Spiky Strings in the SL(2) Bethe Ansatz

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(L. Freyhult, A. Tirziu, M.K.)

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Summary

- **Introduction**

  String / gauge theory duality (**AdS/CFT**)

  Classical strings and field theory operators:
  folded strings and twist two operators

- **Spiky strings and higher twist operators**

  Classical strings moving in AdS and their field theory interpretation

- **Spiky strings in flat space**
  Quantum description.

- **Spiky strings in Bethe Ansatz**
  Mode numbers and BA equations at 1-loop

- **Solving the BA equations**
  Resolvent

- **Conclusions and future work**

Extending to all loops we a find precise matching with the results from the classical string solutions.

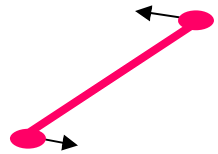
String/gauge theory duality: Large N limit ('t Hooft)

mesons

π, ρ, ...

String picture → Fund. strings

(Susy, 10d, Q.G.)



Quark model → QCD [SU(3)]

q $\bar{q}$

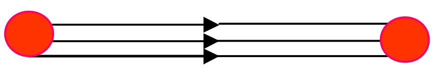
↓

Large N-limit [SU(N)]

↓

Effective strings

q $\bar{q}$



Strong coupling

More precisely: $N \to \infty, \lambda = g_{YM}^2 N \ \ fixed$ ('t Hooft coupl.)

Lowest order: sum of planar diagrams (infinite number)

AdS/CFT correspondence (Maldacena)

Gives a precise example of the relation between strings and gauge theory.

Gauge theory	String theory
$\mathcal{N} = 4$ SYM SU(N) on R^4	IIB on $AdS_5 x S^5$
A_μ , Φ^i, Ψ^a	radius R
Operators w/ conf. dim. Δ	String states w/ $E = \frac{\Delta}{R}$

$$g_s = g_{YM}^2; \qquad R/l_s = (g_{YM}^2 N)^{1/4}$$

$$N \to \infty, \lambda = g_{YM}^2 N$$ fixed ➡

λ large → string th.
λ small → field th.

Can we make the map between string and gauge theory precise? Nice idea (Minahan-Zarembo). Relate to a phys. system

Tr(X X…Y X X Y) ⟷ | ↑ ↑…↓ ↑ ↑ ↓ ›

operator ⟷ conf. of spin chain

mixing matrix ⟷ op. on spin chain

$$H = \frac{\lambda}{4\pi^2} \sum_{j=1}^{J} \left(\frac{1}{4} - \vec{S}_j \cdot \vec{S}_{j+1} \right)$$

Ferromagnetic Heisenberg model !

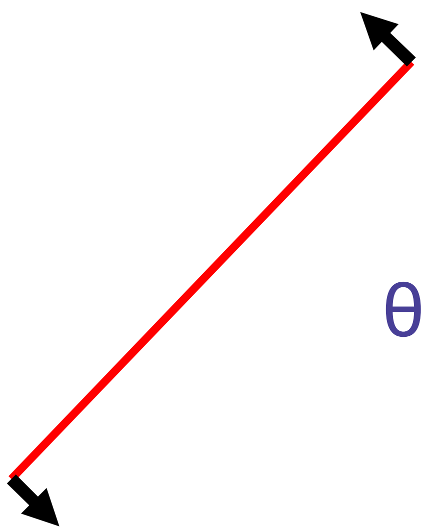
For large number of operators becomes classical and can be mapped to the classical string.
It is integrable, we can use BA to find all states.

Rotation in AdS_5? (Gubser, Klebanov, Polyakov)

$$Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 - Y_5^2 - Y_6^2 = -R^2$$

$\sinh^2\rho;\ \Omega_{[3]}$ $\qquad\qquad$ $\cosh^2\rho;\ t$

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{[3]}^2$$



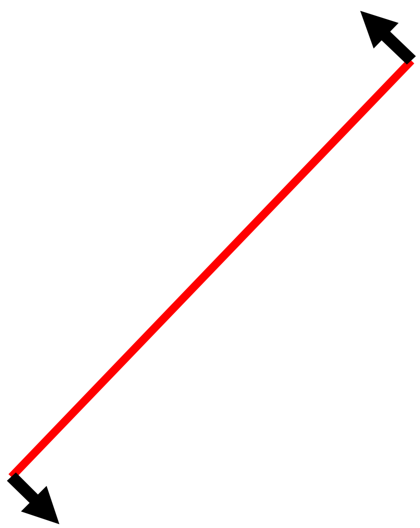
$\theta = \omega\, t$

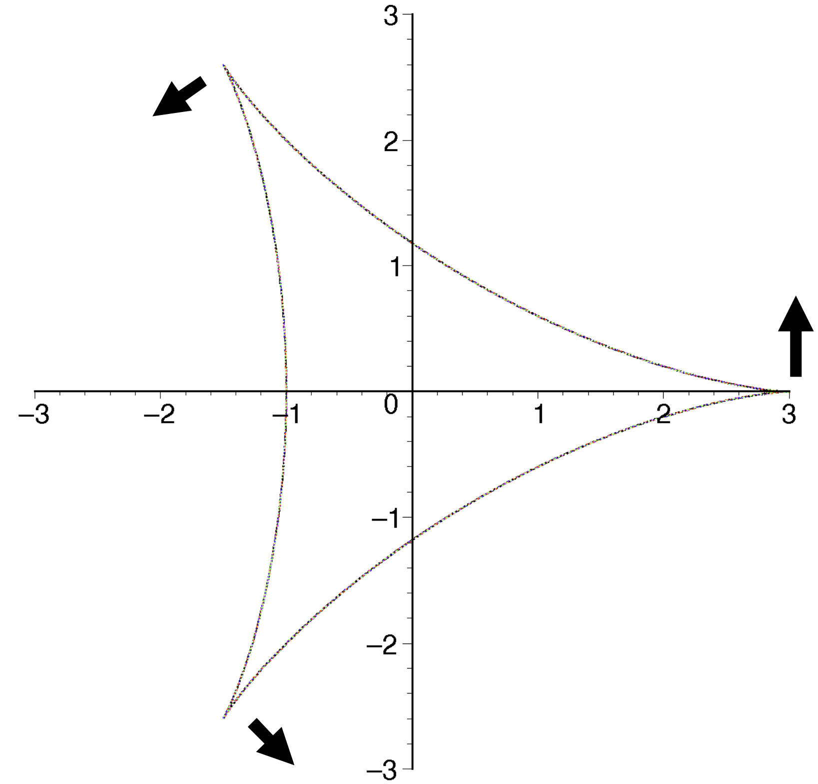
$$E \cong S + \frac{\sqrt{\lambda}}{\pi}\ln S, \quad (S \to \infty)$$

$$O = Tr\left(\Phi\, \nabla_+^S \Phi\right), \quad x_+ = z + t$$

Generalization to higher twist operators (MK)

$$O_{[2]} = Tr\left(\Phi \nabla_{+}^{S} \Phi\right) \quad \longrightarrow \quad O_{[n]} = Tr\left(\nabla_{+}^{S/n}\Phi \, \nabla_{+}^{S/n}\Phi \, \nabla_{+}^{S/n}\Phi \ldots \nabla_{+}^{S/n}\Phi\right)$$



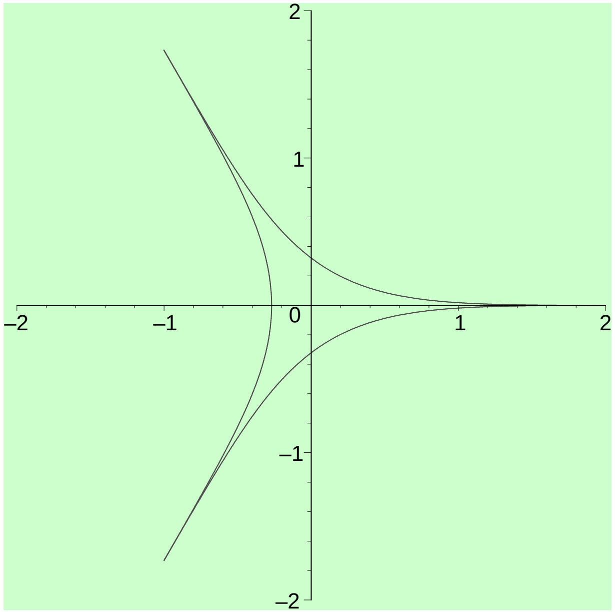


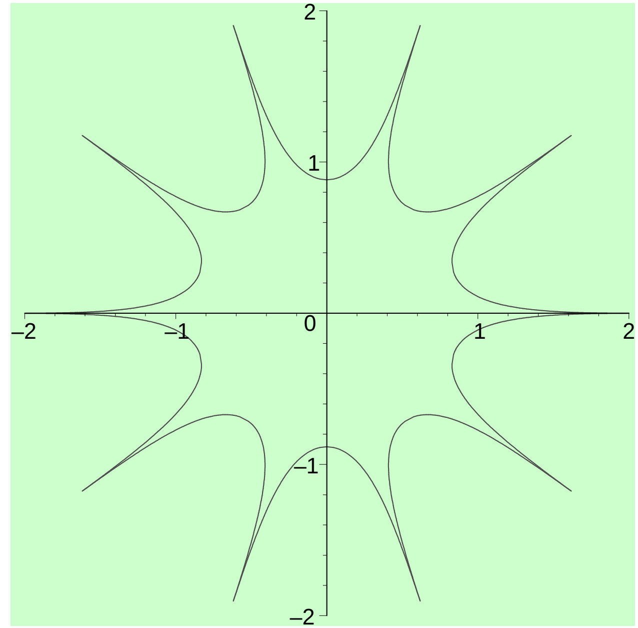
In flat space such solutions are easily found in conf. gaug

$$x = A\cos[(n-1)\sigma_+] + A(n-1)\cos[\sigma_-]$$

$$y = A\sin[(n-1)\sigma_+] + A(n-1)\sin[\sigma_-]$$

Spiky strings in AdS:





$$E \cong S + \left(\frac{n}{2}\right)\frac{\sqrt{\lambda}}{\pi}\ln S\,, \quad (S \to \infty)$$

Beccaria, Forini, Tirziu, Tseytlin

$$O = Tr\left(\nabla_+^{S/n}\Phi\, \nabla_+^{S/n}\Phi\, \nabla_+^{S/n}\Phi \ldots \nabla_+^{S/n}\Phi\right)$$

$$E - S = \frac{n\sqrt{\lambda}}{2\pi}\left(\ln\frac{4\pi S}{\sqrt{\lambda}} + \ln\left(\frac{4}{n}\sin\frac{\pi}{n}\right) - 1\right) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}, \frac{\ln S}{S}\right)$$

Spiky strings in flat space Quantum case

Classical:

$$x = A\cos[(n-1)\sigma_+] + A(n-1)\cos[\sigma_-]$$
$$y = A\sin[(n-1)\sigma_+] + A(n-1)\sin[\sigma_-]$$

Quantum:

$$n_R = A^2(n-1)^2 \qquad k_R = 1$$
$$n_L = A^2(n-1) \qquad k_L = n-1$$

$$n_R k_R = n_L k_L$$

$$E = \sqrt{2(n_L k_L + n_R k_R)} = 2A(n-1)$$
$$S = n_L + n_R = A^2 n(n-1)$$

$$E = 2\sqrt{\frac{n-1}{n}S}$$

Operators with large spin SL(2) sector

Spin chain representation

$$\mathrm{Tr}\left[\nabla_+^{s_1} X \, \nabla_+^{s_2} X \, \nabla_+^{s_3} X \ldots \nabla_+^{s_L} X\right] \quad \rightarrow \quad \left| s_1 s_2 s_3 \ldots s_L \right\rangle$$

s_i non-negative integers.

Spin $S = s_1 + \ldots + s_L$

Conformal dimension $E = L + S + \text{anomalous dim.}$

Bethe Ansatz

S particles with various momenta moving in a periodic chain with L sites. At one-loop:

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{\substack{j=1 \\ j \neq k}}^{S} \frac{u_k - u_j - i}{u_k - u_j + i}, \qquad k = 1 \ldots S$$

$$\prod_{k=1}^{S} \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} = 1 \qquad E_1 = \frac{\lambda}{8\pi^2} \sum_{k=1}^{S} \frac{1}{u_k^2 + \frac{1}{4}}$$

We need to find the u_k (real numbers)

For large spin, namely large number of roots, we can define a continuous distribution of roots with a certain density.

It can be generalized to all loops
(Beisert, Eden, Staudacher $\longrightarrow$ $E = S + (n/2)\, f(\lambda) \ln S$

Belitsky, Korchemsky, Pasechnik described in detail L=3 case using Bethe Ansatz.

Large spin means large quantum numbers so one can use a semiclassical approach (coherent states).

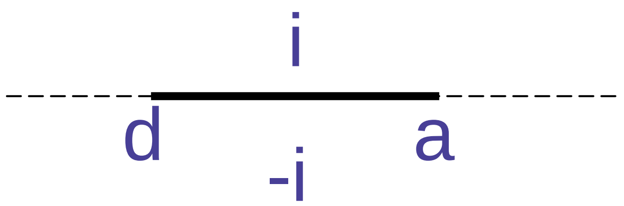
Spiky strings in Bethe Ansatz

BA equations

$$
\begin{aligned}
\fint du \frac{\rho_0(u)}{u'-u} &= \pi n_w(u') \\
\int du \rho_0(u) &= S \\
\fint du \frac{\rho_0(u)}{u} &= 0,
\end{aligned}
$$

Roots are distributed on the real axis between d<0 and a>0. Each root has an associated wave number n_w. We choose n_w=-1 for u<0 and n_w=n-1 for u>0. Solution?

Define $F(w) = \sqrt{w-a}\sqrt{w-d}$

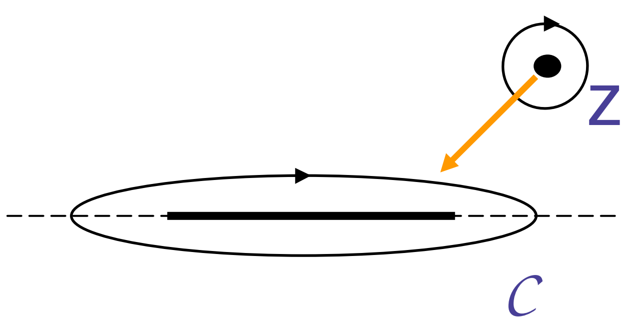


and $$G(w) = \frac{1}{\pi}\int_d^a \frac{F(w)}{F(u+i0^+)}\frac{n_w(u)}{u-w}du$$

We get on the cut:

$$G(x \pm i0^+) = \pm\frac{1}{\pi} -\!\!\!\!\!\!\int_d^a \left|\frac{F(x)}{F(u)}\right| \frac{n_w(u)}{u-x}\, du + i n_w(u) = \pm\rho(x) + i n_w(x)$$

Consider



$$\frac{G(w)}{w-z}$$

We get

$$\fint_d^a \frac{\rho(u)\,du}{u' - u} = \pi n_w(u') - i\pi \mathrm{Res}\left(\frac{G(w)}{w - x'}, w = \infty\right)$$

Also:

$$\int_d^a \rho(u) du = \frac{1}{2}\oint_{\mathcal{C}} G(w) dw = -i\pi \mathrm{Res}\left[G(w), w = \infty\right]$$

$$\fint_d^a \frac{\rho(u)}{u} du = \frac{1}{2}\oint_{\mathcal{C}} \frac{G(w)}{w} dw = -i\pi \mathrm{Res}\left[\frac{G(w)}{w}, w = \infty\right]$$

Since $G \simeq G_0 + \frac{G_1}{w} + \ldots, \quad w \to \infty$ we get

$$G_0 = 0 \quad \Rightarrow \quad -\frac{a}{d} = \tan^2 \frac{\pi}{2n}, \quad S = i\pi G_1 = n\sqrt{-ad}$$

We also have:

$$\frac{1}{2g^2}(E-S) = \int_d^a \frac{\rho(u)\,du}{u^2+\frac{1}{4}} = \pi G(\frac{i}{2}) - \pi G(-\frac{i}{2}) \simeq n \ln\left(-\frac{8ad}{a-d}\right)$$

Finally, we obtain:

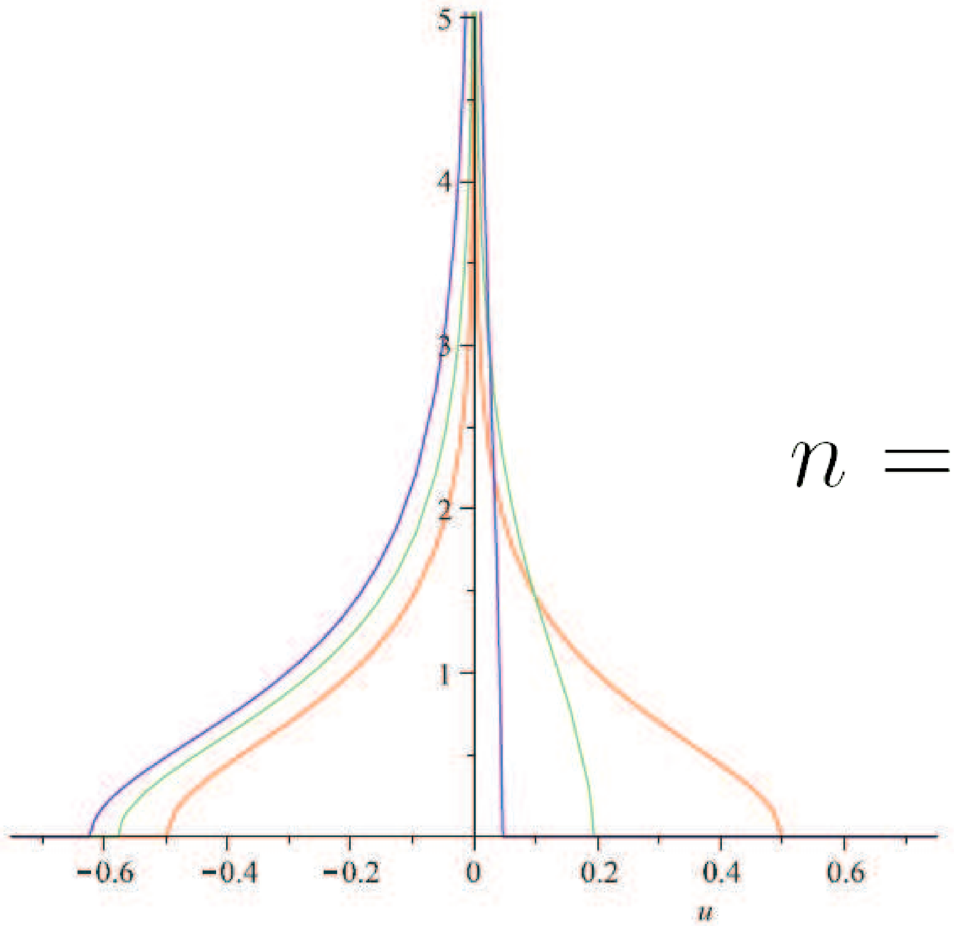
$$a = \frac{S}{n}\tan\frac{\pi}{2n}$$

$$d = -\frac{S}{n}\cot\frac{\pi}{2n}$$

$$\frac{E-S}{2g^2} = 2n\ln\left(\frac{S}{4n}\sin\frac{\pi}{n}\right)$$

Root density

$$\rho_0(u) = \frac{n}{\pi} \ln \left| \frac{\sqrt{a(u-d)} + \sqrt{-d(a-u)}}{\sqrt{a(u-d)} - \sqrt{-d(a-u)}} \right|$$



$n = 2,\ n = 3$ and $n = 6$

We can extend the results to strong coupling using the all-loop BA (BES).

We obtain (see Alin's talk)

$$
\begin{aligned}
E - S \;=\; & \frac{n\sqrt{\lambda}}{2\pi}\left(\ln\frac{4\pi S}{\sqrt{\lambda}} + \ln\left(\frac{4}{n}\sin\frac{\pi}{n}\right) - 1\right) \\
+ \; & \frac{n}{2}\left(1 + \frac{6\ln 2}{\pi} - \frac{3\ln 2}{\pi}\ln\left(\frac{4}{n}\sin\frac{\pi}{n}\right) - \frac{3\ln 2}{\pi}\ln\frac{4\pi S}{\sqrt{\lambda}}\right) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}, \frac{\ln S}{S}\right)
\end{aligned}
$$

In perfect agreement with the classical string result.
We also get a prediction for the one-loop correction.

Conclusions

We found the field theory description of the spiky strings in terms of solutions of the BA ansatz equations. At strong coupling the results agrees with the classical string result providing a check of our proposal and of the all-loop BA.

Future work

Relation to more generic solutions by Jevicki-Jin found using the sinh-Gordon model.

Relation to elliptic curves description found by Dorey and Losi and Dorey.

Semiclassical methods?