charges and the central charge. Everything matches. The parameters can be mapped from one description to the other.

$$a = \frac{\pi^3}{4V} = \frac{1}{4} \frac{8pqr s}{(p+q)^3} \frac{1}{W}$$

Long chiral primaries:



can be matched to geodesics
at the “corners”:



Conclusions

$Y_{p,q}$

- Massless BPS geod. \leftrightarrow Chiral primaries.
- Non-BPS geodesics \leftrightarrow Current insertions.
- Extended strings \leftrightarrow Long operators.
- Effective action emerges in f.t. Similar but not equal to bulk eff. action.

$L_{p,q|r}$

- Found dual gauge theories.
- Matched BPS geodesics with long chiral primaries.
- Strings? Three dimensional quiver?