Spiky strings, light-like Wilson loops and a pp-wave anomaly

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Summary

- **Introduction**

  Twist two operators in gauge theories (**QCD**)
  String / gauge theory duality (**AdS/CFT**)
  AdS/CFT and twist two operators
    - Rotating strings
    - Cusp anomaly

  Higher twist operators: spiky strings
  Other applications / results.

- **Large spin limit of the spiky string**

  Limiting shape and near boundary string

- **Limit in the boundary**

  Gauge theory in a pp-wave

- **PP-wave anomaly**

  Gauge theory in a pp-wave → pp-wave anomaly

  → cusp anomaly / anomalous dim. of twist two ops.

  - **Strong coupling** String calculation:

    Wilson loop in pp-wave w/ AdS/CFT

  - **Small coupling** Field theory calculation:

    Wilson loop in pp-wave (Class. source).

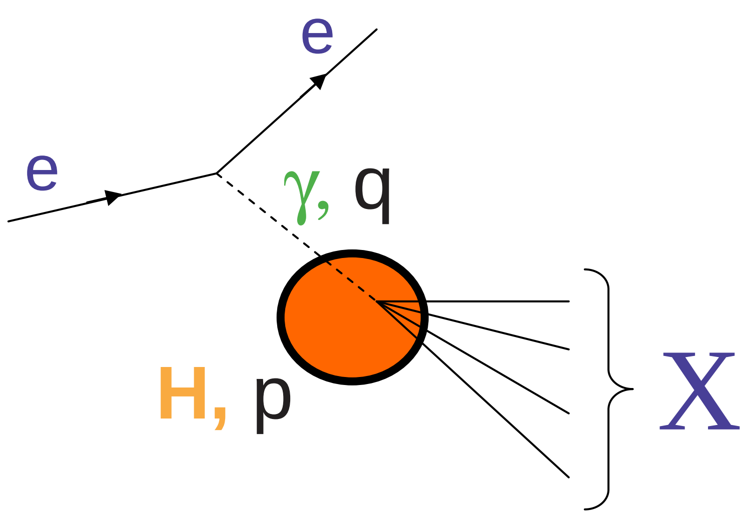
- **Wilson loops in PP-wave**

- **Conformal mapping:** Wilson loops in flat space

- **Other Wilson loops:** New ansatz for Wilson loops

- **Conclusions**

Twist two operators in gauge theories (QCD)



$q^2 \to \infty$,
ω =-2 p.q / q^2 fixed

or $q_+ \to \infty$, q_- fixed
Near l.c. expansion

OPE: ($z^2 \to 0$, light-like, twist), ($z^2 \to 0$, euclidean, conf. dim.)

$$\hat{T} J(z) J(0) = \sum_{\mathcal{O}} \mathcal{O}^{\Delta}_{\mu_1 \mu_2 \ldots \mu_S} z^{\mu_1} z^{\mu_2} \ldots z^{\mu_S} |z|^{\Delta - 6 - S}$$

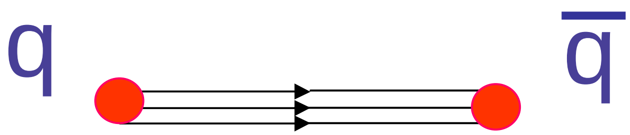
This (after including indices correctly) is plugged into:

$$\int d^4 z e^{-iqz} \langle N | \hat{T} J^{\nu}(z) J^{\mu}(0) | N \rangle = \left(\frac{q^{\mu} q^{\nu}}{q^2} - \eta^{\mu\nu} \right) T_1(q^2, \omega) + \frac{1}{p^2} \left(p^{\mu} - \frac{pq}{q^2} q^{\mu} \right) \left(p^{\nu} - \frac{pq}{q^2} q^{\nu} \right) T_2(\omega, q^2)$$

String/gauge theory duality: Large N limit (‘t Hooft)

QCD [SU(3)] → Large N-limit [SU(N)]

↓



Effective strings

Strong coupling

More precisely: $N \to \infty, \lambda = g_{YM}^2 N \ \ fixed$ (‘t Hooft coupl.)

Lowest order: sum of planar diagrams (infinite number)

AdS/CFT correspondence (Maldacena)

Gives a precise example of the relation between strings and gauge theory.

Gauge theory	String theory
$\mathcal{N} = 4$ SYM SU(N) on R^4	IIB on $AdS_5 x S^5$
A_μ , Φ^i, Ψ^a	radius R
Operators w/ conf. dim. Δ	String states w/ $E = \frac{\Delta}{R}$

$$g_s = g_{YM}^2; \qquad R / l_s = (g_{YM}^2 N)^{1/4}$$

$N \to \infty, \lambda = g_{YM}^2 N$ fixed ➡

λ large → string th.
λ small → field th.

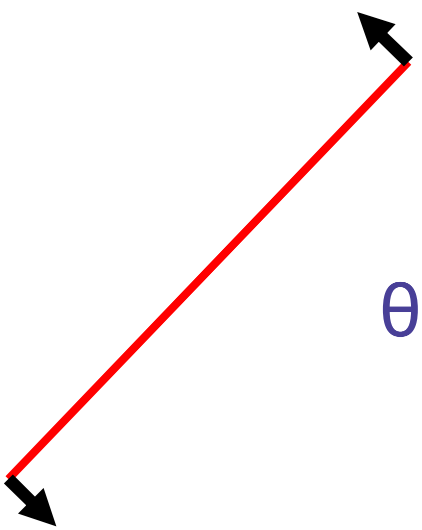
Twist two operators from rotation in AdS_5

(Gubser, Klebanov, Polyakov)

$$Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 - Y_5^2 - Y_6^2 = -R^2$$

$\sinh^2\rho;\ \Omega_{[3]}$ $\qquad$ $\cosh^2\rho;\ t$

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{[3]}^2$$



$\theta = \omega t$

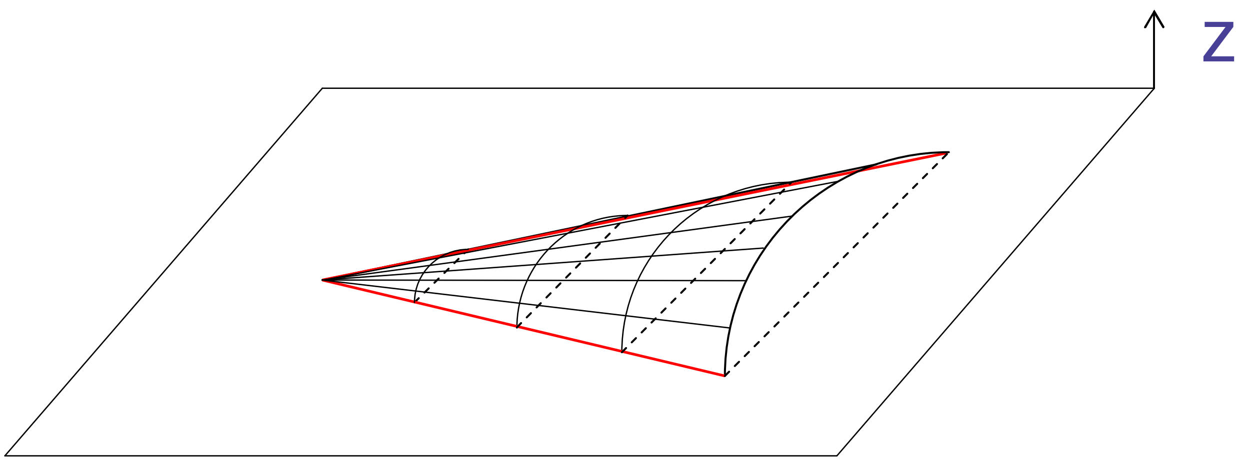
$$E \cong S + \frac{\sqrt{\lambda}}{\pi}\ln S, \quad (S \to \infty)$$

$$O = Tr\left(\Phi \nabla_+^S \Phi\right), \quad x_+ = z + t$$

Twist two ops. from cusp anomaly (MK, Makeenko)

The anomalous dimensions of twist two operators can also be computed by using the **cusp anomaly** of light-like Wilson loops (**Korchemsky and Marchesini**).

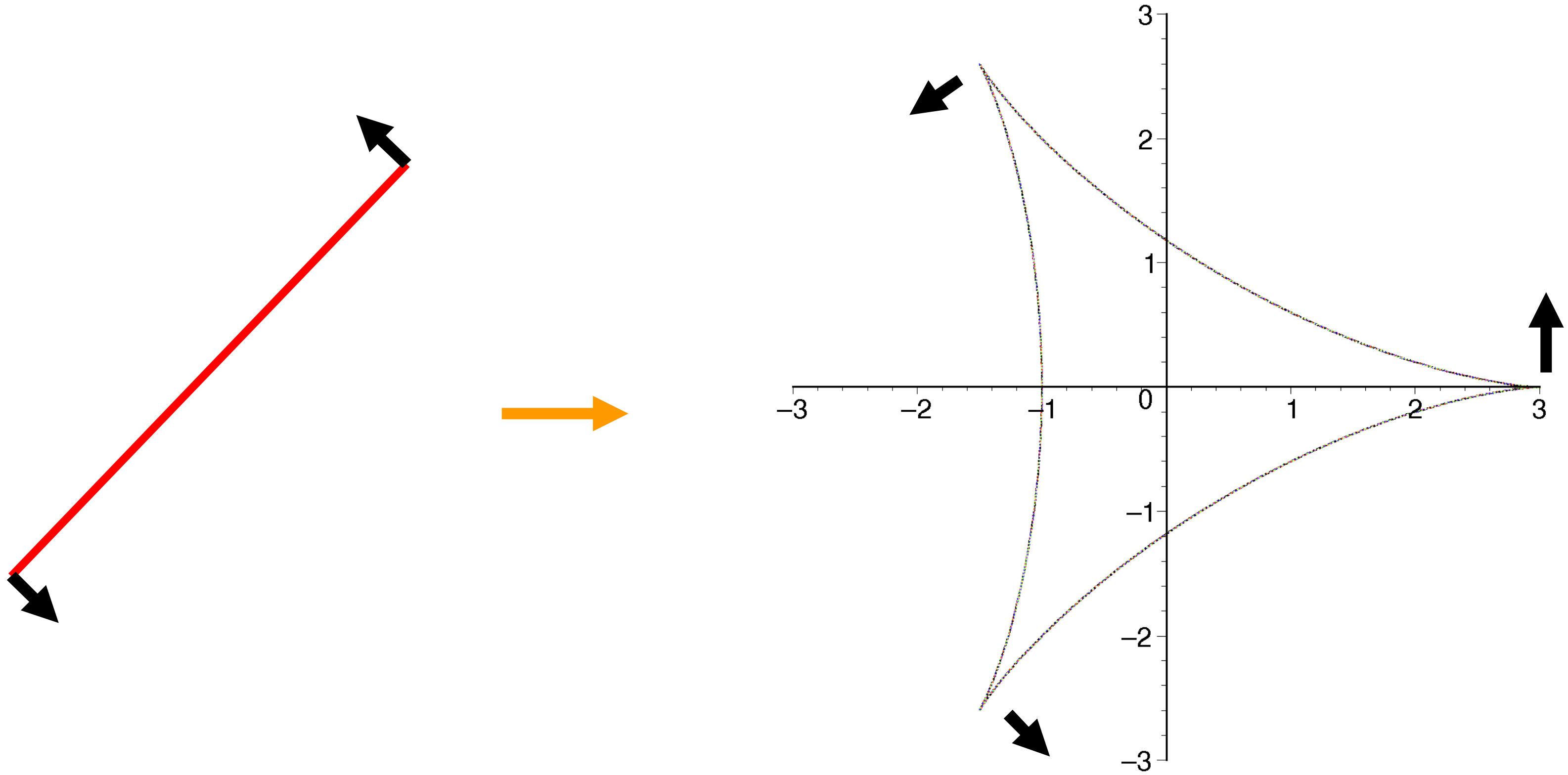
In **AdS/CFT** Wilson loops can be computed using surfaces of minimal area in AdS_5 (**Maldacena, Rey, Yee**)



The result **agrees** with the rotating string calculation.

Generalization to higher twist operators (MK)

$$O_{[2]} = Tr\left(\Phi\, \nabla_+^{S} \Phi\right) \longrightarrow O_{[n]} = Tr\left(\nabla_+^{S/n}\Phi\, \nabla_+^{S/n}\Phi\, \nabla_+^{S/n}\Phi \ldots \nabla_+^{S/n}\Phi\right)$$



In flat space such solutions are easily found in conf. gaug

$$x = A\cos[(n-1)\sigma_+] + A(n-1)\cos[\sigma_-]$$

$$y = A\sin[(n-1)\sigma_+] + A(n-1)\sin[\sigma_-]$$

Spiky strings in AdS:

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\theta^2$$

Ansatz: $t = \tau, \qquad \theta = \omega\tau + \sigma, \qquad \rho = \rho(\sigma)$

Action and momenta:

$$S = T \int d\theta d\sigma \sqrt{\rho'^2(\cosh^2\rho - \omega^2 \sinh^2\rho) + \sinh^2\rho \cosh^2\rho}$$

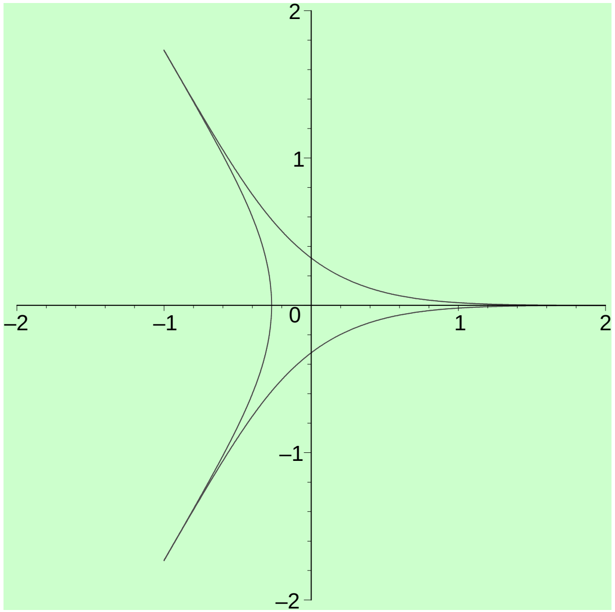
$$T = \frac{\sqrt{\lambda}}{2\pi}$$

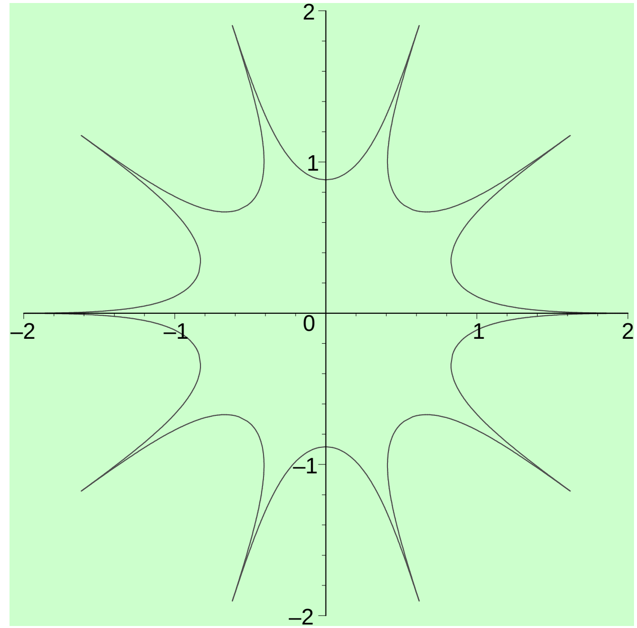
$$P_t = E = T \int d\sigma \frac{\cosh^2\rho\,(\rho'^2 + \sinh^2\rho)}{\sqrt{\rho'^2(\cosh^2\rho - \omega^2\sinh^2\rho) + \sinh^2\rho\cosh^2\rho}}$$

$$P_\theta = -S = -\omega T \int d\sigma \frac{\rho'^2 \sinh^2\rho}{\sqrt{\rho'^2(\cosh^2\rho - \omega^2\sinh^2\rho) + \sinh^2\rho\cosh^2\rho}}$$

Eq. of motion:

$$\rho'^2 = \frac{\sinh^2\rho\cosh^2\rho}{\sinh^2\rho_0\cosh^2\rho_0} \frac{\sinh^2\rho\cosh^2\rho - \sinh^2\rho_0\cosh^2\rho_0}{\cosh^2\rho - \omega^2\sinh^2\rho}$$





$$E \cong S + \left(\frac{n}{2}\right)\frac{\sqrt{\lambda}}{\pi}\ln S\,,\quad (S \to \infty)$$

$$O = Tr\left(\nabla_{+}^{S/n}\Phi\ \nabla_{+}^{S/n}\Phi\ \nabla_{+}^{S/n}\Phi \ldots \nabla_{+}^{S/n}\Phi\right)$$

$$S = \frac{\sqrt{\lambda}}{2\pi}\int dt \sum_j (\cosh 2\rho_1 - 1)\dot{\theta}_j - \frac{\sqrt{\lambda}}{8\pi}\int dt \sum_j \left\{4\rho_1 + \ln\left(\sin^2\left(\frac{\theta_{j+1} - \theta_j}{2}\right)\right)\right\}$$

For all couplings we are lead to define $f(\lambda)$ through:

$E = S + (n/2)\, f(\lambda) \ln S$ (large S)

$$\begin{cases} f(\lambda) = \dfrac{\lambda}{2\pi^2} + O(\lambda^2) \\ f(\lambda) = \dfrac{\sqrt{\lambda}}{\pi} + \text{cst.} + O(\dfrac{1}{\sqrt{\lambda}}) \end{cases}$$

Other applications / results for $f(\lambda)$

- Gluon scattering amplitudes
  (Bern, Dixon, Smirnov …)

- Scattering amplitudes in AdS/CFT
  (Alday, Maldacena,…)

- Anomalous dimension $f(\lambda)$ at all loops
  (Beisert, Eden, Staudacher ,…)

Large spin limit of the spiky string

The limit of large spin (E-S ~ ln S) corresponds to $\omega \rightarrow 1$. In that limit the spikes touch the boundary and the solution simplifies.

$$\coth 2\rho = \frac{\cos\sigma}{\cos\sigma_0}$$

$$\Delta\theta = 2\sigma_0 = \frac{2\pi}{n}$$

$$\cot\sigma_0 = \sinh 2\rho_0$$

In embedding coordinates

$$Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 - Y_5^2 - Y_6^2 = -1 \,,$$
$$Y_1 Y_5 + Y_2 Y_6 - \frac{1}{2}\cos\sigma_0 \left(Y_1^2 + Y_2^2 + Y_5^2 + Y_6^2\right) = 0 \,,$$
$$Y_3 = Y_4 = 0 \,.$$

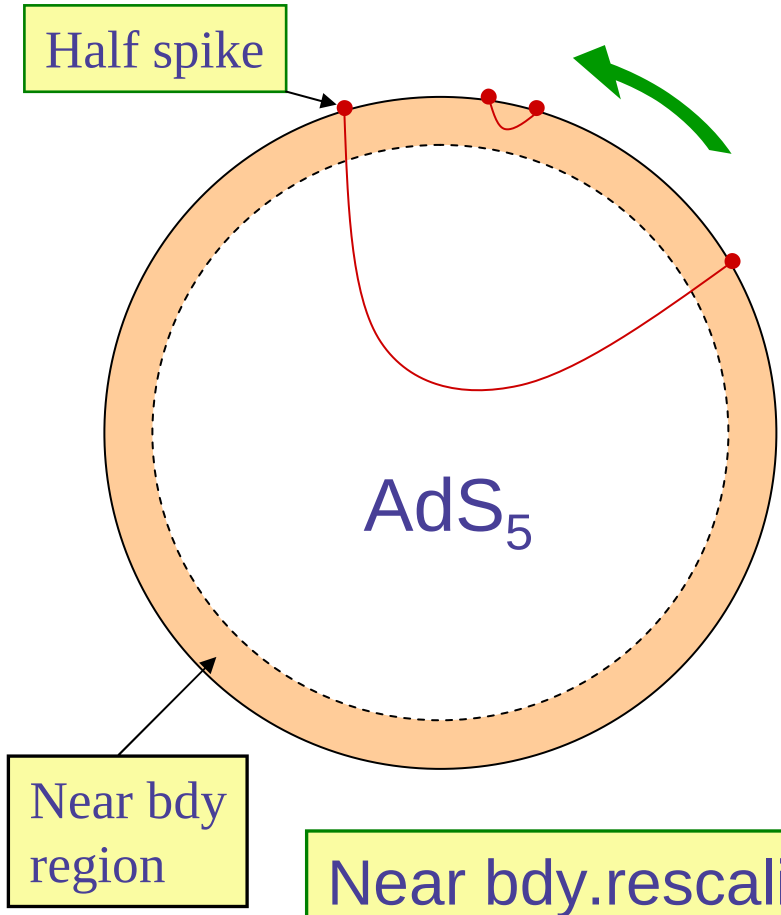
$\Leftrightarrow$

$$Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 - Z_5^2 - Z_6^2 = -1 \,,$$
$$Z_1 Z_5 - Z_2 Z_6 = 0 \,,$$
$$Z_3 = Z_4 = 0 \,.$$

The ***same*** $f(\lambda)$ should appear in all of them!

Near boundary limit

We can take $\rho_0 \to \infty$ and get a solution close to the bdy.



$$
\begin{aligned}
ds^2 &= -\cosh^2\rho \; dt^2 + d\rho^2 + \sinh\rho^2 \; d\Omega^2_{[3]} \\
&= -\cosh^2\rho \; dt^2 + d\rho^2 + \sinh\rho^2 \; \frac{(1-\frac{x_i^2}{4})^2 \; d\theta^2 + dx_i dx_i}{(1+\frac{x_i^2}{4})^2}
\end{aligned}
$$

Define light-like coords.

$$
\rho = -\ln(2z) \; , \qquad t = x_+ - x_- \; , \qquad \theta = x_+ + x_- \; ,
$$

Near bdy.rescaling.
Zooms in the shaded region

$$
\begin{aligned}
x_+ &\longrightarrow \mu^{-1} \; x_+ \\
x_- &\longrightarrow 8\mu \; \varepsilon^2 \; x_- \\
x_i &\longrightarrow 4\varepsilon \; x_i \\
z &\longrightarrow \varepsilon \; z
\end{aligned}
$$

In the limit $\epsilon \to 0$ we get the metric:

AdS!

$$ds^2 = \frac{1}{z^2}\left[2dx_+dx_- - \mu^2\left(z^2 + x_i^2\right)dx_+^2 + dx_i dx_i + dz^2\right]$$

So, the tiny string sees an AdS pp-wave in Poincare coordinates. When $z \to 0$, the boundary metric becomes a pp-wave in usual flat space:

$$ds^2 = 2dx_+dx_- - \mu^2 x_i^2 dx_+^2 + dx_i dx_i$$

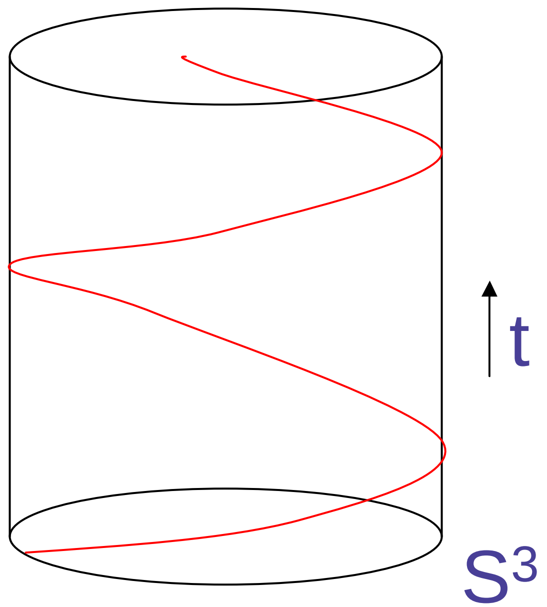
$\mathcal{N} = 4$ SYM SU(N) on R^4 pp-wave dual to

IIB on AdS_5 pp-wave x S^5

According to AdS/CFT

This duality should contain all the information about $f(\lambda)$

Limit in the boundary



In the boundary (RxS^3) the particle is moving along a light-like geodesics so it sees the **Penrose limit** of the metric.

Therefore it suffices to study the gauge theory in a pp-wave:

$$ds^2 = 2dx_+dx_- - \mu^2 x_i^2 dx_+^2 + dx_i dx_i$$

We just argued that to compute $f(\lambda)$ we need to study $\mathcal{N} = 4$ in a pp-wave or strings in an AdS pp-wave.

What should we compute?
Before it was $E\text{-}S = f(\lambda) \ln S$ for the folded string.

Following the change of coordinates we find that

$$P_+ = P_t + P_\theta = E - S, \qquad P_- = -P_t + P_\theta = -E - S$$

Since $E \sim S$ we can say that for a single charge moving along x_+ we should have

$$P_+ = \gamma(\lambda) \ln |P_-| \ , \quad \text{with} \quad \gamma(\lambda) = \frac{1}{4} f(\lambda)$$

<u>This leads us to define a</u> PP wave anomaly

According to the previous calculations we need to compute $P_{\pm}$ in the presence of a charge (WL):

$$P_{\pm} \equiv \int dx_{-} d^2 x_i \sqrt{-g} \, \langle \, T_{\pm}^{+} \, W \, \rangle_{\text{pp-wave}} \, ,$$

$$W = \frac{1}{N} \text{tr} \, \mathcal{P} e^{-i g_{\text{YM}} \int A_{+}^{a} t^{a} dx^{+}} \, .$$

$$P_{+} = \gamma(\lambda) \ln |P_{-}|$$

PP-wave anomaly

Equivalently: $$\gamma(\lambda) = - \lim_{\varepsilon \to 0} \varepsilon \frac{\partial}{\partial \varepsilon} P_{+}$$

$\mathcal{E}$ UV cut-off

PP wave anomaly Strong coupling

We put a particle in a pp-wave background and compute $P_\pm$ which amounts to computing a Wilson loop. We can use AdS/CFT since we know That the dual of the pp-wave is the AdS pp-wave.

The Wilson loop is $x_+=\tau$ and the string solution is simply:

$$x_+ = \tau, \qquad z = \sigma, \qquad x_- = x_i = 0$$

Giving:

$$P_+ = T\int_\epsilon^R dz \frac{\sqrt{\mu^2 z^2}}{z^2} \approx \frac{\mu T}{2} \ln\frac{R^2}{\epsilon^2}\ ,$$

$$P_- = T\int_\epsilon^R dz \frac{1}{z^2\sqrt{\mu^2 z^2}} \approx -\frac{T}{2\mu\epsilon^2}\ .$$

or

$$P_+ \approx \frac{T}{2}\ln|P_-| = \frac{\sqrt{\lambda}}{4\pi}\ln|P_-|$$

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi}$$

OK

PP wave anomaly Small coupling

Again we need to compute a Wilson loop in the pp-wave background. At lowest order it is a classical source in the linearized approximation.

We need to solve Maxwell eqns. in the pp-wave with a source moving according to $x_+ = \tau$.

$$\partial_\mu F^{\mu +} = g_{\rm YM}\ \delta(x_-)\ \delta^{(2)}(x_i)\ , \qquad \partial_\mu F^{\mu -} = \partial_\mu F^{\mu i} = 0$$

$$\begin{aligned} F^{+-} &= F_{-+} = -\frac{2g_{\rm YM}}{\pi^2}\frac{\mu x_-}{\mu^2 r^4 + 4x_-^2}\ , \\ F_{+i} &= \frac{g_{\rm YM}}{\pi^2}\frac{\mu^3 r^2 x_i}{\mu^2 r^4 + 4x_-^2}\ , \qquad F^{-i} = F_{+i} + \mu^2 r^2 F_{-i} = 0 \\ F^{+i} &= F_{-i} = -\frac{g_{\rm YM}}{\pi^2}\frac{\mu x_i}{\mu^2 r^4 + 4x_-^2}\ . \end{aligned}$$

$r^2 = x_1^2 + x_2^2$

We can now compute the energy momentum tensor

$$T^{\mu}{}_{\nu} = F^{\mu\alpha} F_{\alpha\nu} - \frac{1}{4}\delta^{\mu}{}_{\nu} F^{\alpha\beta} F_{\beta\alpha}$$

Obtaining

$$T^{+}{}_{+} = \frac{g_{\rm YM}^2}{2\pi^4} \frac{\mu^2}{\mu^2 r^4 + 4x_-^2} \,, \qquad T^{+}{}_{-} = -\frac{g_{\rm YM}^2}{\pi^4} \frac{\mu^2 r^2}{(\mu^2 r^4 + 4x_-^2)^2}$$

Using

$$P_+ = \frac{N}{2} \int_{-\infty}^{\infty} dx_- d^2x \; T^{+}{}_{+} \,, \qquad P_- = \frac{N}{2} \int_{-\infty}^{\infty} dx_- d^2x \; T^{+}{}_{-}$$

we get

$$P_+ = \frac{\lambda}{4\pi^4} 2\pi \int_{-\infty}^{\infty} dx_- \int_0^{\infty} dr \; r \; \frac{\mu^2}{\mu^2 r^4 + 4x_-^2 + \varepsilon^4} \approx \frac{\lambda}{8\pi^2} \; \mu \; \ln \frac{L^2}{\varepsilon^2} \,,$$

$$P_- = -\frac{\lambda}{2\pi^4} 2\pi \int_{-\infty}^{\infty} dx_- \int_0^{\infty} dr \; r \; \frac{\mu^2 \; r^2}{(\mu^2 r^4 + 4x_-^2 + \varepsilon^4)^2} \approx -\frac{\lambda}{8\pi^2 \varepsilon^2} \,,$$

Or

$$P_+ \approx \frac{\lambda}{8\pi^2} \; \ln |P_-|$$

Giving $$f(\lambda) = \frac{\lambda}{2\pi^2}$$ OK

The symmetry if the pp-wave under

$$\tilde{x}_+ = x_+ \; , \qquad \tilde{x}_- = \xi^2 x_- \; , \qquad \tilde{x}_i = \xi x_i$$

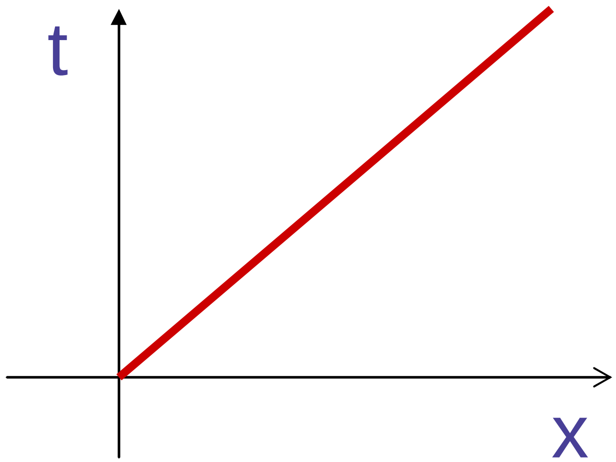
implies that $P_+ \sim \ln P_-$ and therefore that the anomalous dimension grows logarithmically with the spin.

Other Wilson loops in the pp-wave

$$ds^2 = 2\, dx_+\, dx_- - \mu^2 x_\perp^2\, dx_+^2 + dx_\perp^2 ,$$

$$x_\pm = \frac{x \pm t}{\sqrt{2}}$$

- Light-like line: $x_\perp=0,\ x_-=0,\ x_+=\tau$

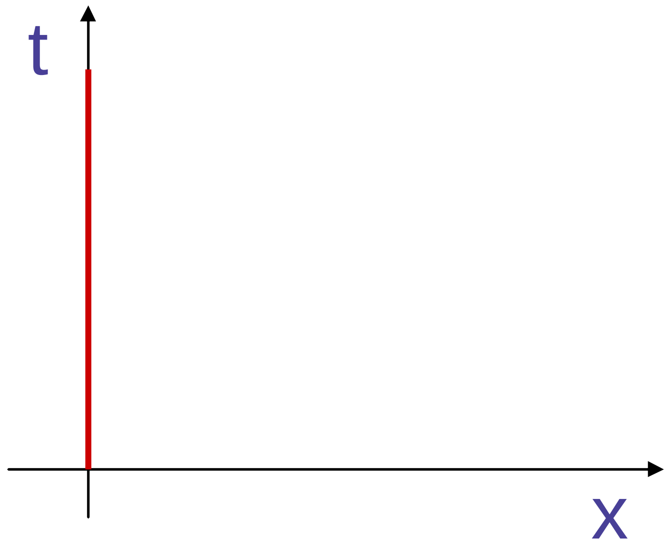


$$x_+ = \tau,$$

$$z = \sigma,$$

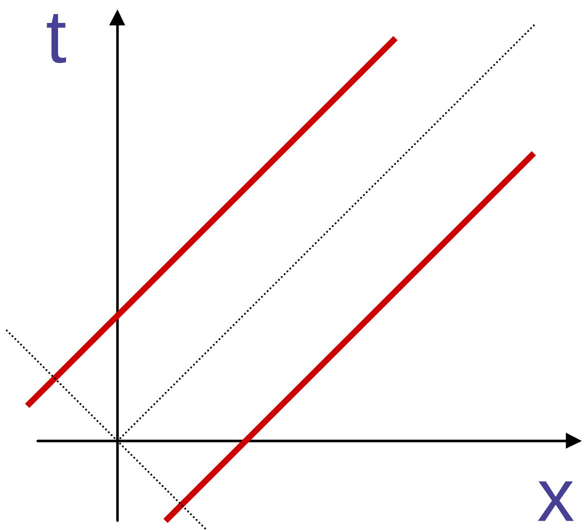
$$x_- = x_i = 0$$

- Time-like line: $x = x_\perp = 0$, $t = \tau$



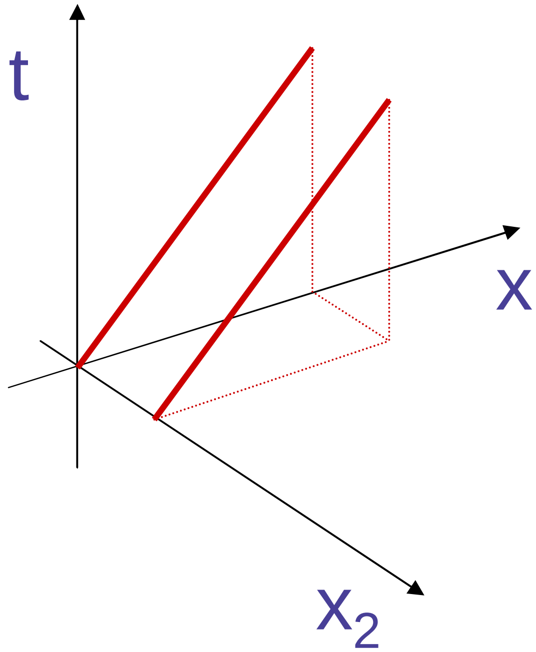
$$t = \tau$$
$$z = \sigma$$

- Parallel lines in x_+ direction: $x_\perp = 0$, $x_- = \pm a$, $x_+ = \tau$



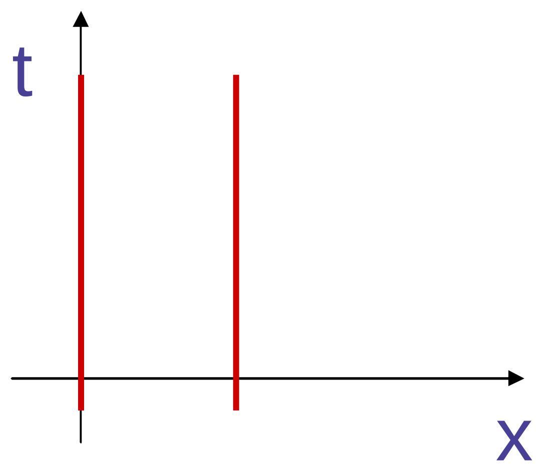
$$x_+ = \tau$$
$$x_- = \sigma$$
$$z = \sqrt{2}(\sigma_0^2 - \sigma^2)^{\frac{1}{4}}$$
$$x_i = 0$$

- Parallel lines in x_+ direction: $x_2=a,b$, $x_- =0$, $x_+=\tau$



$$x_+ = \tau$$
$$x_- = 0$$
$$x_2 = \sigma$$
$$z = z(\sigma)$$

- Parallel line in time-like direction: $x=\pm a$, $t=\tau$



$$t = \tau$$
$$x = \sigma$$
$$z = z(\sigma)$$

Conformal mapping

The metric:

$$ds^2 = 2\, dx_+\, dx_- - \mu^2\, x_\perp^2\, dx_+^2 + dx_\perp^2\,, \qquad x_\pm = \frac{x \pm t}{\sqrt{2}}$$

is conformally flat! But this is a local equivalence. Equivalently the AdS pp-wave is (locally) AdS space in different coordinates. (Brecher, Chamblin, Reall)

We can map the Wilson loop solutions to Wilson loops in ordinary space.

Indeed, the mapping:

$$\tilde{x}_+ = \mu^{-1} \tan \mu x_+ \ , \qquad \tilde{x}_- = x_- - \frac{1}{2}\mu x_i^2 \tan \mu x_+ \ , \qquad \tilde{x}_i = \frac{1}{\cos \mu x_+} x_i$$

gives:

$$2d\tilde{x}_+ d\tilde{x}_- + d\tilde{x}_i^2 = \frac{1}{\cos^2 \mu x_+} \left(2dx_+ dx_- - \mu^2 x_i^2 dx_+^2 + dx_i^2\right)$$

and

$$\tilde{x}_+ = \mu^{-1} \tan \mu x_+ \ , \qquad \tilde{x}_- = x_- - \frac{1}{2}\mu(x_i^2 + z^2) \tan \mu x_+$$

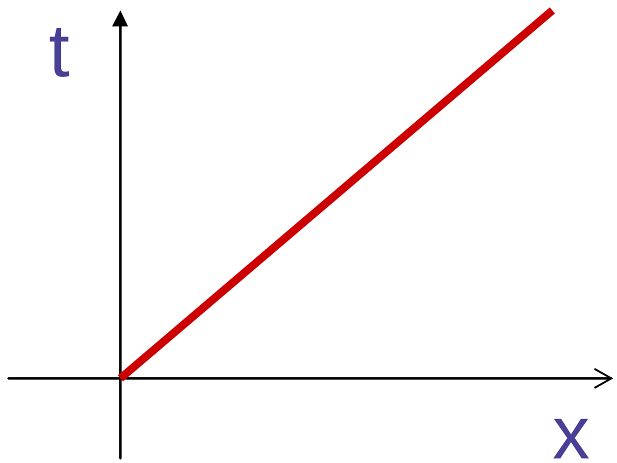
$$\tilde{x}_i = \frac{1}{\cos \mu x_+} x_i \ , \qquad \tilde{z} = \frac{1}{\cos \mu x_+} z \ ,$$

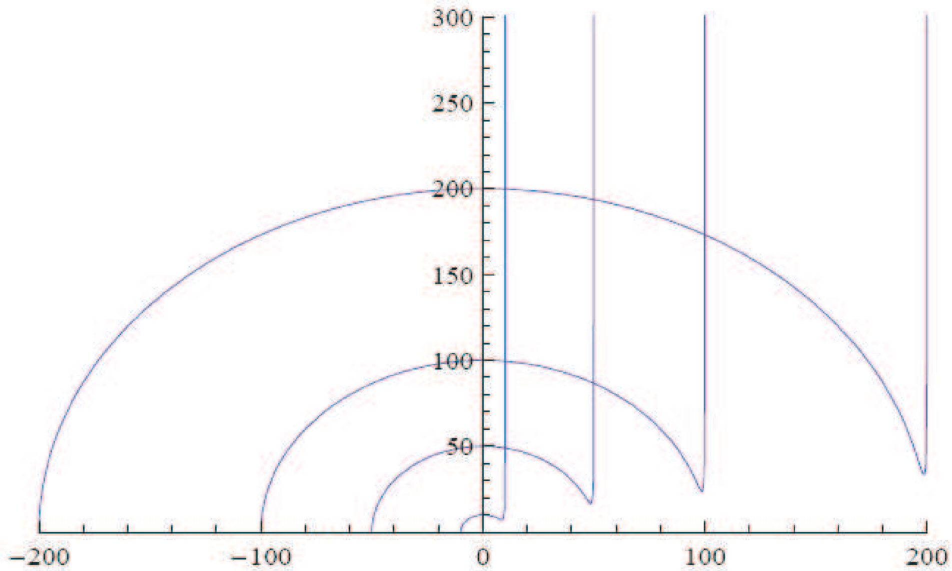
gives:

$$\frac{1}{\tilde{z}^2}\left(2d\tilde{x}_+ d\tilde{x}_- + d\tilde{x}_i^2 + d\tilde{z}^2\right) = \frac{1}{z^2}\left[2dx_+ dx_- - \mu^2 (x_i^2 + z^2) dx_+^2 + dx_i^2 + dz^2\right]$$

This allows us to obtain new Wilson loop solutions in usual AdS in Poincare coordinates.

- Light-like line: $x_\perp=0$, $x_- =0$, $x_+=\tau$



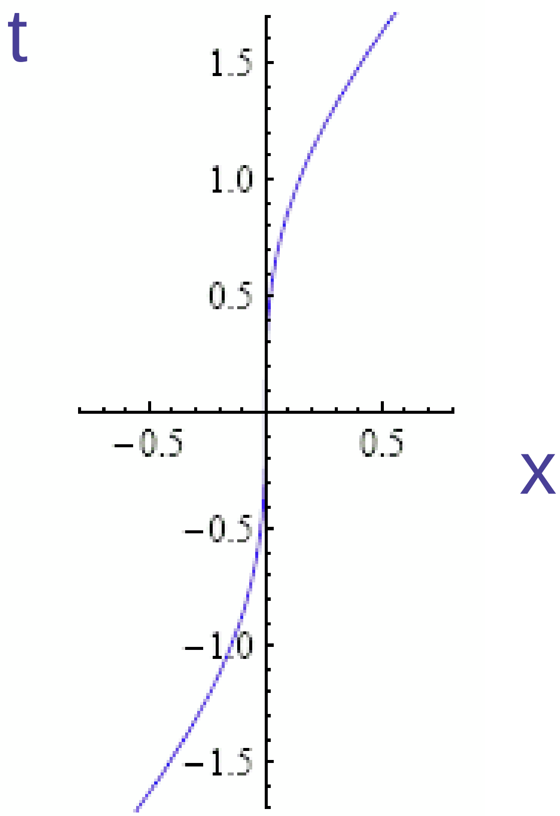


Plot of $\tilde{z}$ versus $\tilde{x}$ for $\mu = 1$, and $\tilde{t} = -200, -100, -50, -10$.

$$\tilde{z} = \sqrt{-\frac{2\tilde{x}_-}{\mu^2\tilde{x}_+}(1+\mu^2\tilde{x}_+^2)}$$

- Time-like line: $x = x_\perp = 0,\ t = \tau$

The Wilson loop now changes shape.
Particle decelerating and then accelerating.



$$\tilde{z} = \sqrt{-\frac{2}{\mu^2 \tilde{x}_+}(\tilde{x}_- + \frac{\arctan \mu \tilde{x}_+}{\mu})(1 + \mu^2 \tilde{x}_+^2)}$$

We can do the same with the other solutions. Generically we obtain surfaces in which z is a function of two variables and therefore very difficult to find directly.

Physically these solution seem to represent Wilson loops in the presence of propagating gluons given by spikes coming out from the horizon.

Also, in general the energy is not conserved so it requires certain power to move the quarks in these particular way.

Other Wilson loop solutions (new ansatz)

$$ds^2 = \frac{1}{z^2}\left(dx_+ dx_- + dw^2 + dy^2 + dz^2\right)$$

We use the ansatz

$$x_+ = u(\sigma,\tau), \quad x_- = 0, \quad w = a, \quad y = \tau, \quad z = \sigma$$

The equations of motion linearize. We get:

$$u''(\sigma,\tau) - \frac{2}{\sigma}u'(\sigma,\tau) + \ddot{u}(\sigma,\tau) = 0$$

The boundary curve (WL) is: $\left\{ \begin{array}{l} x_+ = u(\sigma = 0, \tau), \\ y = \tau, \quad w = a \end{array} \right.$

These solutions are BPS and therefore the area vanishes.

By a conformal transformation we can map them to WL given by closed curves.

$$y = a\cos\tau, \quad w = a\sin\tau, \quad x_+ = \tilde{u}(\tau)$$

Again, $\tilde{u}(\tau)$ is arbitrary. Namely we can find the solution for any such function.

The area is $$-iS = \frac{\sqrt{\lambda}}{2a\epsilon} - \sqrt{\lambda}$$

Conclusions:

- We define a pp-wave anomaly for a gauge theory living in a pp-wave as a (logarithmic) divergence of the energy momentum tensor in the presence of a particle moving at the speed of light.

- We proposed that it is given by the cusp anomaly and verify it at lowest order in the strong (string) and weak (field theory) coupling expansion.

- Following these ideas new open string solutions can be found both, in the AdS pp-wave and, by conformal mapping in ordinary AdS

- We found other solutions using a new ansatz.