Planar diagrams
in light-cone gauge

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Summary

- **Introduction**

  Motivation: large-N, D-branes, AdS/CFT, results

  D-brane interactions: lowest order, light-cone gauge

- **D-brane interactions in planar approximation**

  Dual closed string Hamiltonian: $H = H_0 - \lambda P$

  P: hole insertion

- **Supergravity result:** $H = H_0 - \lambda P'$

- **Calculation of** P **in the bosonic string**

(Neumann coeff. , scattering from D-branes)

- **Comparison of** P **and** P’

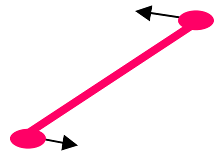
- **Notes on superstring and field theory cases**

- **Conclusions**

Large N limit (‘t Hooft)

String picture → Fund. strings

(Susy, 10d, Q.G.)



mesons

π, ρ, ...

Quark model → QCD [SU(3)]

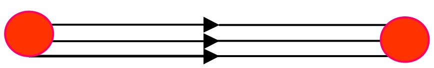
$q\ \bar{q}$

↓

Large N-limit [SU(N)]

↓

Effective strings

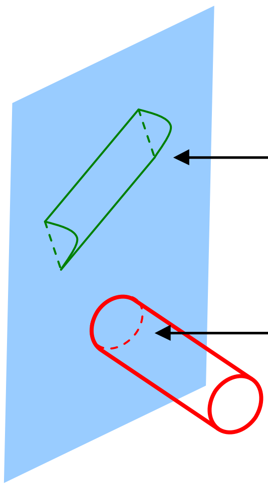
q  $\bar{q}$

Strong coupling

Lowest order: sum of planar diagrams (infinite number)
Suggested using light-cone gauge / frame.

D-branes (Polchinski)

Region of space where strings world-sheets can end.



Open strings. Low energy: SU(N) gauge th.

Emission of graviton → D-branes have mass

If N, # of D-branes is large → mass large → deforms space

e.g. D3- brane:

$$ds^2 = \frac{1}{\sqrt{f}}\left(dX^+dX^- + dX_{[2]}^2\right) + \sqrt{f}\,dY_{[6]}^2, \qquad f = 1 + 4\pi\alpha'^2\frac{g_s N}{Y^4}$$

Suggests an interesting rep. of the large-N limit

AdS/CFT (Maldacena)

We can extract the gauge theory, namely the low energy limit and obtain a duality (no direct derivation)

Open strings

Sugra background

$\boxed{\mathcal{N} = 4 \text{ SYM}}$ $\longleftrightarrow$ $\boxed{\text{II B on } AdS_5 x S^5}$

$$g_s = g_{YM}^2 ;$$

$$R / l_s = (g_{YM}^2 N)^{1/4}$$

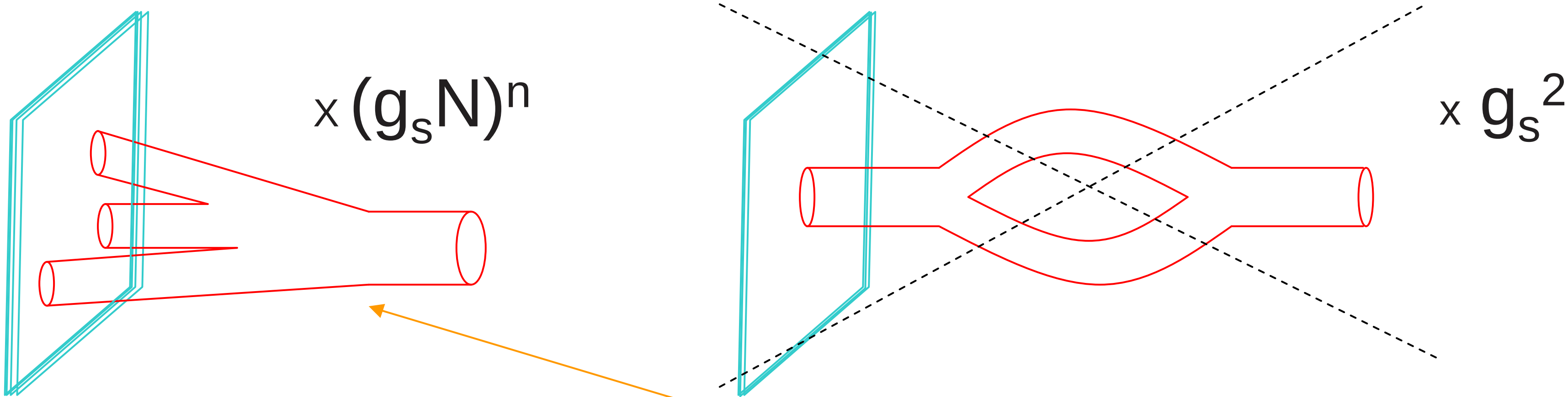
S^5: $X_1^2 + X_2^2 + \ldots X_6^2 = R^2$

AdS_5: $Y_1^2 + Y_2^2 + \ldots - Y_5^2 - Y_6^2 = -R^2$

$N \to \infty, \lambda = g_{YM}^2 N$ fixed ➡

λ large → string th.
λ small → field th.

Planar approximation ($g_s \to 0$, $N \to \infty$, $g_s N$ fixed)



We expect that summing these diagrams gives the propagation of closed strings in the supergravity background. In the bosonic case we get: $H = H_0 - \lambda P$

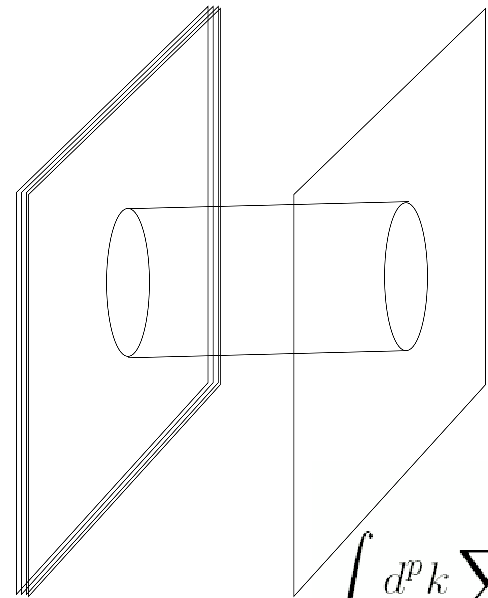
$$\hat{P} \simeq -\int_{-\pi}^{\pi} d\sigma \frac{16\pi^3 c_3}{Y^4} \left((4\pi \Pi_X)^2 + Y'Y' - 4 + 4\frac{YY''}{Y^2} \right), \qquad (q^2 \to 0 \text{ pole part})$$

(open strings)

$$\hat{P} = -\alpha' \int d\sigma \frac{1}{Y^4} \left[(2\pi\alpha' \Pi_X)^2 + Y'^2 \right]$$

(supergravity)

D-brane interactions



N

Open string: zero point energy

$$\omega_k = \sqrt{\vec{k}^2 + m^2}, \quad m^2 = \sum_{n \geq 1, i} N_n^i n - a + L^2$$

$$\int d^p k \sum_{m^2} \omega_k = \int d^p k \sum_{m^2} \int_0^\infty \frac{d\ell}{\ell^{\frac{3}{2}}} e^{-\ell(\vec{k}^2 + m^2)} = \begin{cases} \int d^{p+1}k \sum_{m^2} \int_0^\infty \frac{d\ell}{\ell} e^{-\ell(k^2+m^2)} \\ \int d^{p-1}k_\perp \sum_{m^2} \int_0^\infty \frac{d\ell}{\ell^2} e^{-\ell(k_\perp^2+m^2)} \end{cases}$$

[schematic: divergences have to be regulated]

$$\int d^{p-1}k_\perp \sum_{m^2} \int_0^\infty dp^+ e^{-\frac{1}{p^+}(k_\perp^2+m^2)} = \mathrm{Tr}\, e^{-\tau H_{l.c.}}, \quad H_{l.c.} = P^-, \quad \tau = 1, \quad (p^+ = \frac{1}{\ell})$$

$$\mathrm{Tr}\, e^{-\tau H_{l.c.}} = \int \mathcal{D}X_\perp e^{-\int d\sigma d\tau \left[(\partial_\tau X_\perp)^2 + (\partial_\sigma X_\perp)^2\right]}$$

σ: 0 ⟶ p^+

$$\int \mathcal{D}X e^{-i\int d\sigma d\tau (X^- \Box X^+ + X_\perp \Box X_\perp)}$$

$$\boxed{\begin{aligned} \Box X^+ = 0 \Rightarrow X^+ &= x^+(\tau+\sigma) + x^+(\tau-\sigma) \\ &= \frac{1}{2}(\tilde{\tau}+\tilde{\sigma}) + \frac{1}{2}(\tilde{\tau}-\tilde{\sigma}) = \tilde{\tau} \end{aligned}}$$

$$P^+ = \int d\sigma(\partial_\tau X^+) = \int_0^{p^+} d\sigma = p^+$$

Length = p^+

- **Open string**

$$H_{\rm open} = \frac{1}{4p^+}\left[p_\perp^2 + \sum_{n\geq 1,i} N_n^i n - \frac{d-2}{24\alpha'} + \frac{Y^2}{4\pi^2\alpha'^2}\right]$$

$$Z = \int d^{p-1}p_\perp \sum_{N_n^i} e^{-\tau\, H_{\rm open}}, \quad \tau = 2\pi\sqrt{\alpha'}$$

Agree

- **Closed string**

$$H_{\rm closed} = \frac{1}{\sqrt{\alpha'}}\left[\frac{1}{2}\alpha' p^2 + \sum_{n\geq 1,i}\left(N_n^{Ii} n + N_n^{IIi} n\right) - \frac{d-2}{12}\right]$$

$$Z = \langle B_f|\, e^{-H\tau}\, |B_i\rangle, \quad \tau = 4\alpha' p^+$$

Boundary states

$$X = x_0 + \sum_{n\neq 1} \left[\cos(n\sigma)x_n^I + \sin(n\sigma)x_n^{II}\right],$$

$$P = \frac{1}{2\pi}p_0 + \frac{1}{\pi}\sum_{n\geq 1} \left[\cos(n\sigma)p_n^I + \sin(n\sigma)p_n^{II}\right]$$

$$\boxed{\begin{aligned} X\left|B\right\rangle_D &= 0, \\ P\left|B\right\rangle_N &= 0 \end{aligned}}$$

Conditions for boundary state

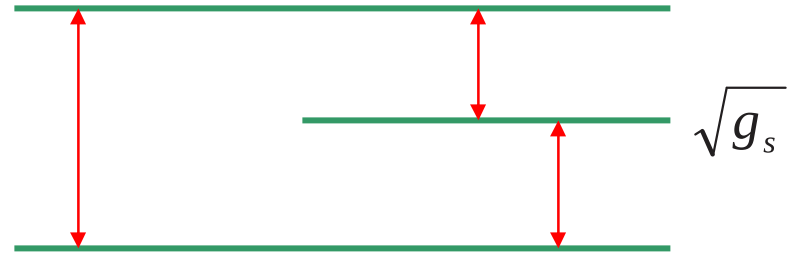
Solutions

$$\left|B\right\rangle_D = \delta(x_0 - Y)e^{-\frac{1}{2}\sum_{n\geq 1}\left(a^\dagger_{In}a^\dagger_{In} + a^\dagger_{IIn}a^\dagger_{IIn}\right)}\left|0\right\rangle$$

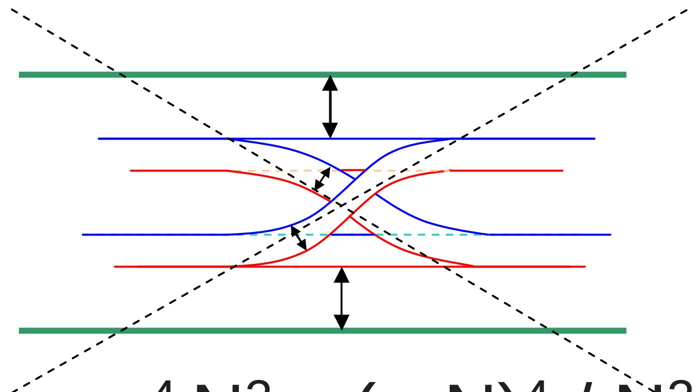
$$\left|B\right\rangle_N = \delta(p_0)e^{\frac{1}{2}\sum_{n\geq 1}\left(a^\dagger_{In}a^\dagger_{In} + a^\dagger_{IIn}a^\dagger_{IIn}\right)}\left|0\right\rangle$$

Higher orders (Include open string interactions)

Open strings can split and join.

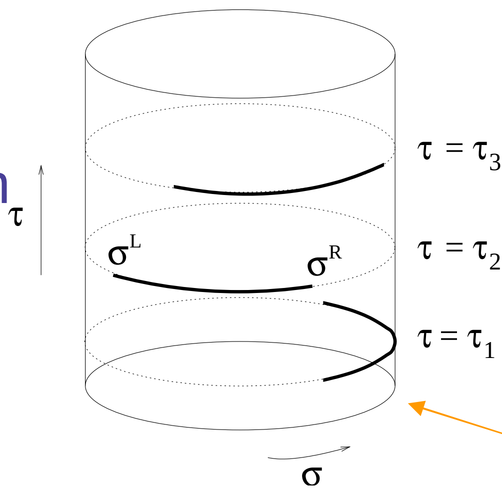


p^+, the length is conserved,



$(g_sN)^4$ $\qquad$ $g_s^4 N^2 = (g_sN)^4 / N^2$

$$\sum_n (g_sN)^n$$



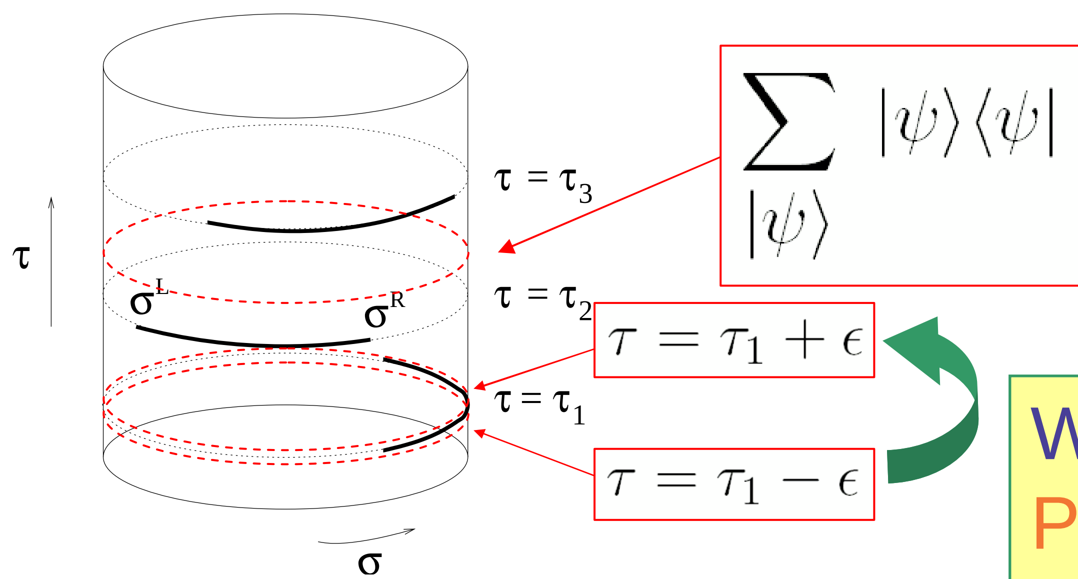
n slits

= propagation of a single closed string

This gives:

$$Z_0 = \sum_{n=0}^{\infty} (g_s N)^n \int \prod_{i=1}^{n} d\sigma_i^L d\sigma_i^R d\tau_i \quad \int \mathcal{D}X_\perp e^{-\int [\dot{X}_\perp^2 + X_\perp'^2]}$$

$$= \sum_{n=0}^{\infty} (g_s N)^n \int \prod_{i=1}^{n} d\sigma_i^L d\sigma_i^R d\tau_i \quad \mu(\sigma_i^L, \sigma_i^R, \tau_i)$$



We define the operator $P(\sigma_L, \sigma_R)$ that maps the string from τ_1-ε to τ_1+ε

To sum we use the closed string point of view

$$Z_0 = \sum_{n=0}^{\infty} (g_s N)^n \int \prod_{i=1}^{n} d\sigma_i^L d\sigma_i^R d\tau_i \;\; \langle B_f| \, e^{-H_0(\tau-\tau_n)} \ldots P(\sigma_2^L, \sigma_2^R) e^{-H_0(\tau_2-\tau_1)} P(\sigma_1^L, \sigma_1^R) e^{-H_0\tau_1} \, |B_i\rangle$$

$$= \sum_{n=0}^{\infty} (g_s N)^n \int \prod_{i=1}^{n} d\tau_i \;\; \langle B_f| \, e^{-H_0(\tau-\tau_n)} \ldots P \, e^{-H_0(\tau_2-\tau_1)} \, P \, e^{-H_0\tau_1} \, |B_i\rangle$$

Where $P = \int d\sigma^L d\sigma^R \, P(\sigma^L, \sigma^R)$. Define $P(\tau) = e^{H_0\tau} \, P \, e^{-H_0\tau}$

$$\boxed{\begin{aligned} Z_0 &= \sum_{n=0}^{\infty} (g_s N)^n \int \prod_{i=1}^{n} d\tau_i \;\; \langle B_f| \, e^{-H_0\tau} P(\tau_n) \ldots P(\tau_1) |B_i\rangle \\ &= {}_I\langle B_f| \, \hat{T} e^{g_s N \int_0^\tau P(\tau)\, d\tau} |B_i\rangle_I = \langle B_f| e^{-(H_0 - \lambda P)\tau} |B_i\rangle \end{aligned}}$$

$$\boxed{\lambda = g_s N}$$

We can define:

$$\boxed{H_{closed} = H_0 - \lambda P}$$

P essentially inserts a hole

Possible problems

We need $\mu(\sigma_i^L, \sigma_i^R, \tau_i) = \int \mathcal{D}X_\perp e^{-\int\left[\dot{X}_\perp^2 + X_\perp'^{\,2}\right]}$

This may need corrections if the path integral is not well defined. For example if two slits collide there can be divergences that need to be subtracted. This can modify P and include higher order corrections in λ .

In fact, at first sight this seems even necessary since the propagation of closed strings in the supergravity bakg. Depends on the metric that has non-trivial functions of λ. We analyze this problem now.

Even if there are extra corrections, P as defined contains important information as we will see.

Closed strings in the D3-brane background

$$ds^2 = \frac{1}{\sqrt{f}}\left(dX^+ dX^- + dX^2\right) + \sqrt{f}\, dY^2, \qquad f = 1 + 4\pi\alpha'^2 \frac{g_s N}{Y^4}$$

$$S = \frac{1}{4\pi\alpha'}\int d\sigma d\tau \sqrt{h} h^{ab} G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$$

Take: $h_{01} = 0, \; X^+ = \tau$

$$S = \frac{1}{4\pi\alpha'}\int d\sigma d\tau \left[E(\dot{X}^- + \dot{X}^2) - \frac{1}{f} E X'^2 + E f \dot{Y}^2 - \frac{1}{E} Y'^2\right]$$

$$\boxed{E = \sqrt{-\frac{h_{11}}{h_{00} f}}}$$

Since $\dot{E} = 0$ we set E=1 and get:

$$S = \frac{1}{4\pi\alpha'}\int d\sigma d\tau \left[-X'^2 + \frac{1}{f}\dot{X}^2 - f Y'^2 + \dot{Y}^2\right]$$

Not good

and

$$H = \frac{1}{4\pi\alpha'}\int d\sigma \left\{(2\pi\alpha'\Pi_Y)^2 + X'^2 + f\left[(2\pi\alpha'\Pi_X)^2 + Y'^2\right]\right\}$$

Which is indeed of the form $\boxed{H_{closed} = H_0 - \lambda P}$

with

$$H_0 = \frac{1}{4\pi\alpha'} \int d\sigma \left\{ (2\pi\alpha'\Pi_Y)^2 + X'^2 + \left[(2\pi\alpha'\Pi_X)^2 + Y'^2 \right] \right\}$$

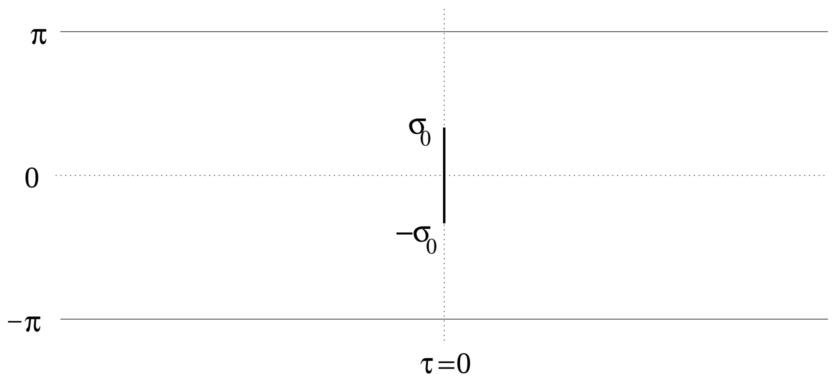
$$\boxed{\hat{P} = -\alpha' \int d\sigma \frac{1}{Y^4} \left[(2\pi\alpha'\Pi_X)^2 + Y'^2 \right]}$$

The near horizon (field th.) limit is:

$$H = \frac{1}{4\pi\alpha'} \int d\sigma \left\{ (2\pi\alpha'\Pi_Y)^2 + X'^2 + \frac{4\pi\alpha'^2 g_s N}{Y^4} \left[(2\pi\alpha'\Pi_X)^2 + Y'^2 \right] \right\}$$

which describes closed strings in **AdS_5xS^5**

What is P in the open string side ?



Identified (cylinder)

Scattering of a closed string by a D-brane

Vertex representation of P: $|V\rangle = \sum_{|1\rangle|2\rangle} \langle 2|\hat{P}|1\rangle \; |1\rangle \otimes |2\rangle,$

Dirichlet:

Neumann

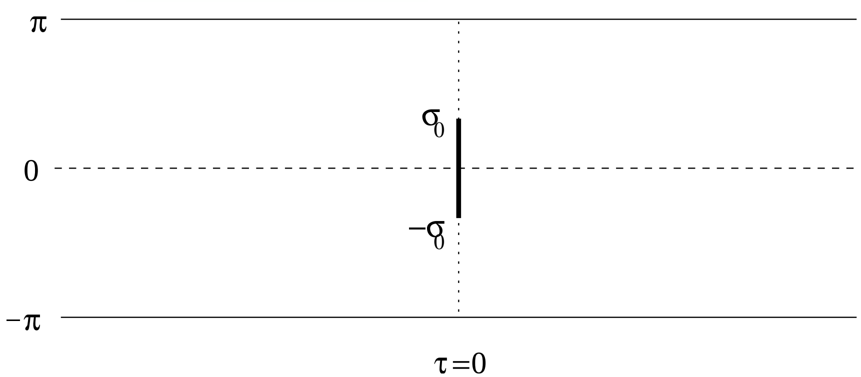
$$(X_1(\sigma,\tau=0) - X_2(\sigma,\tau=0))\,|V\rangle = 0, \quad \sigma_0 \le |\sigma| \le \pi,$$
$$(P_1(\sigma,\tau=0) + P_2(\sigma,\tau=0))\,|V\rangle = 0, \quad \sigma_0 \le |\sigma| \le \pi,$$
$$P_1(\sigma,\tau=0)|V\rangle = 0, \quad |\sigma| \le \sigma_0,$$
$$P_2(\sigma,\tau=0)|V\rangle = 0, \quad |\sigma| \le \sigma_0,$$

Solution:

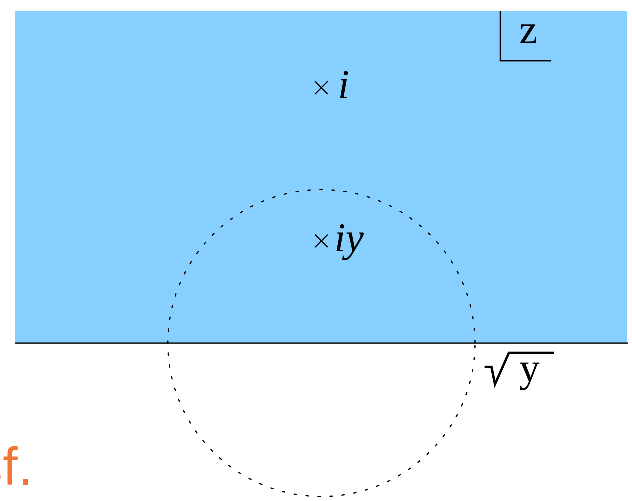
$$|V\rangle = e^{\sum_{rs,imn} N^{rs}_{i,nm} a^\dagger_{irn} a^\dagger_{ism}} \prod_{i/\varepsilon_i=+1} \delta(p^i_1 + p^i_2)|0\rangle,$$ (Neumann)

Compute coefficients.

$\rho = \tau + i\sigma$, $u = e^{\rho}$



Conf. transf.



$$G(z, z') = \ln|z - z'| + \varepsilon_i \ln|z - \bar{z}'|$$

r,s=1,2

$$\mathrm{N}^{rs}(u, u') = G(z_r(u), z_s(u')) - \delta^{rs}\ln|u - u'| = \sum_{m,n=\infty}^{\infty} \underline{\mathrm{N}^{rs}_{mn}(\varepsilon_i, \sigma_0)e^{in\sigma}e^{im\sigma'}}$$

Result:

$$\boxed{\mathrm{N}_{mn}^{rs}(\varepsilon_i, \sigma_0) = -\frac{i}{8}\frac{(1+\varepsilon_i)}{m+n}\left(a_m^r \delta_{n0} + a_n^s \delta_{m0}\right) + \frac{1}{(m+n)\sin\sigma_0}\mathrm{Im}\left(f_m^r f_n^s\right),}$$

with

$$a_m^{1,2} = 2i \pm \frac{i}{\sin\frac{\sigma_0}{2}} \sum_{l=1}^{m} \frac{1}{l}\left(\mathrm{P}_{l-2}(\cos\sigma_0) - \cos\sigma_0 \mathrm{P}_{l-1}(\cos\sigma_0)\right), \quad m > 0,$$

$$f_{m>0}^1 = -\bar{f}_m, \quad f_{m<0}^1 = -\varepsilon_i f_{-m}, \quad f_{m\neq 0}^2 = -\varepsilon_i f_m^1,$$

$$f_{m>0} = -\frac{i}{m} e^{i\frac{\sigma_0}{2}} \sum_{l=1}^{m} \frac{(-i)^l m!}{(m+l)!} l \mathrm{P}_m^l(\cos\sigma_0).$$

$$f_0^1 = \frac{1}{2}\left[(1+\varepsilon_i)(1-\sin\frac{\sigma_0}{2}) - i(1-\varepsilon_i)\cos\frac{\sigma_0}{2}\right],$$

$$f_0^2 = \frac{1}{2}\left[(1+\varepsilon_i)(1+\sin\frac{\sigma_0}{2}) - i(1-\varepsilon_i)\cos\frac{\sigma_0}{2}\right].$$

All together we get:

$$\boxed{\hat{P} = \frac{1}{\pi^3}\int_0^{2\pi} d\sigma \int_0^{\pi} d\sigma_0 \frac{1}{\sin^3\sigma_0} e^{\sum_{rs,imn} N^{rs}_{i,nm} a^\dagger_{irn} a^\dagger_{ism} e^{-i(m+n)\sigma}} \prod_{i/\varepsilon_i=+1} \delta(p_1^i + p_2^i)|0\rangle}$$

Small holes $(\sigma_0 \to 0)$

$$\sum_{rs,imn}{}' N^{rs}_{imn} a^\dagger_{irm} a^\dagger_{isn} \to iq\bar{y} + \frac{i}{4}\sigma_0^2 q.y'' - \frac{\sigma_0^2}{8} y'y' + \sigma_0^2 \bar{p}k - \frac{\sigma_0^2}{2}\bar{p}\bar{p} + \ldots,$$

Gives:

Tachyon pole

$$\hat{P}_0 \simeq 2c_3\left(\frac{\pi}{2}\right)^{q^2-2} \frac{1}{q^2-2} e^{iq\bar{y}} - \frac{8c_3}{q^2}\left(\frac{\pi}{2}\right)^{q^2}\left(\frac{1}{2}(k-\bar{p})^2 + \frac{1}{8}y'y' + \frac{1}{24}q^2 - \frac{1}{2} - \frac{i}{4}qy''\right) e^{iq\bar{y}}$$

$$\boxed{\hat{P} \simeq -\int_{-\pi}^{\pi} d\sigma \frac{16\pi^3 c_3}{Y^4}\left(\underline{(4\pi\Pi_X)^2 + Y'Y'} - 4 + 4\frac{YY''}{Y^2}\right), \qquad (q^2 \to 0 \text{ pole part})}$$

We reproduced the operator P in a certain (small hole) limit. There are extra terms due to the fact that we consider the bosonic string. (tachyon). Should be absent in the superstring.

There are also extra terms which do not correspond to the $q^2 \rightarrow 0$ pole. However we should take into account that the hamiltonian form the background is classical and we should have expected further corrections.

In pple. the Hamiltonian we proposed should reproduce order by order the planar diagrams (by definition).

Notes on superstrings

We need to add fermionic degrees of freedom.
θ^A, λ_A, right moving and $\tilde{\theta}^A$, $\tilde{\lambda}_A$ left moving.
There is an SO(6)=SU(4) symmetry. The index A is in the fundamental or anti-fundamental (upper or lower).

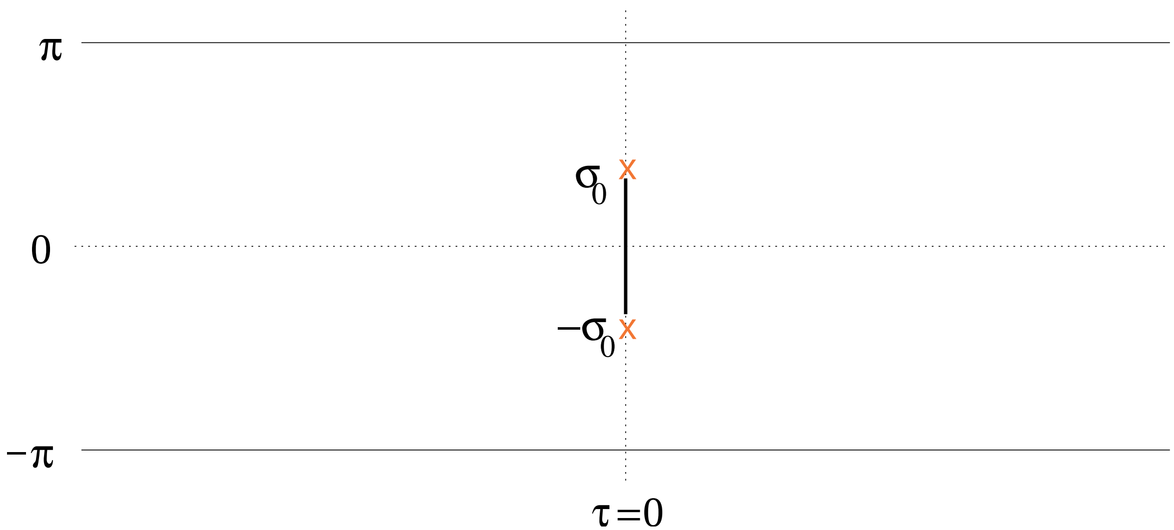
Conditions: (preserving half the supersymmetry)

$$\left(\theta_1^A - \theta_2^A - \tilde{\theta}_1^A + \tilde{\theta}_2^A\right)|V\rangle = 0, \quad -\pi \leq \sigma \leq \pi,$$
$$\left(\lambda_{1A} + \lambda_{2A} + \tilde{\lambda}_{1A} + \tilde{\lambda}_{2A}\right)|V\rangle = 0, \quad -\pi \leq \sigma \leq \pi,$$
$$\left(\theta_1^A - \theta_2^A + \tilde{\theta}_1^A - \tilde{\theta}_2^A\right)|V\rangle = 0, \quad \sigma_0 \leq |\sigma| \leq \pi,$$
$$\left(\lambda_{1A} + \lambda_{2A} - \tilde{\lambda}_{1A} - \tilde{\lambda}_{2A}\right)|V\rangle = 0, \quad \sigma_0 \leq |\sigma| \leq \pi,$$
$$\left(\theta_1^A + \theta_2^A - \tilde{\theta}_1^A - \tilde{\theta}_2^A\right)|V\rangle = 0, \quad -\sigma_0 \leq \sigma \leq \sigma_0,$$
$$\left(\lambda_{1A} - \lambda_{2A} + \tilde{\lambda}_{1A} - \tilde{\lambda}_{2A}\right)|V\rangle = 0, \quad -\sigma_0 \leq \sigma \leq \sigma_0.$$

Can be solved again in terms of a vertex state:

$$|V\rangle = e^{\text{bilinear in creation ops.}} |0\rangle$$

However extra operator insertions are required:

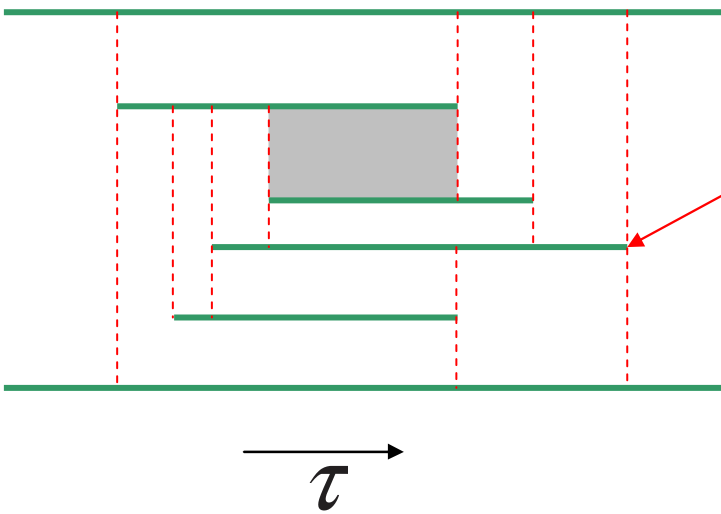


These insertions complicate the calculations.
Further work needed. In this case we should get, at low energy a gauge theory

Notes on field theory

't Hooft:

propagator in light cone frame: $$\frac{1}{|p^+|} e^{-\frac{p_\perp^2+m^2}{2p^+} t_0}$$



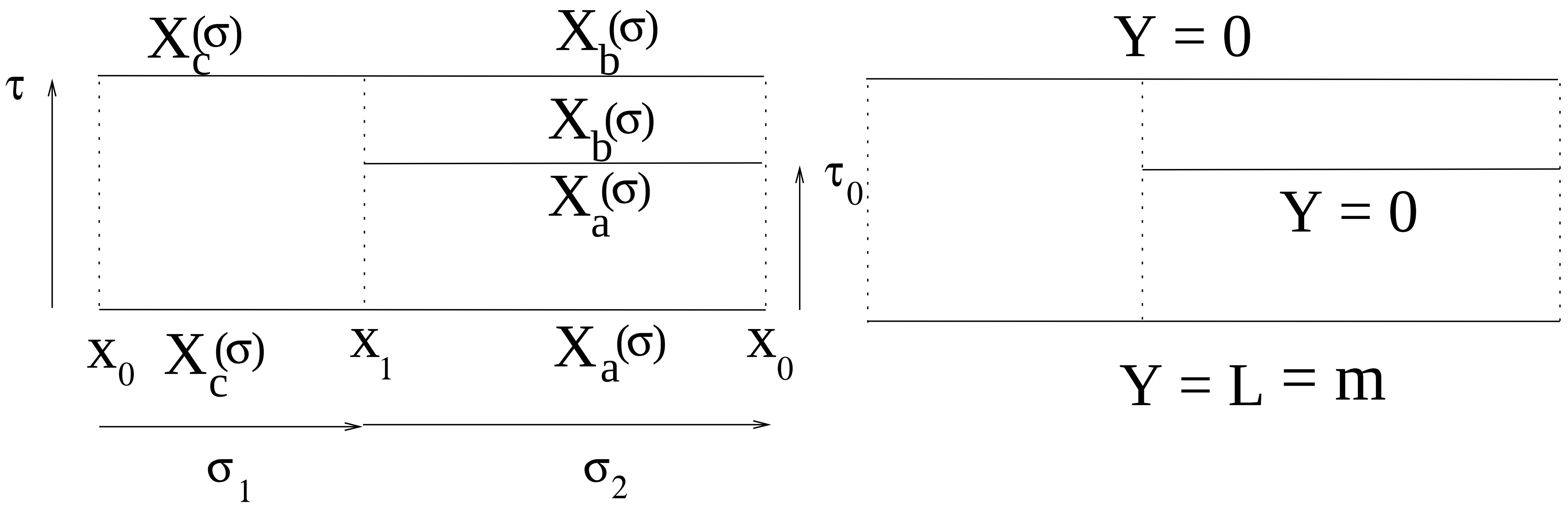
Local in τ and non local in σ. We want to flip $\sigma \leftrightarrow \tau$.
But we should get a local evolution in the new τ.
Not clear if it is possible.

Simple example (Φ^3 theory):

$$H_0 = \frac{1}{2}\int d\sigma \left(\Pi_Y^2 + X'^2\right)$$

$$\langle X_f(\sigma)|e^{-\tau H_{01}}|X_i(\sigma)\rangle = \prod_\sigma \delta\left(X_f(\sigma) - X_i(\sigma)\right) e^{-\frac{1}{2}\tau \int d\sigma X_i'^2}$$

$$\langle Y_f(\sigma)|e^{-\tau H_{01}}|Y_i(\sigma)\rangle = \mathcal{N}_2 e^{-\frac{1}{2\tau}\int d\sigma \left(Y_f(\sigma) - Y_i(\sigma)\right)^2}$$



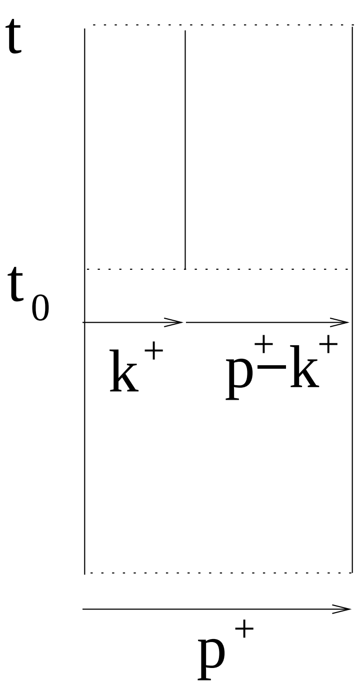
The diagrams equals Z_1 Z_2:

$$Z_1 = \mathcal{N} \int d^2x_1 d^2x_0 e^{-\frac{\tau}{2\sigma_1}(x_1-x_0)^2 - \frac{\tau-\tau_0}{2\sigma_2}(x_1-x_0)^2 - \frac{\tau_0}{2\sigma_2}(x_1-x_0)^2}$$

$$= \mathcal{N} \int d^2X_0 \int d^2x e^{-\frac{\tau}{2\sigma_1}x^2 - \frac{\tau-\tau_0}{2\sigma_2}x^2 - \frac{\tau_0}{2\sigma_2}x^2}$$

$$Z_2 = \mathcal{N}_2 e^{-\frac{\sigma_2 m^2}{2\tau}} e^{-\frac{\sigma_2 m^2}{2\tau_0}}$$

In field theory:



$$Z = \int d^2p_\perp \int d^2k_\perp \frac{1}{|p^+|} e^{-\frac{p_\perp^2+m^2}{2p^+}t_0} \frac{1}{|k^+|} e^{-\frac{k_\perp^2+m^2}{2k^+}(t-t_0)} \frac{1}{|p^+-k^+|} e^{-\frac{(p_\perp-k_\perp)^2+m^2}{2(p^+-k^+)}(t-t_0)}$$

$$k^+ = \tau_0, \quad p^+ = \tau, \quad t_0 = \sigma_1, \quad t = \sigma_1 + \sigma_2,$$

$$Z = \frac{1}{2\pi} \int d^2x \frac{1}{\sigma_1 \sigma_2^2} e^{-\frac{\tau x^2}{2\sigma_1} - \frac{\tau x^2}{2\sigma_2} - \frac{(\tau-\tau_0)x^2}{2\sigma_2}} e^{-\frac{\sigma_1 m^2}{2\tau} - \frac{\sigma_2 m^2}{2\tau_0}}$$

Conclusions

- The sum of planar diagrams is determined by an operator P acting on closed strings. It inserts a hole in the world-sheet.

- The "($\sigma \leftrightarrow \tau$) dual" closed string Hamiltonian is:

$$\boxed{H_{closed} = H_0 - \lambda P}$$

- For bosonic D-branes we obtained P explicitely. From it, after taking a limit we obtained a Hamiltonian similar to the one for closed strings in a modified background.

- There can be corrections to H but, nevertheless, the operator P contains important information (e.g. bkgnd.)

- In field theory we can use a "($\sigma \leftrightarrow \tau$) duality" if we get a representation local in the new τ. In that case we can define a dual $H = H_0 - \lambda P$ that contains the information on the planar diagrams. Less ambitious than obtaining a dual string theory.