Approximately Maximizing Efficiency and Revenue in Polyhedral Environments

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ABSTRACT

We consider a resource allocation game in polyhedral environments. Polyhedral environments model a wide range of problems, including bandwidth sharing, some models of Adwords auctions and general resource allocation. We extend the fair sharing mechanism for such resource allocation games. We show that our mechanism simultaneously creates approximately efficient allocations and approximately maximizes revenue.

We also develop a new approach for analyzing games of these types. At the core of this approach is the relation between the condition for Nash equilibriums of the game and the dual of a certain linear program.

Categories and Subject Descriptors

F.2 [**Analysis of algorithms and problem complexity**]

General Terms

Theory, Algorithm, Economics.

Keywords

Game Theory, Mechanism Design, Efficiency, Revenue.

1. INTRODUCTION

We study a resource allocation game with n players in polyhedral environments. The goal of the game is to determine a real valued outcome $x_i \geq 0$ for each player i, which we think of as the player's level of activity or allocation. Each player is interested in maximizing his value. The simplest polyhedral environment is the sharing of a single resource, where x_i is the amount of the resource allocated to player i. If we have 1 unit of the resource, we require that the allocation satisfies $\sum_i x_i \leq 1$.

We consider games on a general class of polyhedrals. Such environments model a wide variety of games, including bandwidth sharing games [8, 6], some form of Adwords auctions [3, 12], and general resource allocation games, where feasible allocations are constrained by a nonnegative constraint matrix $Ax \leq u$, with each row of the matrix A corresponding to a different resource, or just a different constraint on the allocation.

Bandwidth allocation is naturally modeled via a polyhedral game: assume each player i is associated with a path P_i and each edge e has a capacity u_e. Player i is interested in reserving capacity x_i along the path P_i. The resource constraints are now $\sum_{i:e\in P_i} x_i \leq u_e$, corresponding to the capacity constraint of the edges. A more general resource allocation game can have constraints for different resources, and each resource corresponds to a row in a nonnegative constraint matrix $Ax \leq u$. We will show that polyhedral environments can also be used as a model of certain Adwords auctions, where x_i is the expected number of clicks allocated to bidder i.

In an auction mechanism, buyers submit bids and based on these bids the mechanism decides on the allocation values x_i and the payments w_i, the amount player i has to pay for the allocation x_i he receives. We assume that each user i has a concave utility function $U_i(x_i)$ for the amount x_i he is receiving, and assume that players have linear and separable utility for money, so each player i is interested in maximizing the total utility expressed as $U_i(x_i) - w_i$.

There are two natural, and often competing goals for auction mechanisms: to maximize social welfare and to maximize the auctioneer's revenue. For a given allocation vector x we say that the social welfare is the total $\sum_i U_i(x_i)$, the sum of all user's utilities for the allocation. An alternate goal for a mechanisms is to maximize the revenue, which is the auctioneer's income. In our context, if the mechanism collects payments $w_1, \ldots, w_n$ then the revenue is $\sum_i w_i$. Our question is this: Can we design a simple mechanism that achieves both goals or at least gives an approximate guarantee for both objective functions simultaneously? In this paper we investigate this question in the context of a fair sharing mechanism in a general class of games which we call *allocation games in polyhedral environments*.

In order to obtain good bounds on both efficiency and social welfare, we will need a few assumptions, which we will explain more formally in Section 4.

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- There are many identical or similar players. The type of a player i is described by the utility function U_i and his resource needs for the different resources. We will assume that there are at least k players of each type. Our bounds will improve as k gets larger.
- Users utilities do not decline too fast, namely $U(2x) \geq \alpha U(x)$ for some $\alpha > 1$ and all $x \geq 0$.

Both assumptions are essential for guaranteeing revenue for the type of allocation mechanism used here, where the mechanism is independent of the user types. To see the need for the first assumption, consider the case when there is only a single user with high valuation for the resource, he will have to get most of the resource allocated (due to desiring efficiency), even if he significantly misrepresents his utility. And hence, we cannot expect to extract his valuation as income.

The second assumption is also essential: To gain high revenue in the presence of limited demand for the resource, one has to introduce a reserve price, and possibly leave a large fraction of the resource un-allocated. Our mechanism shares the resource between the users, and hence would necessarily end up with low revenue if the available resource exceeds the high paying interest.

Our Results.

In this paper we adopt Kelly's fair sharing mechanism [8] for general polyhedral environment, and analyze it for both social welfare and revenue of the outcome. For the case of sharing a single resource, when there is a single constraint, say $\sum_i x_i \leq 1$, the fair sharing mechanism assumes that each player bids the amount w_i of money he wants to pay, and the mechanism shared the resource proportional to the payments, allocating $x_i = w_i/(\sum_j w_j)$ to player i. Johari and Tsitsiklis [6] consider this mechanism as a game. They show that the mechanism has a unique Nash equilibrium assuming the utilities U_i are concave. They also show that the Nash equilibrium is approximately efficient, showing that its social welfare is at least 3/4 times the social welfare of the most efficient allocation.

We extend the results of Johari and Tsitsiklis [6] to general polyhedral environments and also show that if there are multiple identical players then this efficiency ratio tends to 1 as the number of identical player increases. Further, we investigate the revenue generated by this game, and show that the game also approximately maximizes the revenue of the auctioneer, with the approximation ratio tending to 1 if players' utilities are linear and the number of identical players increases. In a more general class of utilities satisfying $U(2x) > \alpha U(x)$ for some constant $\alpha > 1$, the approximation ratio of the revenue will tend to $\alpha - 1$.

We also develop a new technique for analyzing games of these types. Our technique uses the the similarity between the condition of Nash equilibriums of the game and the dual of a certain linear program.

Our main theorem can be claimed more precisely as follows:

MAIN THEOREM *Given a constant $\alpha > 1$, and an integer $k \geq 2$, under the assumption that each player's utility satisfies $U(2x) > \alpha U(x)$ and for each player type, there are at least k players (defined formally in section 4), the fair sharing mechanism (defined in section 3) obtains both approximately maximum efficiency, and approximately maximum revenue. The efficiency is at least $(1 - \frac{1}{4k})$ times the optimal efficiency and the revenue is at least $(\alpha - 1)(1 - \frac{1}{k})(1 - \frac{1}{4k})$ the optimal revenue.*

Note that this bound is very strong when utility is linear (and so $\alpha = 2$). For this case we have the revenue of the mechanism is at least $f(k) = (1 - \frac{1}{k})(1 - \frac{1}{4k})$ times the optimal. Already when there are 2 players of each type (when $k = 2$) the mechanism achieves $\frac{7}{8}$ times the optimal efficiency and almost half of the maximum revenue. The following numbers are the approximation ratios of efficiency and revenue in the case of linear utility as k increases.

k	2	3	4	5	6	7
Revenue	0.436	0.611	0.703	0.76	0.8	0.82
Efficiency	0.875	0.91	0.937	0.95	0.958	0.964

Related Work.

Most closely related to our work are the mechanisms of Kelly [8] and Johari and Tsitsiklis [6] considering the social welfare of the fair sharing mechanism. We adopt this fair sharing game for polyhedral environments. The game requires all players to bid money on all the resources and allocates resources proportional to the money offered. Johari and Tsitsiklis consider the efficiency of the resulting allocation. We extend their results, and also consider revenue.

The fair sharing mechanism was motivated by the need for a simple and easy to implement mechanism for the resource sharing problem on the Internet. The proposals vary from using auctions to simple pricing, but they share the basic goal of maximizing social welfare. The idea is to implement a simple lightweight mechanism that helps arrange the socially optimal sharing of resources. Congestion pricing, first proposed by Shenker, Clark, Estrin, and Herzog [11], has emerged as a natural way to decide how to share bandwidth in a congested Internet. The fair sharing mechanism analyzed in this paper is a version of congestion pricing.

While maximizing social welfare is important to keep customers subscribed to the system, and to keep them happy, we believe that revenue should also be considered. Once a mechanism gets implemented, the network managers will try to take advantage of the users, and aim to maximize income, and will no longer only think of the mechanism as a way to arrange the best use of the network by maximizing social welfare. As a result, it is important that we also understand the revenue generating properties of the proposed mechanisms.

The well-known VCG mechanism, due to Vickrey, Clark and Groves is maximizing social welfare (see [10]) in very general settings. This mechanism is truthful, the players report their utility functions to the auctioneer who then decides on the allocation and payments to maximize social welfare. In the context of players with complex utility functions, one difficulty with the VCG mechanism is that it requires users to communicate their whole utility function. Recent papers [7, 13, 14] implement the VCG outcome via a simple mechanism that is analogous to the fair sharing mechanism (though it is less natural). These papers focus on maximizing social welfare, and do not consider revenue.

The issue of maximizing revenue in auctions has been most widely considered in the context of Bayesian games, see for example [9]. In the context of truthful auctions the issue of revenue has been most considered for digital goods, see for example Hartline and Karlin [4, 5].

The fact that properties of some systems improve as the number of users increases has been previously considered in other settings. Edgeworth [2] considers an exchange economy, where users come to the market with a basket of goods and aim to exchange the goods to maximize their utility. He was comparing the concept of Walrasian competitive equilibrium to the notion of the core in this setting. For an exchange economy a competitive allocation is an allocation resulting from market clearing prices p, where all players sell at price p and use the resulting money to buy their optimal set of goods. The core of the exchange economy game is an allocation of goods where no subset of users can re-contract using their initial allocation to improve at least one user's utility without decreasing the utility of any of them. It is not hard to see that all competitive allocations are in the core of the exchange game, but in general the core has other allocations that are not supported by prices. Edgeworth [2] showed that with two different players if the market contains many copies of each player, the set of core allocations converges to the competitive allocation as the number of players grows. More generally, the core in exchange economies with many (small) players is known to converge to the competitive allocations, see Anderson [1] for a survey.

Organization of the paper:.

In section 2 we define a class of resource allocation games, and show some example of games in this class. In section 3 we describe the mechanism in the polyhedral environment introduced in section 2. Section 4 discuss the bound on the revenue and the efficiency of this game.

2. GAMES WITH POLYHEDRAL DOMAIN

The resource allocation problem defined in the introduction has n players, where each player i is to receive an amount of resource x_i. Player i has a concave and monotone utility function $U_i(x)$ for receiving x amount, such that $U_i(0) = 0$. We assume that all players have linear and separable utility for money, and if they have to pay w_i for receiving x_i amount of the resource, they value this at $U_i(x_i) - w_i$.

We call the vector of amounts $x = (x_1, .., x_n) \geq 0$ an *allocation.* Allocations have to satisfy certain constraints, and we say that an allocation x is *feasible* if it is possible to allocate x_i to each player i simultaneously.

DEFINITION 1. *We say that the* resource allocation problem is polyhedral *if there exist a non-negative matrix A and a vector u, such that feasible allocations are the nonnegative vectors $x \geq 0$ satisfying $Ax \leq u$*

The simplest polyhedral allocation problem is bandwidth sharing, where the players share 1 unit of bandwidth, and the constraint for feasibility is that $\sum_i x_i \leq 1$. A more general version of this example is the following bandwidth sharing problem.

Example 1: Bandwidth Sharing.

Polyhedral environments model bandwidth sharing in a network where each user i is sending traffic along a path P_i and x_i is the amount of traffic user i can send along P_i. In this case we have a resource constraint associated with each edge $\sum_{i:e \in P_i} x_i \leq u_e$.

In more general bandwidth sharing, different users can be using the resources at different rates. The rows of the constraint matrix A correspond to different resources, and the coefficients indicate the rate at which the users use the resource.

Example 2: Games with finite outcome sets.

Another interesting example of games in polyhedral environments is the following. Consider a game with a finite set of outcomes, each of which is expressed by an allocation vector x. Let $\{x^1, x^2, .., x^N\}$ be the set of possible outcomes. We consider a game by also allowing mixed outcomes, a probability distribution of the basic outcomes. Choosing between the basic allocations by the probability distribution p we get that the vector of expected allocations to the players is $\sum_j p_j x^j$. Now the set of expected allocation vectors obtained this way is exactly the convex hull $conv(x^1, \ldots, x^N)$:

$$conv(x^1, \ldots, x^N) = \{\sum_j p_j x^j \mid p \geq 0 \text{ and } \sum_j p_j = 1\}.$$

This convex set may not define a polyhedral environment in our sense, as we required that the matrix A has nonnegative coefficients.

In many settings, given a feasible allocation (or expected allocation) x there is a method to reduce the allocation of a player. Therefore we will also consider all the vectors y such that $y \leq x$ feasible allocations. If this is the case, the set of feasible allocations is the following:

$$S = \{y \geq 0 \mid \exists x \in conv(x^1, .., x^N) \text{ s.t } y \leq x\} \qquad (1)$$

This convex set now satisfies the condition of Definition 1, as shown by the following Theorem, that can be proved using standard techniques in convex geometry. We provide a proof in the Appendix.

THEOREM 1. *A set S can be defined by (1) for some non-negative vectors $x^1, .., x^N$ if and only if S is bounded and there exist a non-negative matrix A and a non-negative vector u such that $S = \{x \mid x \geq 0; Ax \leq u\}$.* □

Note that this theorem only claims the existence of a matrix A. Finding such a matrix A algorithmically can be harder. In particular, the size of the matrix A may be exponential in both the dimension and the number of vectors x^j.

Example 3: Adwords as game in polyhedral environment.

An important example of an auction allocating quantities to players with a finite set of outcomes is the Adwords auction. The auction is for a single keyword, and the bidders are bidding to have their bid appear as a sponsored search result. There are finite set of outcomes, depending on which bidder gets displayed in which position. To think of the Adwords auction in our framework, we need to characterize the outcome for each player i with a single value x_i. We use the expected number of clicks the user i receives as his allocation x_i. It is reasonable to model the bidder's utility for this allocation, as a linear function $U_i(x_i) = a_i x_i$, where a_i is the expected revenue received for one click.

Traditional Adwords auctions use an ordering (outcome) that depends on the bids, and do not use randomization [12, 3]. To think of Adwords auctions as a polyhedral environment we need to allow randomization in the allocations of bidders to positions. Our model requires that in any feasible allocation x, one can decrease the allocation of a single player. We can maybe do this by introducing a dummy link on some positions.

Our description of the outcomes associates a nonnegative vector of click-through rates with each selection of bidders allocated to the k positions. This allows us to model some kinds of externalities between the bidders. The valuation of say a bidder for being in position 2 depends on what ad is showing in position 1. For example, NIKE is position 1 makes the value of position 2 less for a query of *sneakers* compared to having an unknown brand name in position 1.

Unfortunately, our allocation mechanism will rely explicitly on the matrix description $Ax \leq u$ of the feasible region, given by Theorem 1. For a general set of allocation vectors $x^1, \ldots, x^N$ the resulting constraints can be rather complex, and the fair sharing mechanism is not a natural auction mechanism for this case. For the case of Adwords auctions we view our results as a sort of "existence proof" that mechanisms that simultaneously maximize revenue and efficiency (in an approximate sense) do exist. We leave it as an open problem to decide if one of the simple and natural Adwords auction mechanisms has this property.

3. THE MECHANISM

In the previous section we introduce the resource allocation game in a polyhedral environment, and showed that this model captures a wide range of problems. We now describe the fair sharing mechanism for this class of games. The mechanism is an extension of the mechanisms introduced by Kelly [8], Johari and Tsitsiklis [6]. Let E denote the set of constraints (the rows of A). For simplicity of notation, we assume that $u_e = 1$ for each $e \in E$ by normalizing each row. We will use α^e to denote the row e of matrix A, which we will also call constraint e. We now have the following description of the set of feasible allocations:

$$\begin{aligned} \sum_i \alpha_i^e x_i \leq 1 \text{ for all } e \in E, \\ x_i \geq 0 \end{aligned} \tag{2}$$

The mechanism.

When sharing a single resource with constraint $\sum_i x_i \leq 1$ the fair sharing [8] mechanism requires that each player j submits a bid b_j, the amount of money she wants to pay, and the resource is allocated proportional to the bids, as $x_j = b_j / \sum_i b_i$. We can think of $\sum_i b_i$ as the unit price p of the good. The allocation is derived from this unit price, as user j gets $x_j = b_j/p$ amount for the cost $w_j = b_j$ at this price.

To extend this mechanism to a single constraint with coefficients $\sum_i \alpha_i x_i \leq 1$, we again require that each player j submit a bid b_j, her willingness to pay, and view $p = \sum_i b_i$ as the unit price of the good. Recall that α_j^e is the rate at which user j uses resource e, so user j needs $\alpha_j x_j$ allocation for a value x_j. At the unit price of p she gets $\alpha_j x_j = b_j/p$ allocation, and hence we need to set $x_j = b_j/(\alpha_j p) = \frac{b_j}{\alpha_j \sum_i b_i}$, and she will have to pay $w_j = b_j = \alpha_j x_j p$.

For environments with more constraints, Johari and Tsitsiklis [6] extends the fair sharing mechanism by requiring that users submit bids b_j^e separately on each resource e. As before, we can view the sum of bids $p^e = \sum_i b_i^e$ as the unit price of resource e, and allocate the resource at this price. This allocation limits the value x_j for user j to at most $x_j^e = b_j^e/(\alpha_j^e p^e)$. The idea is to ask users to submit bids b_j^e for each resource e, allocate the resources separately, make user j pay $w_j = \sum_e b_j^e$, and then set $x_j = \min_{\{e:\alpha_j^e \neq 0\}} x_j^e$.

We need to extend this mechanism to be able to deal with resources that are under-utilized. Some constraints e may not be binding at any solution, and the fair sharing method does not deal well with such constraints: users will want to bid arbitrary small amounts as there is too much of the resource. To deal with such constraints, we allow each player to request an amount r_j^e without any monetary bid. For each resource e if the price is 0 (that is $p^e = \sum_i b_i^e = 0$) and $\sum \alpha_i^e r_i^e \leq 1$ (the requested rates can all be satisfied) then we setting $x_j^e = r_j^e$ for all j.

The mechanism can be described formally as follows:

THE MECHANISM:

Each player j submit a bid b_j^e and a request r_j^e for each resource e. For resource e we use the following allocation:

- *If $\sum_i b_i^e > 0$ then $x_j^e = \frac{b_j^e}{\alpha_j^e(\sum_i b_i^e)}$ for $\forall j$.*
- *If $\sum_i b_i^e = 0$ and $\sum_i \alpha_i^e r_i^e \leq 1$ then $x_j^e = r_j^e$ for $\forall j$*
- *Else, set $x_j^e = 0$ for $\forall j$.*

For each player j, the amount of money that she need to pays is $w_j = \sum_e b_j^e$ and the final allocated $x_j = \min_{\{e:\alpha_j^e \neq 0\}} x_j^e$.

Simplifying assumption.

To simplify the presentation in this extended abstract we will assume that each resource e has at least two dedicated users who only needs resource e, and who have infinitesimally small, but linear utility. These users will guarantee that no resource is under-utilized, but will not change either the optimal allocation of the Nash equilibrium substantially. Using this assumption, we can never have $\sum_i b_i^e = 0$ for any resource e.

Condition for Nash equilibrium.

Next we analyze the condition for an equilibrium for this game. We will use these conditions to show that an equilibrium always exists. Consider a set of bids b_i^e, and a resulting allocation x, where player i gets allocation x_i. When is this allocation at equilibrium? For each resource e we use $p^e = \sum_i b_i^e$, the sum of the bids, as the unit price of the resource (recall that we normalized constraints, so there is 1 unit of every resource available).

Now consider the optimization problem of a player j assuming bids b_i^e for all other players are set. The player j is interested in maximizing her utility at $U_j(x_j) - \sum_e b_j^e$. At equilibrium, it must be the case that $x_j^e = x_j$ for all resources e that costs money, or otherwise player j can reduce her bid b_j^e without affecting her allocation. So we can think of the player's optimization problem as dependent on one variable x_j, the allocation she will receive. What bid does player j have to submit for a resource e to get allocation $x_j^e = x_j$? Bids must satisfy the following condition:

$$\text{If } b_j^e > 0 \text{ then: } \alpha_j^e x_j = \frac{b_j^e}{\sum_i b_i^e}.$$

Assuming all other bids b_i^e are fixed, we can express the bid b_j^e needed as follows.

$$b_j^e(x_j) = \frac{\alpha_j^e x_j \sum_{i \neq j} b_i^e}{1 - \alpha_j^e x_j}.$$

Note that this expression assumes that $\alpha_j x_j < 1$, that is, j is not the only user of the resource at equilibrium. It is not hard to see that this is guaranteed by having at least two dedicated users for each resource.

User j will want to choose x_j to maximize her utility. For this end, it will useful to express the derivative of the bid b_j^e when viewed as a function of x_j. We get the following (again assuming $\alpha_j x_j < 1$):

$$\frac{\partial}{\partial x_j} b_j^e(x_j) = \frac{\alpha_j^e \sum_{i \neq j} b_i^e}{(1 - \alpha_j^e x_j)^2}.$$

Substituting $\sum_{i \neq j} b_i^e = p^e(1 - \alpha_j^e x_j)$ and simplifying we get that

$$\frac{\partial}{\partial x_j} b_j^e(x_j) = \frac{p^e \alpha_j^e}{1 - \alpha_j^e x_j}.$$

Note that a derivative of $p^e \alpha_j^e$ would correspond to a fixed price p^e on resource e, as increasing x_j increases the use of this resource at the rate of α_j^e. In the allocation game, the price is a function of the bids, and his induces the players to "shade" their bid for the resource by a factor that depends on their share of the resource.

Now consider the optimization problem of player j. She wants to maximize her utility $U_j(x_j) - \sum_e b_j^e$, which can now be expressed as

$$U_j(x_j) - \sum_e \frac{\alpha_j^e x_j \sum_{j \neq i} b_i^e}{1 - \alpha_j^e x_j},$$

as a function of the single variable x_j. Note that this is a concave function of x_j. The maximum occurs at a value x_j, where the derivative of this function 0, or if the derivative is negative everywhere, maximum occurs at $x_j = 0$. Using the derivatives we computed above, we get the derivative of user jth utility as a function of her allocation x_j to be

$$U_j'(x_j) - \sum_e \frac{p^e \alpha_j^e}{(1 - \alpha_j^e x_j)}$$

This derivative is a strictly decreasing function, so we have the following Nash condition, which will be used in the analysis of the game.

$$\begin{aligned} & x \text{ is a Nash solution if and only if:} \\ & \sum_i \alpha^e x_i \leq 1; \quad x_i \geq 0 \text{ for all } e \in E \\ & U_j'(x_j) = \sum_e \frac{p^e \alpha_j^e}{(1 - \alpha_j^e x_j)}, \text{ if } x_j > 0 \text{ and} \\ & U_j'(0) \leq \sum_e p^e \alpha_j^e \text{ if } x_j = 0. \end{aligned} \tag{3}$$

Given these equilibrium conditions, we can extend the result of Johari and Tsitsiklis [6] to prove that the game always has a Nash equilibrium.

THEOREM 2 (JOHARI-TSITSIKLIS). *If the utility function of each player is increasing, differentiable and concave, then there always exists a Nash equilibrium. Further more if an allocation x, a bid and a request vector b, r is a Nash equilibrium then they satisfy the condition (3).* □

To evaluate the outcomes of the game, we will compare the social welfare and the revenue with the optimal social welfare, which can be written as an optimum of the following a linear program.

$$\begin{aligned} \max \quad & OPT = \sum_{i=1}^n U_i(x_i) \\ \text{subject to} \quad & \sum_i \alpha_i^e x_i \leq 1; \quad \forall e \in E \\ & x_i \geq 0. \end{aligned} \tag{4}$$

4. REVENUE AND EFFICIENCY

Next we analyze the revenue and efficiency of a Nash equilibrium. We will use the maximum social welfare OPT to measure both efficiency and revenue. Note that OPT is also an upper bound on the revenue.

As already mentioned in the introduction, we need to make two assumptions to be able to get a reasonable bound on the revenue. First we assume that the players' utility functions grow at a reasonably steady rate. Second, we assume that there are at least $k \geq 2$ players of each type.

ASSUMPTION(α, k)

- *The utility function $U_j(x)$ of all users j is non negative, increasing, differentiable, concave, further, U_i satisfies: $U_i(2x) \geq \alpha U_i(x)$ for some $\alpha > 1$.*
- *We say that the* type *of a player j is her utility function $U_j(x)$ and the rate at which she needs the resources, the coefficients α_j^e for all resources e. We assume that there are at least k players of every type: that is for every player j, there are at least $(k-1)$ other players with the same type.*

Notes.

The class of functions we consider captures the linear functions and contains small degree polynomials: $U_j(x) = x^\epsilon$ for any $0 \leq \epsilon \leq 1$.

In the context of bandwidth sharing, the second assumption means that for each player j, there are at least $k-1$ other players with the same utility function and the same path.

We assumed that both the utility and the rate of use is exactly the same for all of the k users. We can relax this conditions to requiring only similar utilities and similar rates, by loosing an additional factor in the approximation bounds for the coefficient of this similarity.

The main result of our paper is the following:

THEOREM 3. *Under Assumption(α, k), the mechanism defined in section 3 approximately maximizes both efficiency and revenue. The loss of efficiency is bounded by a fraction of $\frac{1}{4k}$ and the revenue is at least $(\alpha - 1)(1 - \frac{1}{k})(1 - \frac{1}{4k})$ times the optimal revenue.*

We bound the revenue in two steps. We compare the revenue to the social welfare obtained at equilibrium: $\sum_i U_i(x_i)$, and then compare this value with the optimum social welfare OPT. The final bound is the product of these two ratios.

To make the proof easy to follow, we first consider the simple case of the game, where there is only one constraint on the feasible allocation set. We extend the result to the general case in Section 4.2 using linear programming duality.

4.1 Proof for a simple case

Consider the simplest version of the game, where we have only one constraint for the feasible allocations set: $\sum_i x_i \le 1$. This is the basic bandwidth sharing problem considered also by Johari and Tsitsiklis [6]. Johari and Tsitsiklis show that the social welfare at Nash is at least 3/4 times the optimum. We improve their bound from 3/4 to $(1 - \frac{1}{4k})$ using our assumption. To bound the revenue, we show a bound of $(\alpha - 1)(1 - \frac{1}{k})$ on the ratio between revenue and social welfare at an equilibrium. We start with the second bound.

THEOREM 4. *With Assumption(α, k), in the game whose feasible allocation set is $x \ge 0$ such that $\sum_i x_i \le 1$, the ratio between revenue and social welfare at the Nash equilibrium is at least $(\alpha - 1)(1 - \frac{1}{k})$.*

PROOF. Let x be a Nash equilibrium. According to condition (3) from the previous section, we get:

$$\text{Either } x_i = 0 \text{ or } U_i'(x_i) = \frac{p}{1 - x_i},$$

where we use again the notation that $p = \sum_j b_j$. We can write the bid b_i of player i as $b_i = px_i$. Thus, in both cases we have:

$$b_i = px_i = U_i'(x_i)(1 - x_i)x_i$$

Therefore the revenue of the game, which is the sum of all the bids, is: $\sum_i b_i = \sum_i U_i'(x_i)(1 - x_i)x_i$.

We need to compare this revenue with $\sum_i U_i(x_i)$, the social welfare at the Nash equilibrium. The ratio is

$$\frac{\sum_i U_i'(x_i)(1 - x_i)x_i}{\sum_i U_i(x_i)} \le \max_i \frac{U_i'(x_i)(1 - x_i)x_i}{U_i(x_i)}.$$

We bound this ratio by bounding the individual ratios on the right hand side.

Now, consider two identical players i and j, with identical utility functions $U_i(x) = U_j(x)$. We claim that identical players get equal share of the resource at a Nash equilibrium. To see why, recall that by (3) we either have $x_i = x_j = 0$ (if $U_i'(0) = U_j'(0) < p$) or x_i and x_j are both the unique solution to $p = U_i'(x)(1 - x)$.

Hence for every i, there will be at least $k - 1$ other players getting the same allocation x_i, and hence

$$kx_i \le 1 \Rightarrow (1 - x_i) \ge 1 - \frac{1}{k}.$$

Consider now $\frac{U_i'(x_i)x_i}{U_i(x_i)}$. The utility function U_i is concave function, and hence the slope of $U_i'(x_i)$ is greater than the slope of the line connecting the points $(x_i, U_i(x_i))$ and $(2x_i, U_i(2x_i))$. (See Figure 1). Thus we have the following inequality:

$$U_i'(x_i) \ge \frac{U_i(2x_i) - U_i(x_i)}{2x_i - x_i}.$$

Due to *Assumption(α, k)*, we have $U_i(2x_i) \ge \alpha U_i(x_i)$, thus:

$$U_i'(x_i) \ge (\alpha - 1)\frac{U_i(x_i)}{x_i} \Rightarrow \frac{U_i'(x_i)x_i}{U_i(x_i)} \ge (\alpha - 1)$$

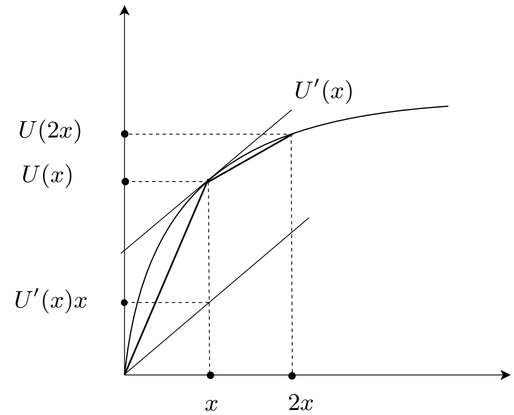

Figure 1: The utility function

Therefore for every player i, the following inequality holds:

$$\frac{U_i'(x_i)(1 - x_i)x_i}{U_i(x_i)} \ge (1 - \frac{1}{k})(\alpha - 1)$$

This inequalities imply that the ratio between the total revenue and the social welfare of this game is also at least $(1 - \frac{1}{k})(\alpha - 1)$. $\square$

Next we bound the ratio between the social welfare of the Nash equilibrium and the optimal social welfare OPT. Johari – Tsitsiklis [6] give a 3/4 bound on this ratio. Using *Assumption(α, k)*, we can improve this bound to $(1 - \frac{1}{4k})$.

THEOREM 5. *Under Assumption(α, k) the fair-sharing mechanism for the simple resource sharing problem $\sum_i x_i \le 1$ obtains a solution with the social welfare at least $(1 - \frac{1}{4k})$ times the optimal.*

PROOF. We will show the result for the special case of pure linear utility functions $U_i(x) = a_i x$ for every players i. Johari – Tsitsiklis [6] showed that the worst social welfare ratio occurs in the case of pure linear utilities with $a_i = U_i'(x_i)$, where x_i is the allocation of player i at the Nash equilibrium. For completeness we include a sketch of the proof in the Appendix.

The maximum social welfare is the optimum of $\sum_i a_i x_i$ where $\sum_i x_i \le 1$. Thus it is equal to $OPT = \max_i a_i$. Let's assume that $a_1 = \max_i a_i$. From the Nash condition:

$$U_i'(x_i) = a_i = \frac{p}{(1 - x_i)} \text{ if } x_i > 0$$

one obtains: if $x_i > 0$ then $a_i > p$ and $a_1(1 - x_1) = p$, and thus if $x_i > 0$ then $a_i > a_1(1 - x_1)$.

By *Assumption(α, k)*, in the original game there are at least k players who have the same utility function as player 1 and hence they get the same allocation x_1 (as the value at equilibrium is unique). These players provide a total utility that is at least ka_1x_1 and all other players fill out the bandwidth of 1, so they have total share of $1 - kx_1$ and have utility coefficients $a_i \ge a_1(1 - x_1)$. This gives us a total utility of at least

$$\sum_i a_i x_i \ge ka_1x_1 + a_1(1 - x_1)(1 - kx_1).$$

Hence the ratio between social welfare at the Nash equilibrium x and the optimum one is:

$$\frac{\sum_i a_i x_i}{a_1} \geq \frac{ka_1x_1 + a_1(1-x_1)(1-kx_1)}{a_1} = 1 - x_1 + kx_1^2.$$

This expression is minimized when $x_1 = 1/(2k)$ when the ratio is $1 - 1/(4k)$ as claimed. □

4.2 Proof of the main theorem

Now we prove the theorem in the general setting, which is a nontrivial extension of the previous result. To do this we need to extend the result to handle multiple resources, and different rates at which players use these resources. Our main tool is linear programming duality.

First we show that under $Assumption(\alpha, k)$, the ratio between revenue and social welfare at Nash equilibrium is bounded by $(\alpha - 1)(1 - \frac{1}{k})$. Consider a Nash equilibrium x. Recall that the equilibrium is not know to be unique in the general case. However, players of identical type must get identical allocation.

LEMMA 6. *If two players i and j have the same type, then in any Nash equilibrium, they get the same allocation.*

PROOF. By the Nash equilibrium conditions (3) both x_i and x_j are 0 if $U_i'(0) = U_j'(0) < \sum_i \alpha_i^e p^e$ and otherwise both are the unique solutions equation of Nash in (3). □

The main observation that allows us to use linear programming is that the condition for a Nash equilibrium (3) is closely related to the dual of a certain linear program related to the Optimum social welfare (4). Recall from the proof of Theorem 4 that because of the assumption about the utility functions, we have: $U_i'(x_i)x_i \geq (\alpha - 1)U_i(x_i)$. We will consider the following primal-dual programs, where $a_i = U_i'(x_i)$:

$$\begin{aligned}
PRIMAL: \quad & \max \sum_{i=1}^n a_i \tilde{x}_i \\
\text{s.t:} \quad & \sum_i \alpha_i^e \tilde{x}_i \leq 1; \;\; \forall e \in E \text{ and } \tilde{x}_i \geq 0. \\
DUAL: \quad & \min \sum_e \tilde{y}_e \\
\text{s.t:} \quad & \sum_e \alpha_i^e \tilde{y}_e \geq a_i \;\; \forall i \text{ and } \tilde{y}_i \geq 0.
\end{aligned}$$

LEMMA 7 (WEAK DUALITY). *If $\tilde{x}$ and $\tilde{y}$, are feasible solutions of the primal and the dual programs above, then $\sum_e \tilde{y}_e \geq \sum_i a_i \tilde{x}_i$* □

THEOREM 8. *Under $Assumption(\alpha, k)$, the ratio between revenue and social welfare at Nash is bounded by $(\alpha - 1)(1 - \frac{1}{k})$.*

PROOF. We want to show that the prices p^e of the resources form a dual solution if we scale them by a factor $(1 - \frac{1}{k})$. The dual objective is $\sum_e p^e$ which is exactly the revenue. All feasible dual solutions have value that upper bounds the primal objective. We will get the claimed bound by considering the scaling needed to make p^e dual feasible, and the relation of $U_i(x_i)$ to the linear objective $U_i'(x_i)x_i$.

In a Nash equilibrium identical players receive the same allocation (by Lemma 6). Because of $Assumption(\alpha, k)$, for every i there are at least $k - 1$ other players with the same resource constraint getting the same value x_i. Therefore: $1 = \sum_j \alpha_j^e x_j \geq k\alpha_i^e x_i$, and so $1 - \alpha_i^e x_i \geq 1 - \frac{1}{k}$. Hence, from the Nash condition (3), one obtains:

$$U_i'(x_i) \leq \sum_e \alpha_i^e \frac{p^e}{1 - \alpha_i^e x_i} \leq \sum_e \alpha_i^e \frac{p^e}{1 - \frac{1}{k}} \;\; , \forall i. \qquad (5)$$

Now, let $a_i = U_i'(x_i)$ and $y_e = \frac{p^e}{1-\frac{1}{k}}$. The inequality (5) shows that the vector y is a feasible solution of the DUAL program. The Nash allocation x is a feasible allocation, and therefore, a feasible solution for the PRIMAL program. By Lemma 7 we have:

$$\sum_e \frac{p^e}{1 - \frac{1}{k}} = \sum_e y_e \geq \sum_i a_i x_i = \sum_i U_i'(x_i)x_i$$

Now we can bound the revenue as follows:

$$\sum_e p^e \geq (1 - \frac{1}{k}) \sum_i U_i'(x_i)x_i \geq (\alpha - 1)(1 - \frac{1}{k}) \sum_i U_i(x_i),$$

where the last inequality follows from:
$U_i'(x_i)x_i \geq (\alpha - 1)U_i(x_i)$, as explained above the linear program. □

Next we consider efficiency. Here we use the improved bound of Theorem 5 for the simple case, and again use linear programming duality to bound the optimal value.

THEOREM 9. *Under $Assumption(\alpha, k)$ the ratio between social welfare at Nash and the optimal one is at least $1 - \frac{1}{4k}$.*

PROOF. We will get the bound from Theorem 5 and linear programming duality by considering separate games for each resource e, along the lines of the proof of Johari and Tsitsiklis [6].

Consider a Nash equilibrium x. As before we can assume without loss of generality that the utility function is linear $U_i(x) = a_i x$ for all players, as was shown by [6] (see the proof in the Appendix for completeness). We use $a_i = U_i'(x_i)$.

With this assumption, social welfare at Nash is $\sum_i U_i'(x_i)x_i$, and the optimal social welfare is the optimal value OPT of the PRIMAL program where $a_i = U_i'(x_i)$.

Consider the Nash condition (3):

$$U_j'(x_j) = \sum_e \frac{p^e \alpha_j^e}{(1 - \alpha_j^e x_j)}, \text{ if } x_j > 0 \text{ and}$$
$$U_j'(0) \leq \sum_e p^e \alpha_j^e \text{ if } x_j = 0.$$

We will consider a separate game for each resource e. In the game corresponding to resource e player i is interested in getting an allocation z_i^e with the constraint $\sum_j z_j^e \leq 1$, and a linear utility function $v_i^e z_i^e$, where $v_i^e = \frac{p^e}{(1-\alpha_i^e x_i)}$. If we set $z_i^e = \alpha_i^e x_i$ then by (3) the allocation vector z^e forms a feasible allocation at equilibrium, with total utility $\sum_i v_i^e z_i^e$.

We want to apply Theorem 5 for each resource e. To be able to do this, we need to see that the new game also satisfies $Assumption(\alpha, k)$. To see why, note that players of identical type will also have identical v_i^e values (as identical players get the same allocation) and hence remain of identical type in the new game.

Now by Theorem 5, the social welfare of each game e at Nash is at least $(1-\frac{1}{4k})$ times the optimal one:

$$\sum_i v_i^e z_i^e \geq (1-\frac{1}{4k})\max_i\{v_i^e\} = (1-\frac{1}{4k})v^e, \qquad (6)$$

where we use $v^e = \max_i\{v_i^e\}$.

Summing the left hand sides over all resources and substituting the values for z_i^e we get:

$$\sum_e\sum_i v_i^e z_i^e = \sum_e\sum_i v_i^e\alpha_i^e x_i = \sum_i x_i\sum_e v_i^e\alpha_i^e.$$

Now, $\sum_e v_i^e\alpha_i^e = \sum_e \frac{p^e\alpha_i^e}{(1-\alpha_i^e x_i)}$, and due to condition (3) of Nash equilibrium, this value is equal to $U_i'(x_i)$ unless $x_i = 0$. Therefore:

$$\sum_i x_i\sum_e v_i^e\alpha_i^e = \sum_i U_i'(x_i)x_i.$$

Combining this with the bound (6) we get:

$$\sum_i U_i'(x_i)x_i \geq (1-\frac{1}{4k})\sum_e v^e$$

We now use the weak duality Lemma 7 to prove that $\sum_e v^e$ is at least the Optimal social welfare, and this will establish the theorem. Because $v^e = \max_i v_i^e$, we have:

$$\sum_e \alpha_i^e v^e \geq \sum_e \alpha_i^e v_i^e = \sum_e \frac{p^e\alpha_i^e}{(1-\alpha_i^e x_i)} \geq U_i'(x_i) \ \ \forall i.$$

The last inequality is due to the condition of Nash equilibriums. This inequality shows that v^e is a feasible solution of the DUAL program where $a_i = U_i'(x_i)$. Therefore $\sum_e v^e$ is at least the optimum of the PRIMAL program OPT, which is the optimal social welfare of the game. □

Combining the bound on revenue in Theorem 8 and the bound on efficiency above, Theorem 3 is proved.

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APPENDIX

PROOF OF THEOREM 1. Given a set

$$S = \{x | x \geq 0; Ax \leq u\},$$

let $x^1,..,x^N$ be the extreme points of S. Because S is bounded we have $S = conv(x^1,..,x^N)$. However, A is non-negative, therefore $x \in S$ implies $y \in S \ \forall \ \ 0 \leq y \leq x$. Thus:

$$S = \{y \geq 0 | \exists x \in conv(x^1,..,x^N) s.t \ \ y \leq x\}$$

We now prove the other direction. Given S defined as above, we will show that for every $y \notin S$, there exists a *non-negative* vector a such that $a^T x \leq 1 < a^T y \ \forall x \in S$. Once we have that, with the observation that the set of the extreme points of S is finite, (since S is defined by a finite set of points $x^1,..,x^N$), we can conclude that there exists a non-negative matrix A and non-negative vector u such that $S = \{x | x \geq 0; Ax \leq u\}$.

Now, we show that there exists such an non-negative vector a for every $y \notin S$. Because of the definition of S: if $x \in S$ then any non-negative vector less or equal to x is also in S, we have: $y \notin S$ implies $y + z \notin S \ \forall z \geq 0$. Both S and $\{y + z | z \geq 0\}$ are convex, furthermore S is compact, there exists a vector a such that $a^T x \leq 1 < a^T(y + z)$; $\forall x \in S$ and $\forall z \geq 0$. But because $a^T(y + z) > 1 \forall z \geq 0$, a has to be a non-negative vector. By this we finished the proof. □

LEMMA 10. *[6] Given an instance of the game with the concave utility U_i, and let x be solution satisfying the Nash condition. Consider the game where the utility U_i is replaced by the function $W_i(z) := U_i'(x_i)z$. The allocation x still satisfies the Nash condition in the new game and the ratio between social welfare at Nash and the optimal does not increase.*

PROOF. We first modify the utility function U_i to the linear function V_i with the slope $U_i'(x_i)$ such that $V_i(x_i) =$

$U_i(x_i)$. That is $V_i(z) = U_i'(x_i)(z - x_i) + U_i(x_i))$. See figure 2. Because the derivative of the new utility function at x_i

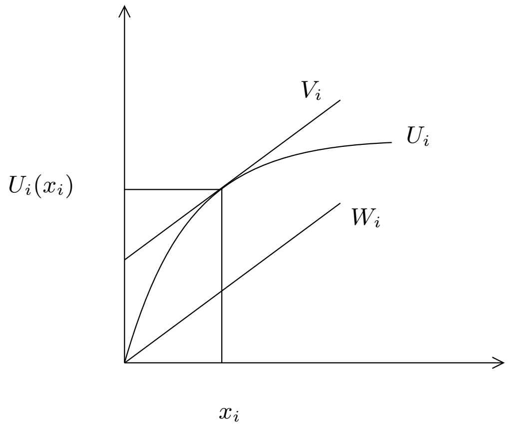

Figure 2: New utility functions

does not change, therefore x still satisfies the Nash condition of the new game. Furthermore, the social welfare of the solution x stays the same. On the other hand, because U_i is concave, thus $V_i(z) \geq U_i(z)\ \forall z$, therefore the new optimal social welfare can only increase. As a result, in the modified game, the ratio between Nash and Optimal social welfare does not increase.

Next we consider new utilities W_i obtained by shifting V_i to the origin. That is $W_i(z) = U_i'(x_i)z$. The difference between V_i and W_i is a constant. Let c be the sum of these differences over all i. If the N and O are respectively the Nash and the optimal social welfare of the game with utility V_i, then the Nash and optimal social welfare of the game with utility functions W_i are $N - c$ and $O - c$, respectively. Since we know $N \leq O$ and $0 \leq c \leq \min\{N, C\}$, we have:

$$\frac{N}{O} \geq \frac{N - c}{O - c},$$

which shows that the ratio also decreases. By this we finished the proof. $\square$