Spiky Strings and Giant Magnons on S^5

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Based on: hep-th/0607044
(Russo, Tseytlin, M.K.)

Summary

- **Introduction**

String / gauge theory duality (**AdS/CFT**)

Classical strings and field theory operators:

folded strings and spin waves in spin chains

folded strings and twist two operators

- **Spiky strings and higher twist operators** **(M.K.)**

Classical strings moving in AdS and their field theory interpretation

- **Spiky strings on a sphere and giant magnon limit**
  (Ryang, Hofman-Maldacena)

- **Spin chain interpretation of the giant magnon**
  (Hofman-Maldacena)

- **More generic solutions:**
  Spiky strings and giant magnons on S^5
  (Russo, Tseytlin, M.K.)

- **Other solutions on S^2** (work in prog. w/ R.Ishizeki)

- **Conclusions**

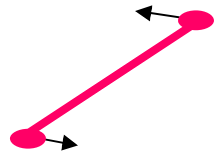
String/gauge theory duality: Large N limit ('t Hooft)

mesons

π, ρ, ...

String picture → Fund. strings

(Susy, 10d, Q.G.)



Quark model → QCD [SU(3)]

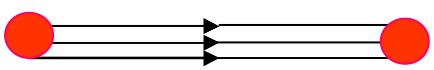
$q \bar{q}$

↓

Large N-limit [SU(N)]

↓

Effective strings

q  $\bar{q}$

Strong coupling

More precisely: $N \rightarrow \infty, \lambda = g_{YM}^2 N \ \ fixed$ ('t Hooft coupl.)

Lowest order: sum of planar diagrams (infinite number)

AdS/CFT correspondence (Maldacena)

Gives a precise example of the relation between strings and gauge theory.

Gauge theory	String theory
$\mathcal{N} = 4$ SYM SU(N) on R^4	IIB on $AdS_5 x S^5$
A_μ , Φ^i, Ψ^a	radius R
Operators w/ conf. dim. Δ	String states w/ $E = \frac{\Delta}{R}$

$$g_s = g_{YM}^2 ; \qquad R / l_s = (g_{YM}^2 N)^{1/4}$$

$N \to \infty , \lambda = g_{YM}^2 N$ fixed ➡

λ large → string th.
λ small → field th.

Can we make the map between string and gauge theory precise?

It can be done in particular cases.

Take two scalars $X = \Phi_1 + i\,\Phi_2$; $Y = \Phi_3 + i\,\Phi_4$

$O = \mathbf{Tr}(XX\ldots Y..Y\ldots X)$, J_1 X's , J_2 Y's, J_1+J_2 large

Compute 1-loop conformal dimension of O , or equiv. compute energy of a bound state of J_1 particles of type X and J_2 of type Y (but on a three sphere)

$R^4 \longleftrightarrow S^3 x R$

$\Delta \longleftrightarrow E$

Large number of ops. (or states). All permutations of Xs and Ys mix so we have to diag. a huge matrix.

Nice idea (Minahan-Zarembo). Relate to a phys. system

Tr(X X...Y X X Y) ⟷ | ↑ ↑...↓ ↑ ↑ ↓ ›

operator ⟷ conf. of spin chain

mixing matrix ⟷ op. on spin chain

$$H = \frac{\lambda}{4\pi^2} \sum_{j=1}^{J} \left(\frac{1}{4} - \vec{S}_j \cdot \vec{S}_{j+1} \right)$$

Ferromagnetic Heisenberg model !

Ground state (s)

$$|\uparrow\uparrow\ldots\uparrow\uparrow\uparrow\uparrow\rangle \longleftrightarrow \mathrm{Tr}(X\,X\ldots X\,X\,X\,X)$$

$$|\downarrow\downarrow\ldots\downarrow\downarrow\downarrow\downarrow\rangle \longleftrightarrow \mathrm{Tr}(Y\,Y\ldots Y\,Y\,Y\,Y)$$

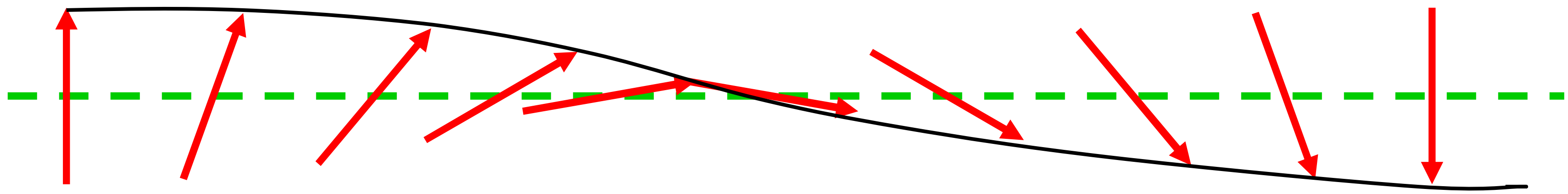
First excited states

$$|k\rangle = \sum e^{ikl}|\uparrow\uparrow\ldots\underset{l}{\downarrow}\ldots\uparrow\uparrow\rangle, \quad k = \frac{2\pi n}{J};\ (J = J_1 + J_2)$$

$$\varepsilon(k) = \frac{\lambda}{4\pi^2}\left(1-\cos k\right) \xrightarrow[k\to 0]{} \frac{\lambda n^2}{2J^2} \qquad \text{(BMN)}$$

More generic (low energy) states: Spin waves **(FT, BFST, MK, ...)**

Other states, e.g. with $J_1=J_2$



Spin waves of long wave-length have low energy and are described by an effective action in terms of two angles θ, φ: direction in which the spin points.

$$S_{eff.} = J\left\{ -\frac{1}{2}\int d\sigma\, d\tau \left[\cos\theta\, \partial_\tau \phi - \frac{\lambda}{32\pi J^2}\left[(\partial_\sigma \theta)^2 + \sin^2\theta (\partial_\sigma \phi)^2 \right] \right] \right\}$$

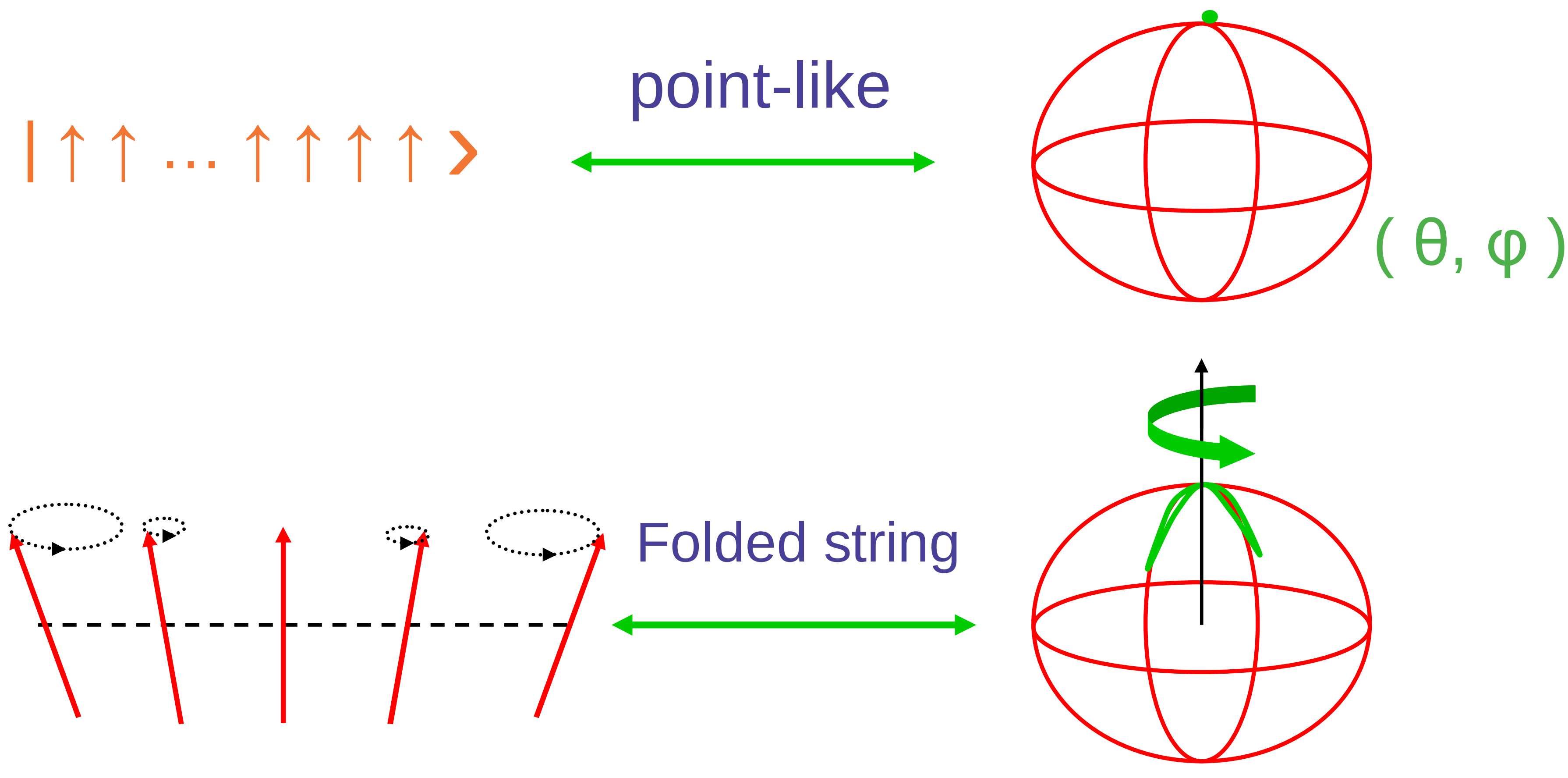
Taking J large with λ/J^2 fixed: classical solutions

Moreover, this action agrees with the action of a string moving fast on S^5.

What about the case k ~ 1 ?

Since (θ, φ) is interpreted as the position of the string we get the shape of the string from $\langle \vec{S} \rangle(\sigma)$

Examples

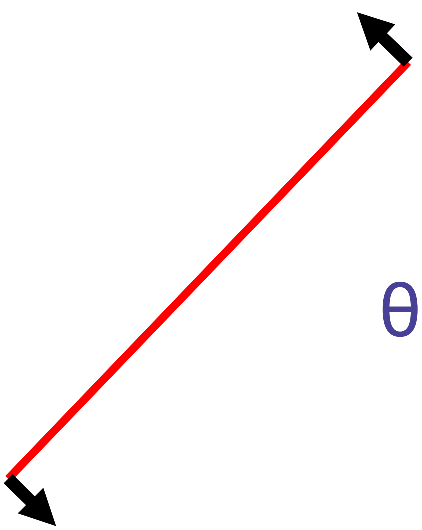


Rotation in AdS$_5$? (Gubser, Klebanov, Polyakov)

$$Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 - Y_5^2 - Y_6^2 = -R^2$$

$\sinh^2 \rho ; \Omega_{[3]}$ $\qquad$ $\cosh^2 \rho ; t$

$$ds^2 = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\Omega_{[3]}^2$$



$\theta = \omega t$

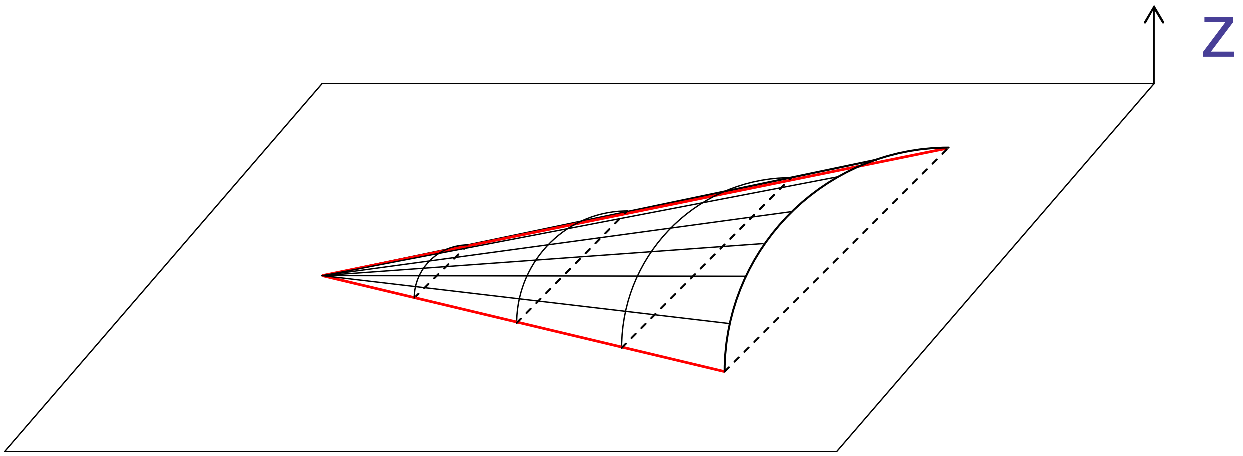
$$E \cong S + \frac{\sqrt{\lambda}}{2\pi} \ln S, \quad (S \to \infty)$$

$$O = Tr\left(\Phi \nabla_+^S \Phi\right), \quad x_+ = z + t$$

Verification using Wilson loops (MK, Makeenko)

The anomalous dimensions of twist two operators can also be computed by using the **cusp anomaly** of light-like Wilson loops (**Korchemsky and Marchesini**).

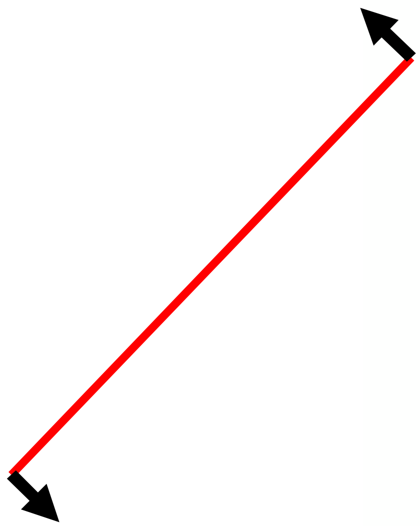
In **AdS/CFT** Wilson loops can be computed using surfaces of minimal area in AdS_5 (**Maldacena, Rey, Yee**)

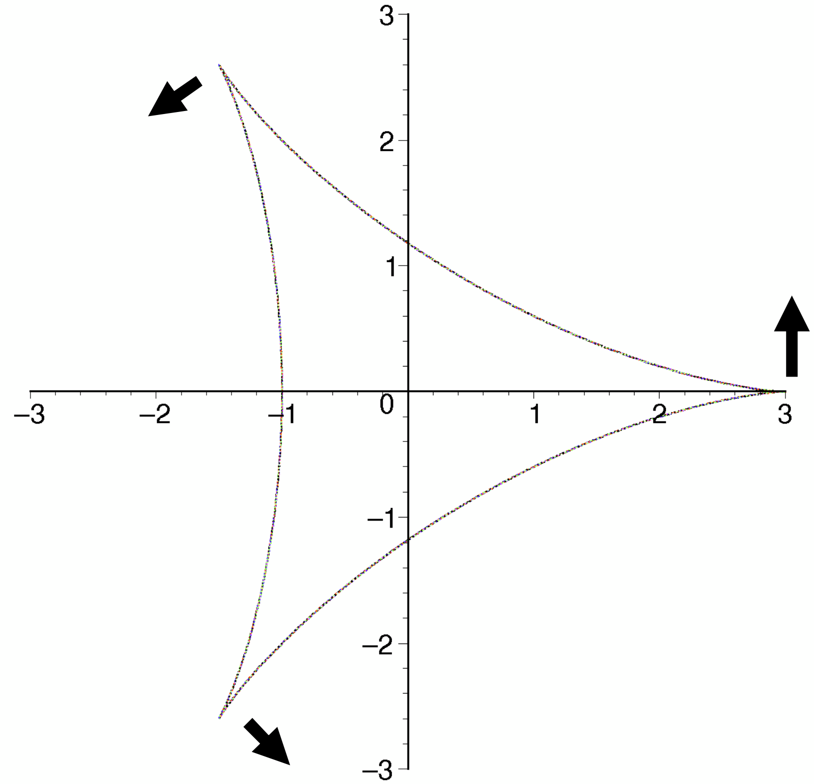


The result **agrees** with the rotating string calculation.

Generalization to higher twist operators (MK)

$$O_{[2]} = Tr\left(\Phi\, \nabla_+^{S} \Phi\right) \quad \longrightarrow \quad O_{[n]} = Tr\left(\nabla_+^{S/n}\Phi\, \nabla_+^{S/n}\Phi\, \nabla_+^{S/n}\Phi \ldots \nabla_+^{S/n}\Phi\right)$$



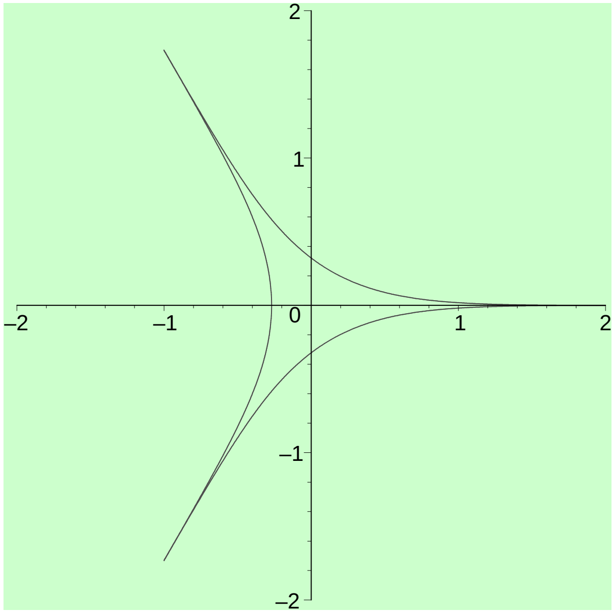


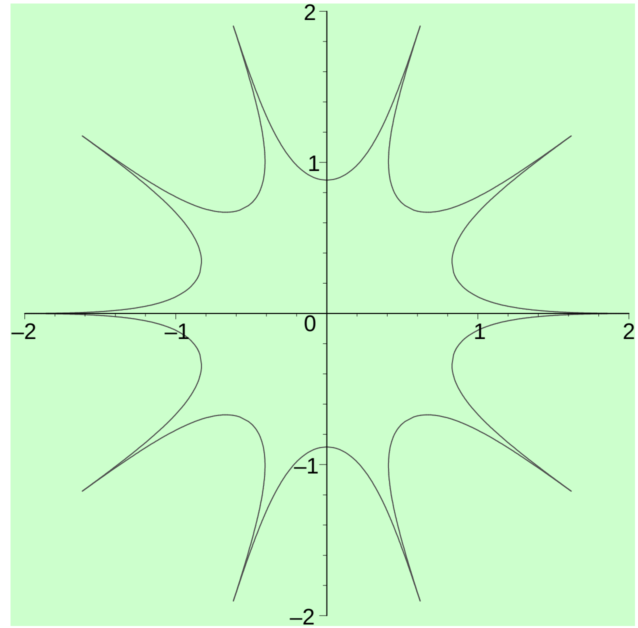
In flat space such solutions are easily found in conf. gaug

$$x = A\cos[(n-1)\sigma_+] + A(n-1)\cos[\sigma_-]$$

$$y = A\sin[(n-1)\sigma_+] + A(n-1)\sin[\sigma_-]$$

Spiky strings in AdS:





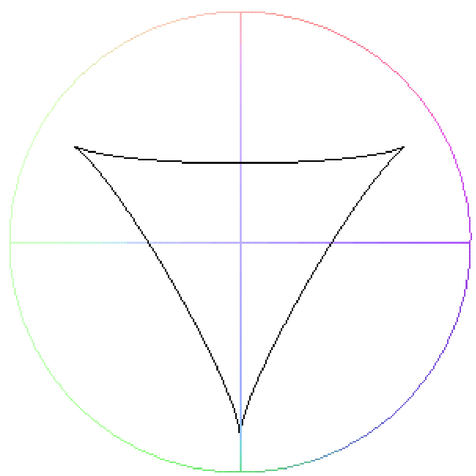
$$E \cong S + \left(\frac{n}{2}\right)\frac{\sqrt{\lambda}}{2\pi}\ln S\,, \quad (S \to \infty)$$

$$O = Tr\left(\nabla_+^{S/n}\Phi\;\nabla_+^{S/n}\Phi\;\nabla_+^{S/n}\Phi\ldots\nabla_+^{S/n}\Phi\right)$$

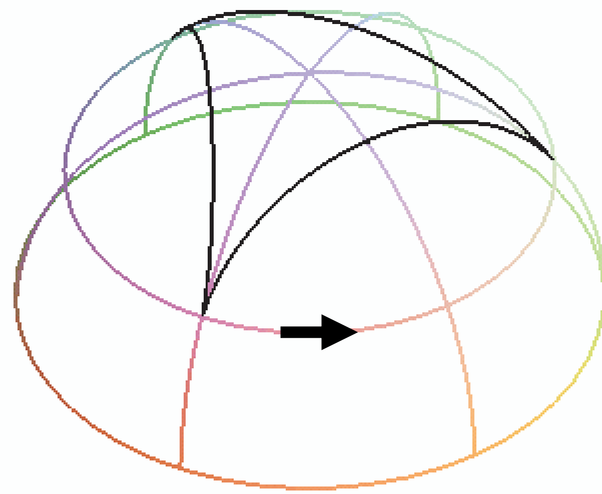
$$S = \frac{\sqrt{\lambda}}{2\pi}\int dt\sum_j(\cosh 2\rho_1 - 1)\dot{\theta}_j - \frac{\sqrt{\lambda}}{8\pi}\int dt\sum_j\left\{4\rho_1 + \ln\left(\sin^2\left(\frac{\theta_{j+1}-\theta_j}{2}\right)\right)\right\}$$

Spiky strings on a sphere: (Ryang)

Similar solutions exist for strings rotating on a sphere:



(top view)



The metric is: $ds^2 = -dt^2 + d\theta^2 + \sin^2\theta\, d\varphi^2$

We use the ansatz: $t = \kappa\tau, \quad \varphi = \omega\tau + \sigma, \quad \theta = \theta(\sigma)$

And solve for $\theta(\sigma)$. Field theory interpretation?

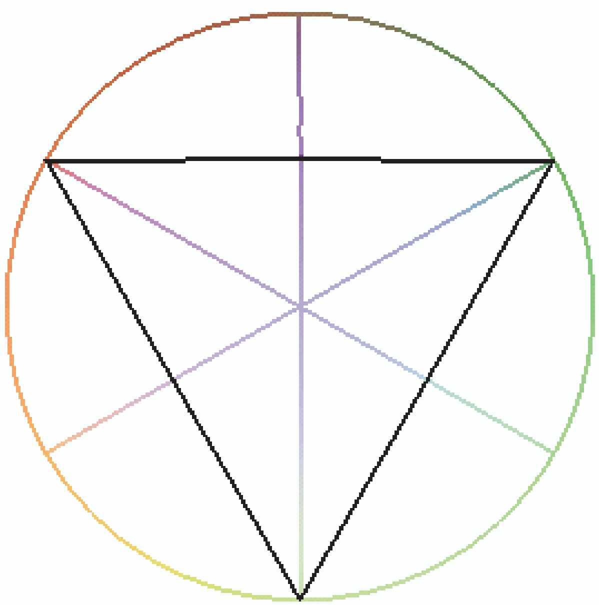
Special limit: (Hofman-Maldacena)

$$\frac{d\theta}{d\sigma} = \frac{\kappa \sin\theta}{A}\sqrt{\frac{\kappa^2 \sin^2\theta - A^2}{\kappa^2 - \omega^2 \sin^2\theta}}$$

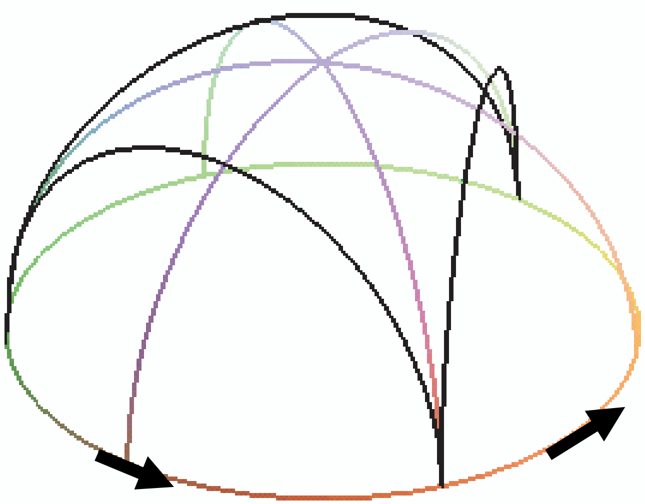
$$\omega = \kappa$$

$$\frac{d\theta}{d\sigma} = \frac{\sin\theta}{A\cos\theta}\sqrt{\kappa^2 \sin^2\theta - A^2}$$

$$\sin\theta = \frac{A}{\kappa}\frac{1}{\sin\sigma}$$



(top view)



giant magnon

The energy and angular momentum of the giant magnon solution diverge. However their difference is finite:

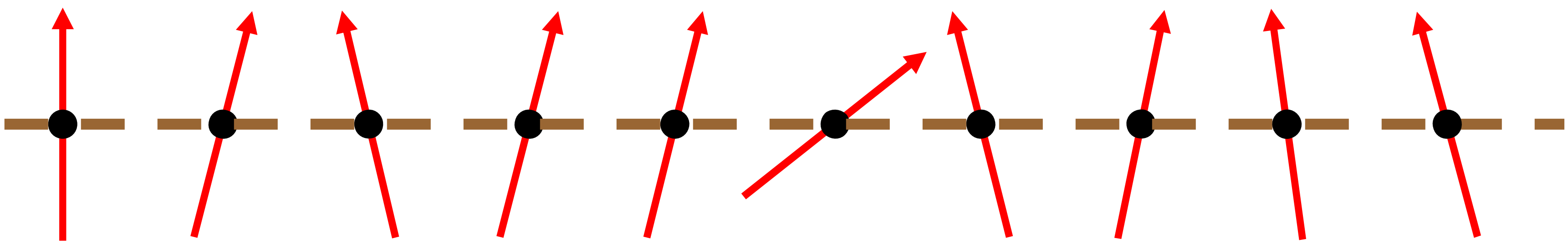
$$E - J = \frac{\sqrt{\lambda}}{2\pi}\frac{A}{\kappa}\int\frac{d\sigma}{\sin^2\sigma} = \frac{\sqrt{\lambda}}{\pi}\sin\frac{\Delta\varphi}{2}$$

$$\cos\frac{\Delta\varphi}{2} = \frac{A}{\kappa}, \quad \Delta\varphi = \text{Angular distance between spikes}$$

Interpolating expression:

$$E - J = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{\Delta\varphi}{2}} \cong \begin{cases} \dfrac{\sqrt{\lambda}}{\pi}\sin\dfrac{\Delta\varphi}{2}, & \lambda >> 1 \\ 1 + \dfrac{\lambda}{2\pi^2}\sin^2\dfrac{\Delta\varphi}{2}, & \lambda << 1 \end{cases}$$

Field theory interpretation: (Hofman-Maldacena)



$$H = \frac{\lambda}{4\pi^2} \sum_{j=1}^{J} \left(\frac{1}{4} - \vec{S}_j \cdot \vec{S}_{j+1} \right)$$

$$|k\rangle = \sum e^{ikl} |\uparrow\uparrow \ldots \downarrow \ldots \uparrow\uparrow\rangle, \quad k = \frac{2\pi n}{J}; \; (J = J_1 + J_2)$$

$$\varepsilon(k) = \frac{\lambda}{4\pi^2} (1 - \cos k) = \frac{\lambda}{2\pi^2} \sin^2 \frac{k}{2}$$

k → Δφ
(The 1 is J_2)

States with one spin flip and k~1 are giant magnons

More spin flips: (Dorey, Chen-Dorey-Okamura)

In the string side there are solutions with another angular momentum J_2. The energy is given by:

$$E - J_1 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2\frac{\Delta\varphi}{2}} \cong J_2 + \frac{\lambda}{2J_2\pi^2}\sin^2\frac{\Delta\varphi}{2}, \quad \lambda << 1$$

Justifies interpolating formula for $J_2=1$

In the spin chain, if we flip a number J_2 of spins there is a bound state with energy:

$$\varepsilon(k) = \frac{\lambda}{2J_2\pi^2}\sin^2\frac{k}{2}$$

$k \rightarrow \Delta\varphi$

(J_2 is absorbed in J)

More general solutions: (Russo, Tseytlin, MK)

Strategy: We generalize the spiky string solution and then take the giant magnon limit.

In flat space:

$$x = A\cos[(n-1)\sigma_+] + A(n-1)\cos[\sigma_-]$$

$$y = A\sin[(n-1)\sigma_+] + A(n-1)\sin[\sigma_-]$$

namely: $x + iy = X = x(\xi)e^{i\omega\tau}, \quad \xi = \alpha\sigma + \beta\tau$

Consider txS⁵: $ds^2 = -dt^2 + \sum_{a=1}^{3} dX_a d\bar{X}_a, \quad \sum_{a=1}^{3} X_a \bar{X}_a = 1$

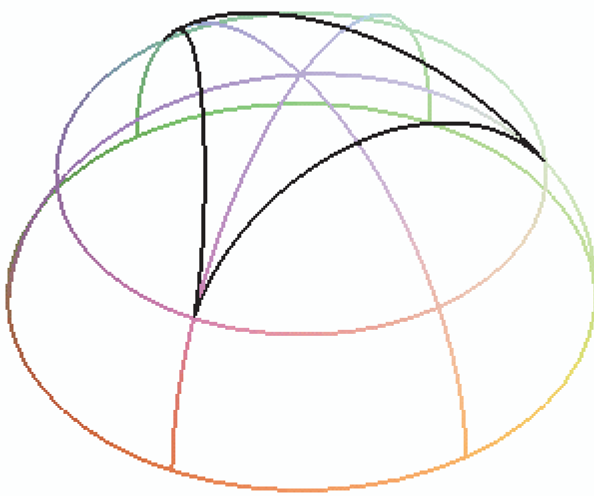
Use similar ansatz:

$$X_a = x_a(\xi)e^{i\omega_a\tau} = r_a(\xi)\, e^{i\mu_a(\xi)+i\omega_a\tau}$$

The reduced e.o.m. follow from the lagrangian:

$$L = (\alpha^2 - \beta^2) x'_a \bar{x}'_a + i\beta\omega_a (x'_a \bar{x}_a - \bar{x}'_a x_a) - \omega_a^2 x_a \bar{x}_a + \Lambda (x_a \bar{x}_a - 1)$$

If we interpret ξ as time this is particle in a sphere subject to a quadratic potential and a magnetic field. The trajectory is the shape of the string



The particle is attracted to the axis but the magnetic field curves the trajectory

Using the polar parameterization we get:

$$L = (\alpha^2 - \beta^2) r'^2_a - \frac{1}{(\alpha^2 - \beta^2)} \frac{C_a^2}{r_a^2} - \frac{\alpha^2}{(\alpha^2 - \beta^2)} \omega_a^2 r_a^2 + \Lambda (r_a^2 - 1)$$

$$\mu'_a = \frac{1}{(\alpha^2 - \beta^2)} \left[\frac{C_a^2}{r_a^2} + \beta \omega_a \right], \qquad x_a = r_a e^{i\mu_a}$$

Constraints: $\omega_a C_a + \beta \kappa^2 = 0\,, \quad H = \dfrac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2} \kappa^2$

Three ang. momenta: $J_a = \int d\xi \left(\dfrac{\beta}{\alpha} \dfrac{C_a}{(\alpha^2 - \beta^2)} + \dfrac{\alpha}{(\alpha^2 - \beta^2)} \omega_a r_a^2 \right)$

Corresponding to phase rotations of $x_{1,2,3}$

Solutions:

- One angular momentum:

  $x_3=0$, x_2 real $(\mu_2=0)$, $r_1^2+r_2^2=1$, one variable.

- Two angular momenta:

  $x_3=0$, $r_1^2+r_2^2=1$, one variable

  Since only one variable we solve them using conservation of H. Reproduced Ryang, Hofman-Maldacena and Chen-Dorey-Okamura

- Three angular momenta: $r_1^2+r_2^2+r_3^2=1$, $r_{1,2}$

Therefore the three angular momenta case is the first “non-trivial” and requires more effort. It turns that this system is integrable as shown long ago by Neumann, Rosochatius and more recently by Moser.

Can be solved by doing a change of variables to ζ_+, ζ_-

$$r_a^2 = \frac{(\zeta_+ - \omega_a^2)(\zeta_- - \omega_a^2)}{\prod_{a\neq b}(\omega_a^2 - \omega_b^2)}$$

In the new variables, the system separates if we use the Hamilton-Jacobi method:

Compute the Hamiltonian: $H(p_{\pm}, \zeta_{\pm})$

Find $W(\zeta_{\pm})$ such that $H\left(p_{\pm} = \frac{\partial W}{\partial \zeta_{\pm}}, \zeta_{\pm}\right) = E = const.$

In this case we try the ansatz: $W = W(\zeta_{+}) + W(\zeta_{-})$
and it works! Variables separate!.

A lengthy calculation gives a solution for ζ_{+}, ζ_{-} which can then be translated into a solution for r_a

The resulting equations are still complicated but simplify in the giant magnon limit in which $J_1 \to \infty$

We get for r_a :

$$r_2^2 = \frac{(\omega_2^2 - \omega_3^2)}{(\omega_1^2 - \omega_2^2)} s_2^2 \frac{1 - A_2^2}{(s_3 A_3 - s_2 A_2)^2}$$

$$r_3^2 = \frac{(\omega_2^2 - \omega_3^2)}{(\omega_1^2 - \omega_3^2)} s_3^2 \frac{A_3^2 - 1}{(s_3 A_3 - s_2 A_2)^2}$$

with $A_2 = \tanh\left(- \frac{s_2 \xi}{1 - \beta^2} + B_2 \right), \quad A_3 = \coth\left(- \frac{s_3 \xi}{1 - \beta^2} + B_3 \right)$

We have r_a explicitly in terms of ξ and integration const.

We can compute the energy and angular extension of the giant graviton obtaining:

$$E - J_1 = E_2 + E_3 = \sqrt{J_2^2 + \frac{\lambda}{\pi^2}\sin^2\frac{\varphi_2}{2}} + \sqrt{J_3^2 + \frac{\lambda}{\pi^2}\sin^2\frac{\varphi_3}{2}}$$

$$\Delta\varphi = \varphi_2 + \varphi_3$$

We get two superposed magnons.
However there is a relation: same group velocity.

$$v_2 = v_3, \qquad v_j = \frac{\partial E_j}{\partial \varphi_j}$$

In the spin chain side we need to use more fields to have more angular momenta.
We consider therefore operators of the form:

Tr(…XXXXYYXYYZZZYYXXYZZZXXXXXXXXXXXX…)

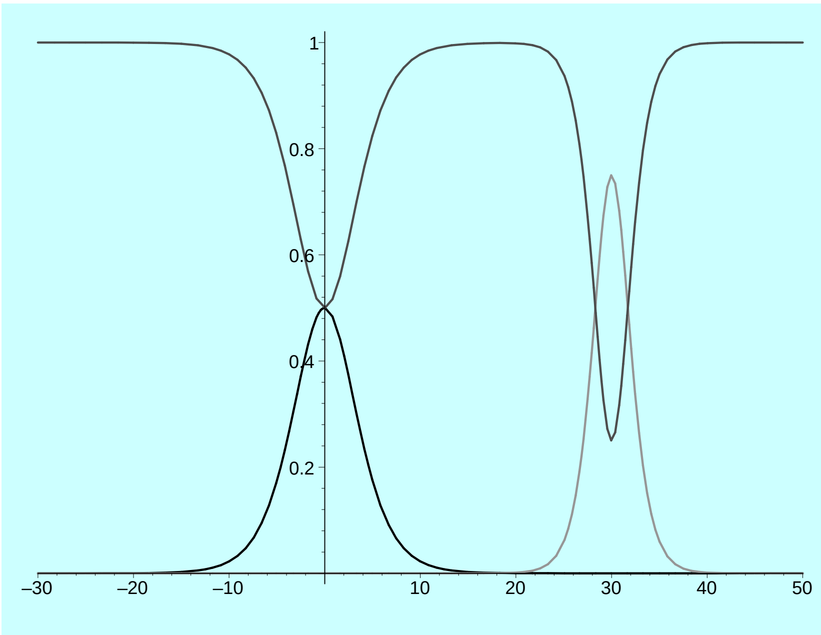
Where $X = \Phi_1 + i\,\Phi_2$; $Y = \Phi_3 + i\,\Phi_4$, $Z = \Phi_5 + i\,\Phi_6$

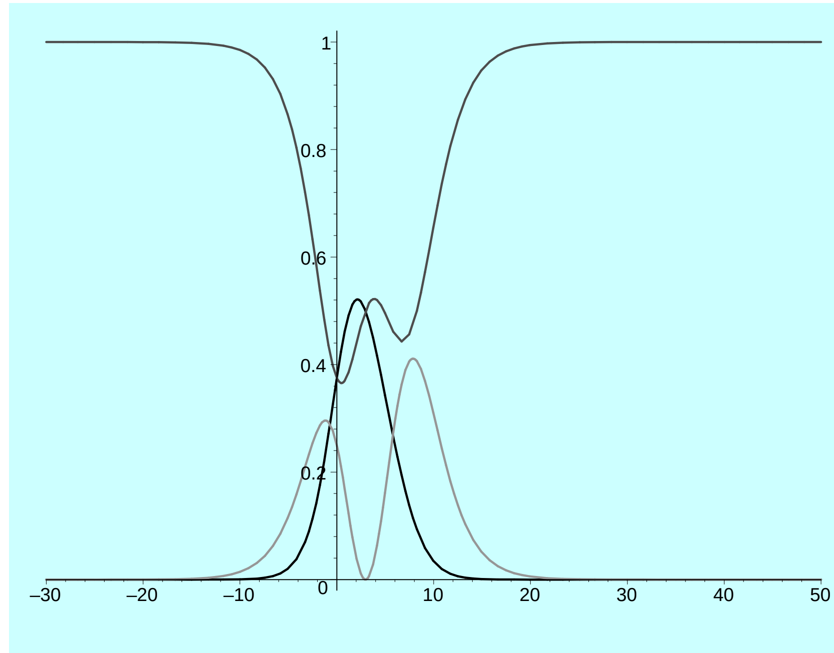
The J_2 Y's form a bound state and the J_3 Z's another, both superposed to a "background" of J_1 X 's ($J_1 \to \infty$)

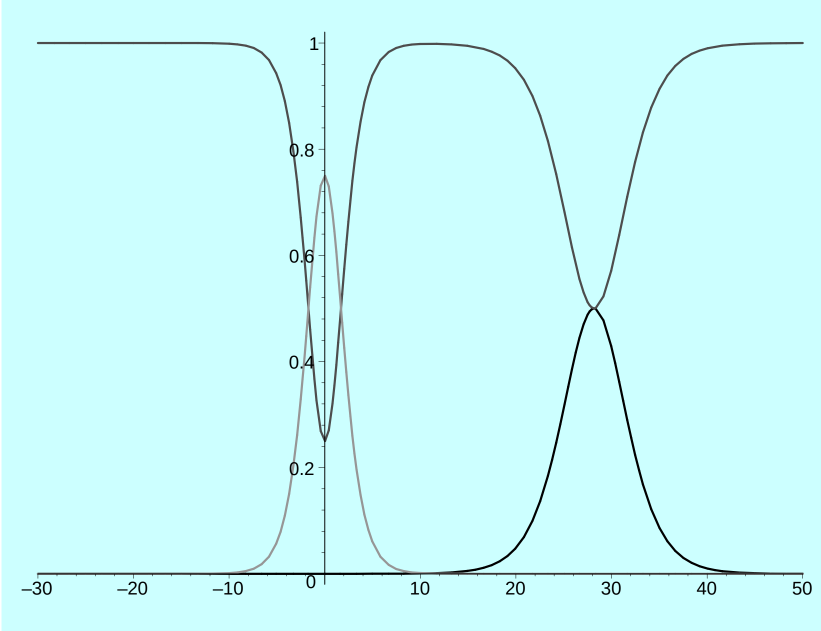
$$E - J_1 \cong J_2 + J_3 + \frac{\lambda}{2J_2\pi^2}\sin^2\frac{\varphi_2}{2} + \frac{\lambda}{2J_3\pi^2}\sin^2\frac{\varphi_3}{2}, \quad \lambda << 1$$

The condition of equal velocity appears because in the string side we use a rigid ansatz which does not allow relative motion of the two lumps.

Some examples of solutions.



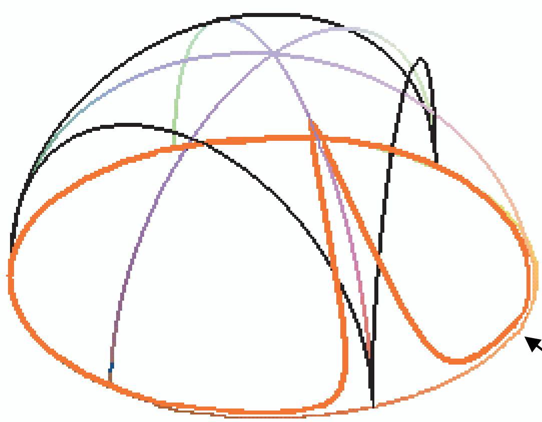




$r_{1,2,3}(\xi)$

Other solutions on S^2: (Work in progr. w/ R. Ishizeki)

It turns out that looking at rigid strings rotating on a two-sphere one can find other class of solutions and in particular another limiting solution:



Antiferromagnetic magnon?
(see Roiban,Tirziu,Tseytlin)

(Goes around infinite times)

Conclusions:

Classical string solutions are a powerful tool to study the duality between string and gauge theory.

We saw several examples:

- folded strings rotating on S^5
- spiky strings rotating in AdS_5 and S^5
- giant magnons on S^2 and S^3
- giant magnons with three angular momenta
- work in progress on other sol. on S^2