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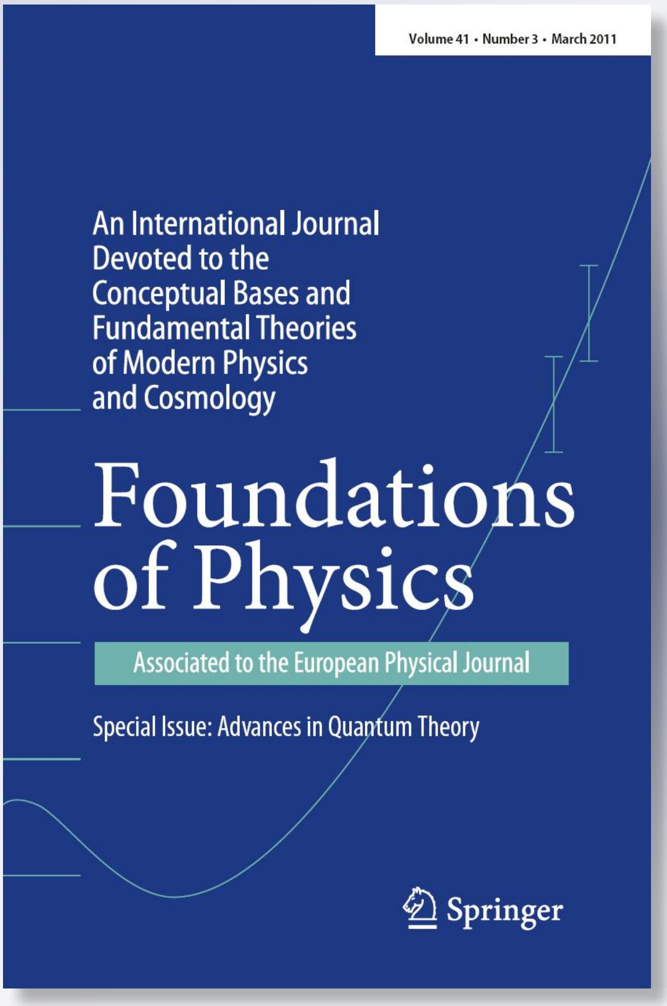

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On the Reasonable and Unreasonable Effectiveness of Mathematics in Classical and Quantum Physics

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Abstract The point of departure for this article is Werner Heisenberg's remark, made in 1929: "It is not surprising that our language [or conceptuality] should be incapable of describing processes occurring within atoms, for . . . it was invented to describe the experiences of daily life, and these consist only of processes involving exceedingly large numbers of atoms. . . . Fortunately, mathematics is not subject to this limitation, and it has been possible to invent a mathematical scheme—the quantum theory [quantum mechanics]—which seems entirely adequate for the treatment of atomic processes." The cost of this discovery, at least in Heisenberg's and related interpretations of quantum mechanics (such as that of Niels Bohr), is that, in contrast to classical mechanics, the mathematical scheme in question no longer offers a description, even an idealized one, of quantum objects and processes. This scheme only enables predictions, in general, probabilistic in character, of the outcomes of quantum experiments. As a result, a new type of the relationships between mathematics and physics is established, which, in the language of Eugene Wigner adopted in my title, indeed makes the effectiveness of mathematics unreasonable in quantum but, as I shall explain, not in classical physics. The article discusses these new relationships between mathematics and physics in quantum theory and their implications for theoretical physics—past, present, and future.

Keywords Classical physics · Quantum mechanics · Epistemology · Mathematical formalism · Prediction vs. description · Probability

1 Introduction

The title of this article is a paraphrase of the title of Eugene Wigner's famous paper, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," a phrase

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that refers primarily to physics, my main concern here as well [1]. My argument, which is different from that of Wigner and which contends in particular that this effectiveness is unreasonable only in quantum but not in classical physics, takes as its point of departure the following remark by Werner Heisenberg in his Chicago Lectures, *The Physical Principles of The Quantum Theory*, given in 1929 and published in 1930. According to Heisenberg: "It is not surprising that our language [or conceptuality] should be incapable of describing processes occurring within atoms, for ... it was invented to describe the experiences of daily life, and these consist only of processes involving exceedingly large numbers of atoms. Furthermore, it is very difficult to modify our language so that it will be able to describe these atomic processes, for words can only describe things of which we can form mental pictures, and this ability, too, is a result of daily experience. Fortunately, mathematics is not subject to this limitation, and it has been possible to invent a mathematical scheme—the quantum theory [quantum mechanics]—which seems entirely adequate for the treatment of atomic processes" [2, p. 11].

Mathematics' freedom from this limitation and Heisenberg's invention of quantum mechanics, which took advantage of this freedom, had an epistemological consequence and, many would say, a cost. For, at least in Heisenberg's and related interpretations of quantum mechanics, such as and in particular that of Niels Bohr, known as complementarity, the mathematical formalism of quantum mechanics, in contrast to that of classical mechanics or relativity, no longer offers a description of quantum objects and processes themselves, however idealized such a description may be. (Theoretical physics, beginning with classical mechanics, generally considers idealized and mathematized models of physical processes in nature, rather than these processes as they actually occur.) Indeed, as both Bohr and Heisenberg argue, unlike quantum mechanics (and in some respects, relativity), classical physics may be seen as a suitably mathematized refinement of our representations and descriptions of "the experiences of daily life" (e.g., [2, p. 11]; [3, v. 2, pp. 69–70]; [4, pp. 56, 91–92]).

From this perspective, the effectiveness of mathematics in classical physics is not unreasonable, at least no more unreasonable than our capacity to develop mathematics itself. For, in this case, we use our mathematical idealization of certain features of objects and motions phenomenally perceived by us, and disregard those features of both that are not thus mathematically idealized. As used by the ancient Greeks, (Euclidean) geometry was, arguably, the first science of this kind: it was the science of space and spatial measurements (geo-*metry*). Indeed, until the rise of what can be called modern mathematics, more abstract in nature, in the second half of the nineteenth century (just in time for quantum mechanics), mathematics itself was closely connected to classical physics in its functioning just described.

By contrast, again, in the interpretations just mentioned, the quantum-mechanical formalism (which is mathematically different from that of classical mechanics, in part by virtue of its more abstract nature) only enables correct predictions of the outcomes of quantum experiments, defined by the interactions between quantum objects and measuring instruments. Moreover, as against those of classical mechanics, these predictions are in general probabilistic. However, they properly correspond to what is observed in quantum experiments, since identically prepared experiments in general lead to different outcomes. Crucially, unlike in classical physics, this difference is not a matter of the accuracy of our measuring instruments, at least not beyond the limit defined by Planck's constant, h. At least as things stand now, no improvement of our experimental technology, not even if we had ideal instruments, would allow us to override this limit, which is correlative to the uncertainty relation, $\Delta q \Delta p \cong h$. In classical physics, if we have good instruments, experimental errors become negligible and can be ideally neglected, and repeated experiments can be considered as having the same outcomes and, correlatively, as possible to predict (ideally) exactly. In quantum physics, this is not possible, at least, again, as things stand now, which make the recourse to probabilities unavoidable.

As a result, as Bohr noted in the wake of Heisenberg's introduction of quantum mechanics in 1925, an entirely new type of relationships between mathematics and physics (vis-à-vis that found in classical physics) is established [3, v. 1, p. 51]. Indeed, both, and correlatively, mathematics and physics are different. Given the difference between classical and quantum physics, just outlined, one might also argue that, while the effectiveness of mathematics is not unreasonable in classical physics, this effectiveness may be seen, in accordance with Wigner's contention, as unreasonable in quantum physics, in the following sense. We may not be able to offer the same physical reasons as in classical physics, or possibly any conceivable reasons, for why the mathematics of quantum theory works, and works so effectively. This effectiveness is enigmatic.

While the new epistemological situation in physics that quantum phenomena and quantum mechanics gave rise to has been seen as scientifically productive and intellectually liberating to some, Bohr and Heisenberg among them, it has been troublesome and even unacceptable to many others, including Erwin Schrödinger, and, most prominently, Albert Einstein. Those from this second group would question, at least on philosophical grounds, Heisenberg's contention that "a mathematical scheme—the quantum theory [quantum mechanics]—... *seems* entirely adequate for the treatment of atomic processes," and they would welcome Heisenberg's cautious "seems." Unlike in the case of Heisenberg, Schrödinger's invention of the same or, in any event, a mathematically equivalent formalism (in the course of his attempt to develop, as an alternative to Heisenberg's theory, a wave quantum mechanics) was a product of a more classical way of thinking. Schrödinger, however, eventually came to accept (at least until late in his life) that it would be difficult and perhaps impossible to physically interpret this formalism along the classical epistemological lines [5].

The debate concerning this (im)possibility has defined the history of quantum mechanics and its interpretation ever since, and, still as intense as ever, this debate does not appear likely to end anytime soon. The reasons for the apparently interminable character of this debate is a worthy subject of reflection in its own right, and it has thus far received far less attention than it deserves, in part undoubtedly because our philosophical positions tend to govern our interests and concerns, a predicament difficult to avoid or to be certain to avoid. Accordingly, while I cannot be certain that I will avoid this predicament either, and while I do find the radical epistemology in question more productive than inhibiting, it is not my aim to deny that there might be good reasons for alternative positions concerning the situation in question. Such alternatives, for example, those based on Einstein's more classically oriented view, may even ultimately prove to provide better ways for understanding how nature works. On the other hand, it is also possible that nature does require from physics this type of epistemology or even something more radical. Given that our present theories of the ultimate constitution of nature are manifestly incomplete (we do not know, for example, whether this constitution is quantum in the current sense of the term), it is difficult to predict what the future holds in store for us. On the other hand, it may be instructive to revisit the past, which I would like to do here before I proceed with my main argument concerning quantum theory in Sect. 4 of this article. First, in Sect. 2, I travel back about 2500 years, to the pre-Socratics and then Plato and Aristotle, and, second, in Sect. 3, I discuss the epistemological underpinnings of classical physics.

2 Physics, Mathematics, and Epistemology in Ancient Greek Thought

The earliest known pre-Socratics, such as Thales, Anaximander, Democritus, and Heraclitus, may be seen (at least in our reconstitution of their thought) not only as philosophers but also as *physicists*. That is, they were observers and describers of nature—of things and motions found there. Thales was also a mathematician and is sometimes credited with having proved the first mathematical theorem and even with establishing the idea of the mathematical proof. *Physis* was the Greek word for nature, both as it would appear to our phenomenal perception and as the possibly hidden way in which things and motions exist. Thales believed that water was the origin of all things. Heraclitus, a great philosopher of becoming, the all-pervasiveness of which, he famously said, would prevent us from ever being able to step into the same river twice, went further when he said: "Nature [*physis*] loves to hide" [6, p. 106; Fragment 10]. My main point here is that, while pre-Socratic physics was not primarily mathematical, as modern classical physics is (although one might locate certain elements of mathematical thinking about nature in the pre-Socratics as well), their physics was more akin to classical than quantum physics as concerns the phenomenology and epistemology involved. It is true that Heraclitus's "hidden" may in principle be interpreted along the lines of the hidden one finds in quantum physics. Given, however, the fragmentary and famously "obscure" nature of Heraclitus's writings (he is sometimes called "Heraclitus, the Obscure"), it is difficult to be certain, and it is more likely that he saw the strife-like play of the elements (earth, water, air, and fire) as being at the origin of things. How the pre-Socratics may have ultimately conceived of the elements may also have further complexities: in particular, these "elements" may have been only metaphorically modeled on what appears to us as the elements of nature. The present account is far too schematic to do justice to their thought. It appears, however, that, for most of the pre-Socratics, observed phenomena, such as those of the elements, were the primary source for defining, even if, again, metaphorically, what nature was and how it behaved. Thus, these phenomena also provided the source for such physical concepts as "physical bodies" and "motion," through which the hidden strata of nature could be discovered to be similar to what is apparent to us.

Although this last point also applies to the ancient atomists, Leucippus and, especially, his student Democritus, their view of the situation has additional dimensions that are worth commenting upon here. They thought, in a proto-Kantian manner, that the ultimate nature of things is not as it appears to our senses. They argued that nature ultimately consisted of atoms, invisible and physically, but not geometrically, indivisible, and (in our language) massive entities. In this view, the ultimate constitution of nature was something that, akin to Kant's things-in-themselves or noumena, we cannot know but that we can think about logically. However, as just explained, this logical thinking by ancient atomists depended on ideas, such as those of "body" and "motion," derived from experience, as in the case in modern classical physics, including in classical atomic theories, such as the kinetic theory of gases. Indeed, this dependence is unavoidable, unless atoms are considered as *strictly* mathematical entities (rather than both physical and mathematical as in the ancient and modern classical atomism), as elementary particles of the present-day quantum theory are sometimes, although usually with further qualifications. As will be seen below, in his later thinking, in the wake of quantum field theory, which he helped to develop, Heisenberg took a similar view, in turn, however, in a qualified way. On the other hand, although this type of view would also illustrate Heisenberg's contention in this 1929 statement, under discussion here, that mathematics is not subject to the limitations of our phenomenal intuition and language, his overall point in that statement is different. His statement does not imply, quite the contrary, that the quantum-mechanical formalism *describes* quantum processes. In particular, no matter how mathematically idealized, the idea of motion, particle-like or wave-like, remains physical in nature; and this idea, in whatever form, may not apply to quantum objects, although we can ascertain, on statistical grounds, that we can find the same quantum object in different places. Also, the currently accepted treatment of elementary particles as massive dimensionless points (the idea introduced only in 1920s), or more recently as one-dimensional strings, is usually considered to be a mathematical idealization rather than to correspond to how they exist in nature.

Now, Leucippus and Democritus had good reasons for their conjecture. But so did Parmenides and Zeno, and then Socrates and Plato, who, on different but equally logical grounds, conversely denied the reality of motion and proclaimed the indivisible and unmovable character of the ultimate reality, which they, moreover, saw as that of disembodied ideas rather than as that of a material world. Neither set of reasons itself is that important at the moment, since my primary concern here is how we fundamentally think about the world and its ultimate constitution, and both types of thinking reflect how we do so.

Plato is the next (and unavoidable) juncture in my brief excursion into ancient-Greek physics. More properly, one should speak of Socrates and Plato. I shall, however, only refer to Plato, given the firmer textual evidence for his argumentation and the fact that Socrates' key ideas have mostly reached us via Plato in any event. Plato agreed with Democritus that our senses "deceive" us, insofar as they do not give us correct evidence concerning the ultimate nature of reality. However, Plato also realized that, as any human idea concerning this nature, the idea of invisible and (physically) indivisible atoms is only *an idea*, which may or may not be correct, even as concerns its consistency (although it appeared sufficiently consistent), let alone as concerns its correspondence to the ultimate reality of things. As I said, following the Pythagoreans and Parmenides, Plato denied the existence of the material world altogether, in any event as anything other than a deception of our senses. One of his reasons, which is of particular interest here, appears to have been that an argument concerning the nature of this existence cannot be definitively established as correct, unlike the ideal world to which we can, at least in principle, relate by mathematically and, thus, use mathematical proof to assure the correctness of our thought concerning this ideal world. The fact that Democritus was compelled to describe his indivisible atoms as geometrical (hence mathematically divisible) figures and derive their properties or those of composite bodies from geometrical considerations is significant in this context as well. Democritus's theoretical physics was already mathematical! In his *Timaeus*, Plato developed, in part against Democritus, his own mathematical (rather than physical, mechanical) "atomic" theory of the unchangeable eternal primary mathematical elements (more primordial than physical ones), out of which God created the world, a theory based on regular geometrical solids. It is, accordingly, not surprising that, while Aristotle, a physicist and a thinker of continuity, rejected atomism in general, he was more critical of Plato's attempt to derive physical substances from mathematical abstractions than of Democritus's physical atomism, where the main problem for Aristotle was discontinuity. Technically, Plato's argument concerning the ultimate nature of reality is subtler. For, while he saw the ultimate architecture of the ideal world of thought as *mathematical*, he thought that only philosophy and specifically dialectic, as developed by Parmenides and Zeno, could achieve the *maximally close proximity* to this architecture and expressed doubts that mathematics, as a technical discipline, or at least mathematicians could do so. Plato did not think that it was possible for the human mind to ever reach this architecture and the truth, but he thought that philosophy could bring us close to it. Plato's model of the ultimate architecture of the world was mathematical, nevertheless.

It may be added that, grounded in the idea of proportion, the Pythagorean idea of the harmony of the spheres (which was a major influence upon Plato) was mathematical as well. The idea allowed Pythagorean to fuse mathematics and aesthetics in their vision of the cosmic order, or indeed *cosmos*, since the word itself means order. This concept or ideal of mathematical-aesthetical order has had a strong hold upon mathematics, physics, and philosophy, and defined the corresponding conceptions of the ultimate nature of reality (whether the latter was conceived of as that of ideas or as that of material entities) throughout Western intellectual history. In more recent history, this ideal was expressed arguably most famously by Paul Dirac's much cited statement, initially written on a blackboard at the University of Moscow in 1956, "a physical law must have mathematical beauty."[1]

It is difficult to deny an appeal or (reasonable) effectiveness of this view in building our physical theories. On the other hand, it is far from clear why nature, especially at the ultimate level of its constitution, should conform to this view, any more than, to return to the starting point of this article, to other concepts derived, as that of beauty is, from our daily concepts and language. If Bohr (a rare exception in this respect) did not appear to have seen aesthetic considerations as important in quantum theory, one of the reasons might well have been that he did not think that human thought and

[1] The statement is cited, for example, and discussed in detail in [7, p. 275]. For a prominent recent advocacy of this idea, see [8].

language, where the idea of beauty originates, are likely to be capable of describing the ultimate constitution of nature. On the other hand, Heisenberg, while adopting the same or (in his later view) similar epistemology, highly valued aesthetic aspects of both mathematics itself and its use in physics.

The concept of beauty is difficult to define with a sufficient degree of even philosophical exactitude, and the term is often used differently in mathematics and science than in art or in considering the beautiful in nature. It is true that, along with that of order, the concept of beauty is sometimes associated, in various domains, with the concept of symmetry, which can be given a precise mathematical definition, as invariance under a symmetry group. Apart from its great and wide spread significance in mathematics itself, this concept has been fundamental to quantum theory since the introduction of quantum mechanics. It is of some interest (although not surprising) that Hermann Weyl made seminal contributions in addressing all three subjects—group theory, the idea of symmetry in general, including in art, and the role of group theory in quantum mechanics [9, 10]. It does not appear that Dirac, who, as was customary to him, remained (wisely?) cryptic on the subject, had symmetry groups in mind. On the other hand, the role of symmetry group in quantum theory, especially in the quantum field theory, appears to have influenced Heisenberg's more Platonist later views (e.g., [11, pp. 71–88]).

But then, as I argue here, it is also far from clear why mathematics, including group theory, works as concerns quantum phenomena, defined by the ultimate constitution of nature. And yet, it does work, at least predictively, even if not descriptively. Dirac was, again, wisely cryptic only to say that a physical law must have mathematical beauty and not that it must *describe* nature, in its ultimate constitution, as something beautiful. A physical law may be a mathematically beautiful predictive law; and certain laws of quantum mechanics (the uncertainty relations, for instance, or Dirac's equation) may indeed be seen as mathematically beautiful laws.

One can perceive in Plato's thinking one of the roots of modern mathematical physics, which, beginning with Galileo, aims to bring physics and mathematics together in part for the same reasons that guided Plato. For mathematics provides a way of more rigorously establishing the logical truth of our physical theories, even though the latter are, in classical physics, concerned with objects and motion along Aristotelian rather than Platonist lines. The genealogy of Heisenberg's argument in question in this article has a Platonist component as well. Plato's influence on Heisenberg's thinking is well known, beginning with the fact that Heisenberg's father was a classics teacher and that Heisenberg was exposed to Plato early on in his intellectual development. As is also well known, his earlier (around 1919) reading of Plato's *Timaeus* and specifically of Plato's discussion, mentioned above, of "atomism," based on the concept of regular solids, had impact on his thinking concerning atomic theory. Heisenberg also reflected philosophically on the ancient Greek history in question here later on (e.g., [4, pp. 59–75], [11, pp. 71–88]). This influence of Plato does not mean that Heisenberg subscribed to Plato's views, especially (he had other reservations as well) given that Heisenberg's thinking was very different epistemologically, at least at the time of his earlier work on quantum mechanics, as against his more Platonist later views ([4, pp. 59–75], [11, pp. 71–88]). This epistemology, influenced perhaps more by Kant and Ernst Mach (with both of whom Heisenberg was familiar, albeit, it appears, superficially), was defined by the view, under discussion here, that mathematics of quantum mechanics does not map, even ideally or approximately, the ultimate *physical* reality but only predicts, probabilistically, what would happen in experiments. Plato, again, rejected that the ultimate reality could be physical. And then, we no longer take the consistency of mathematics for granted either, at least not since Kurt Gödel's theorems, demonstrated by him in 1931, concerning the existence of undecidable propositions and the incompleteness of the system of axioms of arithmetic.

Plato's thinking on the subject was more complicated as well, since, as indicated above Plato's concept of proof, at least as concerns the ultimate nature or truth of ideal reality, was defined by a complex and uneasy balance of mathematics and dialectic. The first "undecidable" propositions, albeit of the philosophical, dialectical kind, was advanced at that time as well, it appears, for the first time, by Parmenides, and such propositions are found in Plato's dialogues, for example, in *Parmenides*. Democritus, too, questioned the consistency of the most basic mathematical reasoning on the ground of the difficulties of the concept of (in our language) the continuum, since two immediately adjacent parallel sections of a triangle could not be properly distinguished in length. A logical consequence is that the triangle as a continuous figure is an illusion. The problem is obviously resolved if the ultimate nature of things is discrete—atomic. I shall, however, bypass these nuances and move directly to my main point in this section, which is as follows.

The ancient Greek thinking considered here reflects what may be seen as a "conceptual enclosure" of our theoretical thinking, which enclosure has also shaped the history of modern physics, from Galileo to quantum theory, in part of course because this history was significantly influenced by ancient Greek thought. This enclosure is defined by a set of concepts and relationships among them, on which ancient and then classical physics depended, and which is at work in quantum physics as well. However, unlike ancient Greek or modern classical physics, quantum physics (at least, again, in some interpretations) also reveals, from within this enclosure, something that is beyond the reach of our thought and hence beyond this enclosure. The primary features of this enclosure are as follows: (a) the presence of observable entities or phenomena, including in their relation to our language and concepts; (b) the possible existence of the underlying hidden or, in Kant's language, noumenal entities ("nature likes to hide"); and (c) the role of both logic and mathematics in our understanding of the world. Mathematics may serve to describe phenomena or even (hypothetically) noumena, and, through this description, to predict phenomena, for example and in particular, astronomical phenomena—eclipses of the Sun, the motion of planets, and so forth. These key aspects of physical thought and the interactions between them came to define the nature of modern (classical) physics, with the addition of probability, as a crucial new element of theoretical physics, in the nineteenth century. Chance does play a significant role in Aristotle's *Physics* [12], as it does throughout his work, where one also finds certain intimations of probability, more along Bayesian lines of "degree of belief." However, as elsewhere in modern physics, vis-à-vis Aristotle's physics, at stake is the mathematical use of probability and of the mathematics of probability. It is worth noting, however, that while he did not use mathematics in the way modern physics does, Aristotle's thinking was not as non-mathematical as is sometimes claimed (e.g., [12, v. 1, p. 366]).

It is, I argue, only with quantum mechanics (and then only in certain interpretations) that physics breaks with this enclosure, or rather relates to what is beyond this enclosure and human thought itself, but from within this enclosure—unless Heraclitus's "hidden" already anticipated the "hidden" of quantum epistemology (which "hidden" is, however, quite different from Bohmian "hidden variables"). This is not altogether inconceivable, but, as I said, appears to be unlikely, given that the latter emerged from considering phenomena, such as those encountered in the double-slit and other paradigmatic quantum experiments, that are far stranger than anything we had ever confronted or even could have imagined. In John A. Wheeler's words, "What could one have dreamed up out of pure imagination more magic ... than this?" [13, p. 189]. It is remarkable and, one might agree with Heisenberg, fortunate that mathematics allows us to meaningfully respond to this strangeness, even if not to explain it.

3 Phenomenology, Epistemology, and Mathematics in Classical Physics

Heisenberg's statement with which I began here implies, then, the essential difference between classical and quantum physics, a difference that both Heisenberg and Bohr explained in detail on many occasions (e.g., [2, p. 11]; [3, v. 2, pp. 69–70]; [4, pp. 56, 91–92]). Just as our everyday or, for the most part, philosophical language and concepts, those of classical physics, including its mathematics, serve to describe physical bodies and processes "involving exceedingly large numbers of atoms," which generally behave classically.[2] In other words, classical physics may be seen as a suitable philosophical and then mathematical refinement of concepts and language of daily life, beginning with the concepts of physical "object" (or "body") and "motion," and the phenomenal representations of objects and motions. But then, we have no other language and concepts, and mathematics may indeed be the only domain that is not subject to the limitations imposed by this language and conceptuality, in contrast, for example, to our description of experiments and our experimental instruments themselves, which depends on the enclosure of classical *physical* concepts. I might add, that this last observation is essentially the reason for Bohr's often misunderstood argument to the effect that our *description* of our measuring instruments is physically classical—a *description*, but *not constitution*, and then only a description of certain observable parts of these instruments (e.g., [3, v. 3, pp. 3–6]).[3] Modern classical physics and (with further qualifications) relativity are different from Aristotle's physics by virtue of being concerned strictly with *material bodies* and motions, and by virtue of its *mathematical* character. However, they remain Aristotelian as

[2]It is not a matter of the "smallness" of quantum systems, since they could be large, but only of the "smallness" of their ultimate constituents, as Heisenberg indicates in his statement by speaking of "processes occurring *within atoms*." However, just as small quantum systems, large quantum systems cannot be observed as *quantum systems* without using suitable measuring instruments. In particular, they cannot be observed without being "disturbed," in the way we observe classical systems, by disregarding the role of Planck's constant, h. While classical systems, or what we observe as such, ultimately have a quantum constitution as well (at least, according to most views of the situation), it is difficult to observe them as quantum. Certain systems can, however, be observed as both classical and quantum (e.g., [14]).

[3]I have considered Bohr's argument in detail in [15, pp. 32–35, 307–310].

concerns the fundamental *physical* concepts, such as motion, that modern physics continues to rely upon, with the exception of quantum theory, again, in certain interpretations, in which such concepts are only used to describe the (classical) behavior of certain parts of measuring instruments. In classical physics, specifically classical mechanics, we start with objects and processes that are phenomenal representations of actual objects and processes, similar to those representations that we form concerning material bodies in our daily life. Then we refine these representations in order to be able to create and use mathematical models to describe these phenomena and to predict them on the basis of such descriptions. Thus, the idealizations and models of classical physics, at least, again, of classical mechanics, are those of objects and motions that we actually observe in nature.

For simplicity, from now on I shall mostly deal with classical mechanics, but my argument applies elsewhere in classical physics, for example, in classical electrodynamics or in classical statistical physics, with some adjustments but without essential modifications of the scheme in question. Thus, although we do not in the cases considered in classical statistical physics, say, the kinetic theory of gases, directly observe the motion of individual atoms and molecules, the models used by the theory still assume that classical mechanics apply to this motion. (Both relativistic and quantum effects are considered negligible.) Additional adjustments are necessary in relativity, given, for example, the peculiar behavior of light. For, considering for the moment only special relativity theory, were it possible to put a clock and a measuring rod on a moving photon (there are no other photons), this clock would stand still and the photon would be in all places at once. General relativity contains yet further complexities, such as those introduced by the behavior of black holes. Nevertheless, the essential features of classical-physical representation, including causality, would still apply in relativity. It is only with quantum mechanics that the adequacy of this scheme began to be questioned, since quantum objects and their behavior cannot be either observed in the way classical physical objects are or, it appears, modeled on the scheme of classical mechanics, or classical electrodynamics.

Of course, things are not quite so simple in classical physics either, and, as explained earlier, already the pre-Socratics realized some of the complications involved in the epistemological scheme eventually adopted by classical physics. Einstein once famously asked [Abraham Pais], in the context of quantum theory, "whether he [Pais] really believed that the moon exists only when he looked at it" [16, p. 907]. Well, rigorously speaking, the answer should in fact be "yes" even if we remain within the limits of classical physics! The moon exists *as the moon* only when there is somebody who can look at it. It does not exist as the moon if there is no one capable of looking at it. This does not of course mean that nothing exists where we see the moon. But whatever it is—and we don't know and perhaps cannot conceive of what it ultimately is—it would not be what we see as the moon, and we certainly *do not see* the moon in its ultimate constitution, especially as a quantum object. Given his philosophical acumen and sophistication, as a reader of Kant in particular, Einstein must have realized that such is the case. Indeed he insisted that physical theory could only approach nature through the mediation of concepts, indeed based on a free choice of concepts, a position that is closer to that of Kant or Hegel [17, p. 47]. His question must have been partly rhetorical and asked with this realization in mind. On the other hand, he appears to have believed (in this regard closer to Hegel than to Kant) that our concepts should allow us to at least approach the actual nature of physical reality, which quantum theory makes far less certain.

The nature of all such concepts would depend on our perception and thus on the particular constitution of our bodies and brains, which enables us to see the moon as the moon, or for that matter to see anything in the first place and in particular ways. To cite Marcel Proust, "the trees, the sun and the sky would not be the same as what we see if they were apprehended by creatures having eyes differently constituted than ours, or else endowed for that purpose with organs other than eyes which would furnish equivalents of trees and sky and sun, though not visual ones" [18, v. 3, p. 64]. Perhaps such "organs" (if this concept could even apply) would not even furnish that much, or in any event nothing equivalent. This, it might be noted in passing, is why the dream of communicating with an extraterrestrial intelligence is, in my view, unlikely to prove to be anything but a dream, because there may be no message we could construct that such extraterrestrial beings could meaningfully *receive*, that is, this incommensurability is well beyond the question of deciphering a message.

Observed *phenomena* are representations, not nature in itself; that is, they are something that *appears to our mind*—to our conscious perception or thought. On the other hand, certain *objects* that are thus phenomenally represented, either directly or, as in classical physics, in an idealized manner, are assumed to exist in nature, and not necessarily in the way of representations of them as phenomena suggest to us. This distinction between objects, as things-in-themselves or noumena, and phenomena, as appearances, was the starting point of Kant's philosophy, devoted to the exploration of the difficulties involved in the representation of the objects, material or mental, with which a philosophical or physical (and possibly, when dealing with mental objects, even mathematical) inquiry may be concerned. These difficulties were recognized before Kant, even, as I have explained, already by the pre-Socratics. In the history of modern physics, both Descartes and Newton pondered these difficulties (specifically in their optical studies), and Galileo was aware of them as well. In Kant's scheme, phenomena, as "object[s] of sensible intuition, i.e. as appearance[s]" are *what we know experientially through the senses* [19, p. 115]. Phenomena are opposed by Kant to *noumena*, each of which is seen as "ein *Ding an sich*" or a thing-in-itself. Corresponding to the ancient Greek etymology of the word (something "thought-of"), noumena are those objects, material or mental, that we can only *think about*, rather than cognize as something represented in our mind and present to our intuition. Our knowledge concerning noumena can be at most indirect or inferential, and is often provisional, since our inferences may be proven wrong at a later point.

The concept of noumena applies in classical physics. As I explained, the actual objects and processes in nature, to which classical physical objects and processes, as *phenomenal idealizations*, correspond, are noumenal in Kant's sense. However, while noumena are, by definition, not knowable, they are, at least in principle, thinkable; and our conceptions of them may in principle prove to be correct at least within certain practical limits, which, as Kant indeed say, is sufficient [19, p. 115]. This is indeed the case in classical mechanics (one of Kant's models) or in the case of our conception, on the model of classical mechanics, of molecular motion in the kinetic theory of gases. Their behavior is not known directly, but there is indirect evidence for this assumption as workable within classical limits. This approach, however, does not appear to be effective, if workable at all, in quantum theory. The "old" (pre-quantum-mechanical) quantum theory of Planck, Einstein, Bohr, and Sommerfeld was still grounded in this type of approach, albeit only partially, since there was no classical model for electrons' discontinuous transitions ("quantum jumps") from one orbit to another around atomic nuclei. Quantum mechanics, at least, again, in the interpretations of the type considered here, abandons the descriptive program of classical physics altogether. It retains Kant's concept of phenomenon but radicalizes the difference between quantum phenomena (i.e., those observable phenomena in considering which Planck's constant, h, cannot be neglected) and quantum objects beyond Kant by making quantum objects and processes literally unthinkable, inconceivable.

While, however, the objects and processes considered by classical physics are noumenal in Kant's sense, classical physics, and in particular classical mechanics, can circumvent the difficulties of their noumenal nature. It can do so by virtue of being able to idealize them for the purposes of the mathematical descriptions of observed phenomena that classical mechanics provides, via, on the one hand, the data obtained in the experiments and, on the others, the equations of classical mechanics. (Technically speaking, the processes in question are also idealized *objects* of the theory involved, insofar as these processes are theorized by this theory just as material bodies are, but the distinction between "objects" and "processes" is useful in physics.) Such descriptions also enable predictions of the behavior of actual objects in nature considered in classical physics, such as, again, planets moving around the sun. Classical physics can bypass the fact that it ultimately deals not with natural objects but only with phenomena arising by virtue of our *interaction* with nature, which can in principle have a totally different constitution than our phenomenal experience of them may suggest. In other words, it can disregard the distinction between physical objects and their phenomenal representations. This is primarily because we can observe the behavior of the corresponding objects in nature "without disturbing them appreciably," as against the situation that obtains in quantum physics, a point that became crucial for both Heisenberg and Bohr, whom I cite here [3, v. 1, p. 52].

It is true that, say, the Copernican picture of the solar system does not correspond to our direct phenomenal experience, since it is an observable fact that the Sun moves relative to the ground, which fact indeed created difficulties for Galileo in advocating the Copernican system. One could, however, still intuitively visualize the Copernican picture by forming the corresponding mental picture, and one could actually draw such a picture, just as in the case of the motion of other physical bodies. The observations involved, again, do not affect the actual behavior of the planets observed, and we can make our predictions on the basis of this model, just as we can on the basis of the previous geocentric one. With Kepler's discovery that planets move along elliptical orbits, these predictions were improved considerably. Later on, in the case of the kinetic theory of gases, it was, as I said, assumed by physicists, just as it was by the ancient atomists, that the atomic constitution of nature was not directly available to us. Nevertheless, individual atoms and their behavior would still be conceived on the model of classical mechanics, which enabled an effective statistical physical theory, even if it was not completely free of conceptual and epistemological complexities. One could, again, intuitively visualize this individual atomic behavior by forming a mental image of it or, again, draw a corresponding picture. Accordingly, such objects could be effectively considered not only as thinkable (as Kant's noumena are) but also as, in principle, knowable, even if indirectly and within the scope of a given theory. In any event, these objects and their behavior are available to physical modeling, which could, through *descriptive* idealizations, enable excellent predictions concerning the observed phenomena and, hence, of the behavior of the objects considered. This view represents what Schrödinger, well aware of the complexities just described, called the "classical ideal" in physics [5, p. 152]. In sum, classical physics predicts because it describes, even if by way of idealized models, and in modern physics we, again, only deal with such models. By contrast, in quantum theory such descriptive models do not appear to work, and quantum mechanics predicts in the absence of these models, at least, again, in certain interpretations. In other words, it predicts without describing, which is, again, remarkable.

From this perspective, the effectiveness of mathematics in classical physics would appear to be reasonable: it is a reasonable outcome of our phenomenal experience of the world, to which classical physics essentially relates. Classical physics deals with the suitable mathematized refinement of objects and processes in nature or rather of their general phenomenal representations, and disregards other properties of these objects and effects these properties are responsible for. Indeed, even mathematics itself may, historically, be seen as a similar type of refinement of ordinary thought and language, and indeed both of these refinements, physical and mathematical, come together and in many respects have emerged together, as in Euclid's geometry or differential calculus and classical physics. However, in the case of mathematics, this refinement may lead and has led mathematics quite far from our daily experience, concepts, and language, including specifically those that found their way into classical physics, which does not depart as far from our daily life. It is this situation that enables Heisenberg's observation with which I began here and his very discovery of quantum mechanics, which this observation reflects. This move away from any connections to our daily intuition or, in part correlatively, from mathematical idealizations of classical physics characterizes the development of *modern mathematics*, essentially more abstract mathematics, roughly from the mid-nineteenth century on.[4] As Weyl noted, a propos the mathematical concept of the continuum (or real numbers), one of the most striking examples of this kind, introduced in the late-nineteenth century, this refinement can reach the point of a *nearly* complete break between our mathematical and everyday or even philosophical intuition and ways of thinking [21, p. 108]. I qualify Weyl's even stronger assessment, which implies a *complete* break, whereby "the demand for the coincidence between the two must be dismissed as absurd" [21, p. 108], because we might still use our everyday intuition and thought in comprehending and working with such concepts. In addition, some among these concepts, including the continuum, can be used in mathematical idealization of physics, descriptive (as in classical physics and relativity) or strictly predictive (as in quantum mechanics). As Weyl himself adds: "Nevertheless, those abstract schemata

[4] See [20] for a comprehensive historical discussion of the subject.

supplied us by mathematics must underlie the exact science of domains of objects in which continua [or other abstract schemata of modern mathematics] play a role" [21, p. 108]. Quantum mechanics gave Weyl's statement, made in 1917, a new meaning and significance. For, the abstract mathematical schemata used there, such as those of the Hilbert-space mathematics, no longer appears capable of offering descriptive idealizations of the type found in classical physics or relativity.

4 Experimental Technology, Mathematics, and Probability in Quantum Physics

It is hardly surprising that the highly abstract and nonvisualizable mathematics adopted by quantum mechanics, such as that of Hilbert spaces or other currently available versions of the formalism, say, C^*-algebras, is unsuited for the kind of description of physical behavior that defined the mathematical models of classical physics or relativity, although relativity, again, already exhibits significant complexities in this regard. For one thing, Hilbert spaces used in quantum mechanics are generally of higher dimensions, and in the case of continuous variables they are infinite-dimensional. Hilbert spaces of finite dimensions are used in the case of discrete variables, such as spin, which does not have classical analogues that would help us to visualize it in physical terms in the first place. It does not help either that we deal, it appears unavoidably, with Hilbert spaces over *complex numbers* rather than with the *real-number* mathematics used in classical physics and defining its descriptive model. In addition, Hilbert-space mathematics is continuous, while observable quantum phenomena are discrete. This fact posed major difficulties for Schrödinger in his attempt to offer a continuous descriptive theory of quantum processes by means of his wave mechanics, which, as a descriptive theory, assigned certain unobservable (continuous, wave-like) properties to these processes. Schrödinger was never able to satisfactorily resolve these difficulties, although it is of course conceivable that continuous quantum-level processes can give rise to discrete quantum phenomena that we observe. In any event, it is not surprising that the type of mathematics currently used in quantum mechanics or in higher-level quantum theories poses difficulties for interpreting it in descriptive terms, even though attempts to do so have never waned. What is surprising is how well this mathematics works in predictive terms, since somehow—mysteriously or unreasonably—this mathematics correctly predicts the outcomes of relevant experiments in spite and perhaps because of the absence of its descriptive capacity as concerns the behavior of quantum objects.

This predictive capacity is all the more enigmatic given that quantum mechanics relates this mathematics to the outcome of quantum experiments by means of, unreasonably effective, *ad hoc* rules (such as Born's rule) that are not justified either on physical grounds or by the formalism of the theory. These rules are reasonably natural mathematically insofar as they (by means of conjugation of complex quantities) allow us to move from complex to real quantities, necessary for our physical predictions. On the other hand, there are no physical or mathematical reasons why such quantities should correspond, as they do, to the probabilities of the outcomes on the relevant experiments. It is no wonder then that Heisenberg speaks of the quantum-mechanical situation as fortunate. Of course, one might also say that we are equally fortunate that classical physics works so well in its domain. However, as I argue here, in this case one can at least positively connect classical physics to the way we experience the world, rather than divorce this experience from nature, in the way quantum theory or, to some degree, relativity, does. We are, to begin with, fortunate to exist, to have evolutionarily emerged and developed, as what we are, which enables us to invent all the physics or mathematics that we have.

It is this abandonment of offering a descriptive theory of quantum phenomena (in this case, spectra) that defined Heisenberg's approach and proved to be essential to his discovery of quantum mechanics. The situation was unprecedented and unforeseen at the time, given the preceding history of the relationships between classical physics and mathematics, as discussed earlier. To recapitulate briefly, classical physics and then (again, in a more complex and qualified way) relativity are mathematically descriptive idealizations of observed phenomena, and predict numerical data associated with these phenomena by virtue of being such mathematically descriptive idealizations. They predict because they describe. Heisenberg's theory predicted without physically describing anything.

As noted earlier, the old, semi-classical, quantum theory, especially the theory of the atom, introduced by Bohr in 1913, already departed from both classical mechanics and classical electromagnetism insofar as its descriptive capacity as regards quantum processes was only partial, which is why the theory was, in retrospect, called semi-classical. However, the old quantum theory was still pursued in the *descriptive spirit* of classical physics. Indeed, it was expected at the time that a proper mechanics of quantum processes would remedy the nonclassical aspects of the theory. With Heisenberg's discovery the opposite had happened: Heisenberg's matrix mechanics was fully nonclassical in that it offered no description of quantum processes at all. This is in part (there were also more direct physical reasons) why Heisenberg's approach did not encounter the kind of problems that haunted Schrödinger's wave mechanics. As indicated above, the latter aimed to bring its mathematics into accord with classical (wave) physics and, more generally, offered the hope that a classical-like (ideally) descriptive theory would be possible, which is why wave mechanics was welcomed by Einstein, who was, by contrast, deeply suspicious of matrix mechanics. This hope never died, and, as I said, it might as yet prove to be justified by the future development of physics.

One might say that Heisenberg's approach *benefited* from the split between our phenomenal world and the world of modern, more abstract mathematics, given that the mathematics used in classical physics for the purposes of its descriptive idealization did not appear to offer effective, if any, algorithms for predicting quantum phenomena. It can hardly be seen as coincidental that quantum mechanics was introduced in the wake of the development of modern mathematics, defined by its break with, interactively, both our daily phenomenal intuition and with the descriptive needs of physics. Physics could take advantage of this new mathematics and in Heisenberg's case it did, as Bohr was quick to note [2, v. 1, p. 48]. It is true that Heisenberg reinvented (infinite-dimensional) matrix algebra, *from physics itself*, in the course of his discovery of quantum mechanics. Nevertheless, this reinvention did not occur in a vacuum. New mathematics and a new way of thinking about mathematics were around and by then even dominant, especially in Göttingen, where Heisenberg was at the time. Max Born's immediate realization of the proper mathematical nature of Heisenberg's formalism as that of infinite-dimensional matrix algebra clearly reflected this new, more abstract spirit of mathematics. So did the ease with which Paul Dirac was able to adopt and develop Heisenberg's scheme into an even more abstract one (that of q-numbers). Ironically, Heisenberg himself was initially more apprehensive concerning his mathematics. So was also, intriguingly, Pauli. Remarkably, unlike Born and Dirac, Pauli was troubled by the noncommutative nature of Heisenberg's formalism and even thought that the final version of the theory should be commutative. He also did not like an excessively abstract character of the full-fledged version of matrix mechanics, as developed by Born and Jordan. Finally, he thought that the ultimate version of quantum theory should be free of probability. These were, however, the noncommutativity and abstraction of Heisenberg's mathematics and the irreducible probability introduced by his physics that were to carry the day.[5] If mathematics outpaced Heisenberg in inventing matrix algebra, Heisenberg's *strictly predictive* way (in the absence of any description) of connecting mathematics and physics was truly revolutionary. But then the old ways, physical or mathematic, no longer worked by this point, and, along with many others, Heisenberg, too, had tried and tried hard to make them work for a few years.

In Heisenberg's original approach, then, and in a certain type of interpretation of the theory ever since, quantum mechanics shares with classical physics its experimental and mathematical character as concerns its predictive capacity, in spite of lacking a descriptive capacity as concerns the behavior of quantum objects. It is true that these predictions, unlike those of classical mechanics, are inherently probabilistic or statistical. As I explained, however, our experimental data does not allow us to do better, because identically prepared experiments (such a preparation is possible) in general lead to different outcomes. In other words, quantum mechanics predicts as much as we can observe, even though it does not describe the behavior of the ultimate objects and the processes that are responsible for these outcomes or what we observe in quantum experiments more generally. But then, this behavior as such, as independent behavior, is not observable either. Nobody has ever seen a moving electron or photon, but only observed the traces of their interactions with measuring instruments.[6]

At the same time, and perhaps correlatively, the observable effects or traces of the "behavior" of quantum objects, as manifest in the paradigmatic quantum experiments, such as the double-slit experiments, are remarkable. They are, as I said, nothing like anything we have ever encountered before them in any domain. Other standard locutions include strange, puzzling, mysterious (and sometimes mystical), and incomprehensible. Thus, how do particles "know," individually or (which may indeed be even more disconcerting) collectively, that both slits are open and no counters are installed or, conversely, that counters are installed to check which slits particles

[5] I have discussed Pauli's initial response to matrix mechanics in [15, pp. 117–119].

[6] Of course, as noted earlier, this does not mean that arguments concerning the independent behavior of quantum objects cannot be or have not been developed, either by way of interpreting the standard quantum mechanics or otherwise, for example, in Bohmian mechanics, which, however, is nonlocal.

pass through, and modify their behavior accordingly? This behavior is all the more remarkable given the fact that the interval between emissions could be made large enough for the preceding quantum object to be destroyed before the next one is emitted, without affecting the probability counting or the appearance or disappearance of the interference pattern in a given set-up. Attempts to conceive of this behavior in terms of physical attributes of quantum objects themselves appear to lead to unacceptable or at least highly problematic consequences. Among such consequences are logical contradictions; incompatibility with one aspect of experimental evidence or the other; a behavior of quantum objects themselves, based on such difficult assumptions as attributing volition or personification to nature in allowing quantum objects individual or collective "choices;" or the spatial or temporal nonlocality of the situation, in the sense of its incompatibility with relativity, a possibility first broached by Einstein. It must be acknowledged, however, that some physicists and philosophers do not find this last possibility troublesome and even argue (in my view, problematically) that it is a necessary consequence of quantum phenomena, for example, in view of Bell's and related theorems.

On the other hand, one can consistently (and locally) account for the situation by suspending any account of the independent quantum behavior, and instead follow Bohr's logic of complementarity, grounded in the fact that these two set-ups and the two types of phenomena occurring in the double-slit or related experiments are always mutually exclusive. Bohr appears to have been the first to take advantage of this fact. It allowed him to contend that, since the features of quantum phenomena exhibited in the double-slit experiment or other key quantum experiments are mutually exclusive, they need not be seen as paradoxical, which was also the starting point for his concept of complementarity. As he says: "[I]t is only the circumstance that we are presented with a choice of *either* tracing the path of a particle *or* observing interference *effects*, which allows us to escape from the paradoxical necessity of concluding that the behavior of an electron or a photon should depend on the presence of a slit in the diaphragm through which it could be proven not to pass" [3, v. 2, p. 46; emphasis added]. Our tracing of the path of any quantum object could only amount to, by classical standards, incomplete and indirect information. Indeed, "tracing the path" (not the best expression here) only means that we can know, at least with good probability, which slits the particle has passed through. This information is, however, sufficient to avoid the paradoxes in question.

This argumentation can be suitably adjusted to apply to such more recent developments as Bell's and the Kochen-Specker theorems (and their extensions) and related experimental findings, which are, admittedly, under debate as concerns their epistemological meaning and implications. These developments are beyond my scope here, as are the debates around them, which are part of the apparently interminable debates concerning quantum phenomena and quantum theory.[7] I would like, instead, to consider, from this perspective, Heisenberg's invention of quantum mechanics—

[7] I have, however, considered the quantum-mechanical situation from the perspective offered in the present article in detail in [15, pp. 237–278], which does include a discussion of these developments and debates, and related subjects, in particular the Einstein-Podolsky-Rosen (EPR) thought-experiment, introduced earlier, in 1935 [22], but still largely responsible for the more recent developments just mentioned.

the inaugural instance of the type of mathematical response to quantum phenomena under consideration in this article [23].

Heisenberg starts with *formally*, but only formally, adopting the classical equations of motion in Fourier's representation. This purely formal correspondence, in the absence of a classical-like physical mechanical description of quantum processes, sometimes compelled Bohr to speak of quantum mechanics, in either version, as "symbolic," also in view of the role of complex number there (e.g., [3, v. 1, pp. 75–76]). Heisenberg's approach was prompted by what may be seen as a mathematical form of Bohr's correspondence principle. The correspondence principle more or less tells us that in the regions where classical physics can be used (and sometimes it can) for predicting quantum phenomena, the predictions of quantum and classical theory should coincide. In particular, classical equations, applied to the standard classical physical variables, work well for the large quantum numbers (when electrons are far from the nucleus). Importantly, as Bohr often stressed and as Heisenberg explains in his uncertainty-relations paper, this does not mean that the physical processes themselves in question may be seen as classical and hence that classical equations correctly describe these processes, but only that one can use classical equations to make correct predictions [24, pp. 72–76]. Classical equations, however, do not work at all for small quantum numbers because they do not satisfy Bohr's rules for frequencies and the Rydberg-Ritz combination rules, which fact led Bohr to his 1913 semi-classical theory of the hydrogen atom.

Bohr's theory described the orbital motion of electrons on classical lines but also postulated discontinuous quantum jumps between such orbits, each accompanied by either an emission or absorption of the radiation quantum, $h\nu$, corresponding to the difference between the corresponding energy levels, in accordance with Planck's hypothesis. By the same token, only a discrete set of orbits ("stationary states") was allocated to electrons, and Bohr also postulated a minimal energy level, from which electrons would not radiate. While these postulates were manifestly in conflict with both the classical mechanics of motion and classical electrodynamics of radiation, they allowed Bohr and his followers to account for the atomic spectra. However, although quite successful vis-à-vis previous attempts, the theory had proven to be deficient, in some respects gravely deficient, on several grounds by the time of Heisenberg's discovery. For example and in particular, it became clear that one could no longer apply the concept of classical orbits even to electron's behavior in stationary states, but could only speak of energy levels corresponding to these states. In addition to manifest physical difficulties, the old quantum theory exhibited a peculiar ambiguity as concerned the mathematics involved. On the one hand, classical *equations* could be used to predict quantum processes in certain quantum regions, which fact was reflected in the correspondence principle; on the other hand, these equations manifestly failed in other quantum regions. The theory tried (with some successes), but ultimately failed, to handle the situation by adjusting the equations in these other regions, while retaining classical physical variables and, hence, to the degree possible, the descriptive approach of classical mechanics.

Heisenberg, by contrast, made a truly radical move both, and correlatively, in physical-mathematical and in epistemological terms. In physical-mathematical terms, he formally retained the equations of classical mechanics, while fundamentally changing the variables to which these equations applied. In epistemological terms, he abandoned the idea that the equations of the theory should describe the quantum processes in question. In other words, he realized that, if one used a different type of variables than those used in classical physics or even relativity, it was possible to predict observable quantum phenomena (specifically, spectra) without describing the quantum objects and processes responsible for the appearance of these phenomena. His equations were no longer equations of motion. Heisenberg's discovery was thus the discovery or invention of a new type of physical variables—infinite-dimensional matrix variables, whose elements were also complex rather than real quantities. It may be noted that, while complex numbers enter the classical equations of motion in Fourier's representation as well, they cancel out and do not appear in the solutions of these equations. By contrast, applied to complex matrix or Hilbert-space-operator variables, the mathematical solutions of quantum-mechanical equations retain complex numbers. The connections to real numbers, obtained in measurements, is established by means of Born's or related rules, defined via the conjugation of complex numbers, which, for any complex number $z = (a + ib)$ is $(a + ib) \times (a - ib)$. The latter, this is part of the beauty of the theory, is always a positive real number, $a^2 + b^2$, known as the square modulus, $|z|^2$ of z. The procedure can also be normalized so that $|z|^2$ lies in the interval between zero and one, and hence can be interpreted as the probability of the outcome of the corresponding experiment.

In his paper announcing his discovery of quantum mechanics, Heisenberg states: "[I]n quantum theory it has not been possible to associate the electron with a point in space, considered as a function of time, by means of observable quantities. However, even in quantum theory it is possible to ascribe to an electron the emission of radiation" [23, p. 263]. Heisenberg then says: "In order to characterize this radiation we first need the frequencies which appear as functions of two variables. In quantum theory these functions are in the form [originally introduced by Bohr]:

$$v(n, n - \alpha) = 1/h\{W(n) - W(n - \alpha)\} \tag{1}$$

and in classical theory in the form

$$v(n, \alpha) = \alpha v(n) = \alpha/h(dW/dn)\text{''} \quad [23, \text{p. } 263].$$

This difference leads to a difference between classical and quantum theories as concerns the combination relations for frequencies, which correspond to the Rydberg-Ritz combination rules. However, "in order to complete the description of radiation [in accordance with the Fourier representation of kinematic formulas] it is necessary to have not only frequencies but also the amplitudes" [23, p. 263]. The crucial point is that, in Heisenberg's theory and in quantum mechanics since then, these "amplitudes" are no longer amplitudes of any physical, such as orbital, motions, which makes the name "amplitude" itself an artificial, *symbolic* term. Instead, quantum amplitudes were to be linked to the probabilities of transitions between stationary states; in other words, they are what we now call probability amplitudes. "The amplitudes may be treated as complex vectors, each determined by six independent components, and they determine both the polarization and the phase. As the amplitudes are also functions of the two variables n and α, the corresponding part of the radiation is given by the following expressions:

$$\text{Quantum-theoretical:} \quad \mathrm{Re}\{A(n, n-\alpha)\mathrm{e}^{i\omega(n,n-\alpha)t}\}$$
$$\text{Classical:} \quad \mathrm{Re}\{A_\alpha(n)\mathrm{e}^{i\omega(n)\alpha t}\}.$$

The problem—a difficult and, "at first sight," even insurmountable problem—is now apparent: "[T]he phase contained in A would seem to be devoid of physical significance in quantum theory, since in this theory frequencies are in general not commensurable with their harmonics" [23, pp. 263–264]. Heisenberg now proceeds by inventing a new theory around the problem that appears to be insurmountable and is insurmountable within the old theory. It is a question of changing the perspective completely. Most of all, the new theory offers the possibility of rigorous predictions of the outcomes of the experiments, even if at the cost of abandoning the physical description of the ultimate objects considered, which is no longer seen as a problem but instead as a way to the solution. It is no longer a cost but a benefit. Heisenberg now says: "However, we shall see presently that also in quantum theory the phase had a definitive significance which is *analogous* to its significance in classical theory" [23, p. 264; emphasis added]. "Analogous" could only mean here that the way it functions mathematically is analogous to the way the classical phase functions mathematically in classical theory, or analogous in accordance with the *mathematical* form of the correspondence principle, as defined above. Physically there is no analogy. As Heisenberg explains, if one considers "a given quantity $x(t)$ [a coordinate as a function of time] in classical theory, this can be regarded as represented by a set of quantities of the form

$$A_\alpha(n)\mathrm{e}^{i\omega(n)\alpha t},$$

which, depending upon whether the motion is periodic or not, can be combined into a sum or integral which represents $x(t)$:

$$x(n,t) = \sum_{-\infty}^{+\infty}{}_{\alpha} A_\alpha(n)\mathrm{e}^{i\omega(n)\alpha t}$$

or

$$x(n,t) = \int_{-\infty}^{+\infty} A_\alpha(n)\mathrm{e}^{i\omega(n)\alpha t}\,d\alpha\text{''} \quad [23, \text{p. } 264].$$

Heisenberg is now ready to introduce his most decisive and most extraordinary move. He first notes that "a similar combination of the corresponding quantum-theoretical quantities seems to be impossible in a unique manner and therefore not meaningful, in view of the equal weight of the variables n and $n-\alpha$" [23, p. 264]. "However," he says, "one might readily regard the ensemble of quantities $A(n, n-\alpha)\mathrm{e}^{i\omega(n,n-\alpha)t}$ [an infinite square matrix] as a representation of the quantity $x(t)$" [23, p. 264]. The arrangement of the data into square tables is a brilliant and—in retrospect but only in retrospect (especially because it also changed our view of what is natural in quantum physics)—natural way to connect the relationships (transitions) between two stationary states, and it is already a great concept.

However, it does not by itself establish an *algebra* of these arrangements, for which one needs to find the rigorous rules for adding and multiplying these elements—rules without which Heisenberg cannot use his new variables in the equations of the new mechanics. To produce a *quantum-theoretical interpretation* (which, again, abandons motion and other physical concepts of classical physics at the quantum level) of the classical equation of motion that he considered, as applied to these new variables, Heisenberg needs to be able to construct the powers of such quantities, beginning with $x(t)^2$. The answer in classical theory is of course obvious and, for the reasons just explained, obviously unworkable in quantum theory. Now, "in quantum theory," Heisenberg proposes, "it seems that the simplest and most natural assumption would be to replace classical [Fourier] equations ... by

$$B(n, n-\beta)\mathrm{e}^{i\omega(n,n-\beta)t} = \sum_{-\infty}^{+\infty}{}_{\alpha} A(n, n-\alpha)A(n-\alpha, n-\beta)\mathrm{e}^{i\omega(n,n-\beta)t}$$

or

$$= \int_{-\infty}^{+\infty} A(n, n-\alpha)A(n-\alpha, n-\beta)\mathrm{e}^{i\omega(n,n-\beta)t}\, d\alpha\text{''} \quad [23, \text{p. } 265].$$

This is the main postulate, the (matrix) multiplication postulate, of Heisenberg's new theory, "and in fact this type of combination is an *almost* necessary consequence of the frequency combination rules" (equation (1) above) [23, p. 265; emphasis added]. "Almost" is an important word here. While Heisenberg, to some degree, arrives at this postulate in order to get the combination rules right through a complex process of "guessing" (not the best word here)—by manipulating, among other things, the correspondence principle and the data—the justification or derivation is not strictly mathematical, but it corresponds to the observable phenomena and numerical data. As he explained later in his Chicago lectures, this rule can only be justified by an appeal to experiments, which is the case even for a fully developed matrix (or wave) theory, and in some respects, it is still a guess, both systematic and lucky [2, p. 108]. This combination of the particular arrangement of the data and the (re)invention through physics of an algebra of multiplying his new variables is his great invention. This multiplication is of course in general noncommutative, and the scheme essentially amounts to the Hilbert space formalism, with Heisenberg's matrices serving as operators.

The key point in the present context is that, while the classical equations of motion are retained, the variables and rules of mathematically manipulating them are replaced, as are by the rules of relating the resulting equations to the experiments. These latter relations are no longer based on describing the physical behavior (motion) of quantum objects, but only relating to the probabilistic outcome of the corresponding experiments. Planck's constant, h, too, enters the scheme as part of this new relation between the data in question and the mathematics of the theory. As I argue here, the nature of the mathematics used, that of the infinite-dimensional Hilbert spaces over complex numbers, already makes it difficult to establish such descriptive relations in terms of these mathematics. On the other hand, it is possible to establish the relation to probabilities of the outcomes of the relevant experiment, via Born's or related rules, which allows us to move from complex quantities of the formalism to positive real numbers between zero and one.

To help drive my argument home, let us imagine that the data in question in quantum mechanics, for example, as manifest (quantitatively) in spectra or, to make the case at bit more dramatic, in the double-slit experiment, were given to a mathematician on the cutting edge of mathematics at the time of the invention of quantum mechanics. One can think of a doctoral student of Hilbert, or somebody like von Neumann or Amy Noether, but, unlike them, unfamiliar with quantum or even with classical physics, although familiar with probability theory, since we really deal here with an invention of a new algorithm for counting probabilities. This mathematician would then have been asked to develop a mathematics that would enable one to *predict* these data. This would have been a formidable problem, especially in the absence of physics, although, as things actually happened, physics (as it existed then) was almost more inhibiting than helpful. The main difficulty would have been the existence of two mutually exclusive patterns depending on the corresponding set-up, since either distribution by itself would not have been difficult to handle. The mathematician needed not to think in terms of describing mathematically some moving classical-like objects, particles, that hit the screen, but only in terms of two different patterns that are produced. The mathematician then only needs to find the mathematics that predicts them in terms of two different probability distributions. (There are no waves either.) This mathematician would have had to have made two extraordinary guesses—one truly extraordinary and, with the first guess in hand, the other somewhat less so. The first guess is that one can use a Hilbert space over complex numbers, and along with it what we now call observables, which are operators, and then probability amplitudes, which are vectors in this space. This is difficult, but for a doctoral student of Hilbert it would not have been impossible. The second guess would have been Born's rule, and while not easy either, it would be almost natural, because probabilities are real numbers and, as indicated above, moving from complex to real numbers, the moduli of complex numbers is the most obvious way to do so. One would need a square moduli, but that could be figured out by trial and error, and Born's first guess was just moduli, too. Indeed, Hilbert's student could have thought of von Neumann's projection postulate as well. Finally, Planck's constant could have been established from the data as well, at least in principle.

This fable is not that far from how Heisenberg made his discovery of quantum mechanics as described above, since to a large degree he had to suspend, to "forget," classical physics or the old quantum theory to arrive at his new mathematical scheme. His physics was defined by finding this predictive mathematics from the available data. Of course, unlike in the case of my fictional mathematician, the equations of classical mechanics and the correspondence principle guided his reinvention of the wheel of (in effect) Hilbert space formalism. Still, his process was deeply mathematical and, as such, quite analogous to that described in my fable. The crucial point was, again, that while he adopted the *form* of classical equations, he did not use them as equations of *motion*, and he guessed new variables that were necessary to predict the data, which was a mathematical guess. This is why he made his statement, with which I started here and which merits to be repeated now, with the preceding analysis in mind: "it is very difficult to modify our language so that it will be able to describe these atomic processes, for words can only describe things of which we can form mental pictures, and this ability, too, is a result of daily experience. Fortunately, mathematics is not subject to this limitation, and it has been possible to invent a mathematical scheme—the quantum theory [e.g., quantum mechanics]—which seems entirely adequate for the treatment of atomic processes" [2, p. 11]. One has to find or, again, invent the right mathematics, however, which is not easy.

The new character of quantum theory introduced by Heisenberg was also bound to have an impact on the very practice of theoretical physics in the quantum domain, as it in fact did, beginning with Heisenberg's and, perhaps especially, Dirac's work. Indeed, it may be argued that a new way of doing theoretical physics has effectively taken quantum theory over ever since, whatever the philosophical attitudes of the practitioners themselves may be. In this new paradigm, the practice of theoretical physics is transformed into working with the mathematical apparatus of the theory (while building upon the preceding mathematical architecture) to make this apparatus enable correct predictions, rather than trying to develop an idealized mathematical description of the physical processes considered. Dirac spoke, including in describing his discovery of his even more famous equation for the (free) relativistic electron, of most of his work as "playing with equations," to which expression the present analysis gives a more rigorous meaning [25]. This "playing with equations," as a mathematical *work*, is then related to the outcome of experiments in terms of actual numbers, by applying these equations to the numerical data obtained in the previously performed experiments. This is in particular how Dirac discovered his equation [25].

There is also a deeper philosophical point here. In my fable, the mathematician still invents the standard formalism of quantum mechanics in whichever of its several equivalent forms, all of which imply or allow for a Hilbert space of formalism over complex numbers, cum Born's rule for probabilities. That, however, does not mean that another formalism cannot be invented, since, unlike in classical physics, we are no longer constrained by descriptive imperatives tied to our phenomenal representations of physical processes and concepts and language defined by this representation. Any mathematics will do, if it works predictively, although I do suspect (but cannot, it follows, be certain either) that complex numbers are unavoidable, and their mathematical nature and their use in quantum mechanics fit nicely given the epistemology of the situation, as against that of classical physics where real numbers are naturally used descriptively with respect to observed phenomena. All versions of formalism hitherto available use complex numbers (and indeed all these versions are mathematically equivalent); real numbers only correspond to probabilities of the events. Probabilities themselves are, again, by definition unavoidable given the data, which, unlike quantum processes, we can observe and describe in real, actually rational, numbers.[8] This approach may be and was, by Heisenberg in his later works, seen

[8] My suggestion that complex numbers may be unavoidable need not imply that the probability-amplitude approach is unavoidable, since, while related, these two aspects of the standard formalism are not the same. Indeed, my point here implies a possibility of an alternative as concerns the use of amplitudes. See [26] and, in the context of quantum information theory, [27] (and further reference there). The latter article, by C.A. Fuchs and R. Schack, pursues the approach that aims to bypass "amplitudes" and establishing quantum theory as working directly with probabilities in the case (it is worth qualifying) of discrete variables. If successful, this approach would enable one to relate more directly to quantum experiments, as

as closer to Plato, as opposed to Aristotle, whose physics, as I explained, is a model for the classical physics of objects and motion. Heisenberg's view is quite different from Plato in other respects, in particular by virtue of assigning any specific actual or imaginable form, mathematical or other, to the quantum-level reality, beyond a weak ontological assumption that quantum objects exist and exist independently of our existence. The actual character of their existence is not only beyond our knowledge but also beyond our thought; hence, our thought cannot approach their reality in the way Plato believes to be possible. Both views, however, share the emphasis on the role of mathematics in our thought concerning the world, or indeed, "reality," again, meaning by the term that which exists, while, by necessity, abandoning any pursuit of quantum-level physical description, which pursuit defines Einstein's vision of and hope for physics.

This, again, does not mean that the future development of quantum theory, for example, quantum gravity will necessarily conform to the paradigm here outlined, as against, say, Einstein's hope for a more classical-like approach. Even in the case of quantum mechanics, the spectrum of alternative expectations is wide—from the hopes that the theory might be proven incorrect even within its proper scope to more classical views of it (or of quantum phenomena) to the possibility of more classical alternative theories of quantum phenomena. Nevertheless, there does not appear anything thus far that substantively contradicts the present view of quantum mechanics or, as quantum field theory would suggest, quantum theory in general. On the other hand, there appears to be much to support such a view. Once things become quantum, epistemological complexities appear to show up, making the use of mathematics here discussed at least compelling; and although one cannot be certain on that score either, there is thus far no physical reason to believe that it will be otherwise in the case of string or brane theories, quantum gravity, or quantum cosmology.

5 Conclusion

Quantum mechanics, thus, and then higher-level quantum theories continue classical physics insofar as it is, just as classical physics, from Galileo on, and then relativity have been, the experimental-mathematical science of nature. However, quantum theory, at least, again, in the interpretations of the type discussed here, breaks with both classical physics and relativity by establishing radically new relationships between mathematics and physics, or mathematics and nature. The mathematics of quantum theory is able to predict correctly the experimental data in question without offering and even preventing the description of the physical processes responsible for these data. Indeed, taking advantage of and bringing together both main meanings

opposed to a more artificial machinery of the present version of quantum mechanics. The mathematics of both approaches is equivalent to that of the standard formalism, which suggests that, although we may be able to avoid amplitudes, we might need the mathematics that involves complex numbers. The argument of Fuchs and Schack expressly depends on Hilbert spaces (in this case of finite dimensions) over complex numbers and would not work with Hilbert spaces over real numbers. This fact as such does not prove that a real-number quantum mechanics (which could, accordingly, not be mathematically equivalent to the standard version) is impossible, but it does indicate that it might be difficult to develop it.

of the word "experiment," I would argue, that, while not without some, indeed indispensable, help from nature, quantum mechanics was the first physical theory that is both, and jointly, truly experimental and truly mathematical. It is (I am indebted to G. Mauro D'Ariano on this point) truly experimental because it is not, as in classical physics, merely the independent behavior of the systems considered that we track, but what kinds of experiments we perform, how we *experiment* with nature, that defines what happens. Of course, we experiment, often with great ingenuity, in classical physics as well. There, however, our experiments do not define what happens, but merely track what would have happened in any event. By the same token, quantum mechanics is truly mathematical because the mathematical formalism of the theory is not in the service of such a tracking, by way of auxiliary description of what would have happened anyhow, but is in the service of predictions defined by our experiments. Quantum theory makes us *know* less than we used to think it possible to know. On the other hand, we can *do* more, indeed for the first time we can do something in *defining* the world by our experiments, and our experiments cannot avoid doing so. This point is crucial because some of our classical experiments may also change the world if our interference is sufficient to significantly disturb the classical configuration involved.

Indeed, it also follows that we experiment with mathematics as well, in any event more so than in classical physics, since we invent mathematical schemes unrelated to any reality rather than refine our phenomenal perceptions or representations, which constrain us in classical physics. Heisenberg's discovery of quantum mechanics was a remarkable product of this type of mathematical experimentation. One might say that experimentally and mathematically, or experimentally-mathematically, quantum mechanics is essentially *compositional*. It is, I think, fitting to borrow a term from music or abstract painting. Both, with Arnold Schoenberg and Igor Stravinsky, modernist music and, with Wassily Kandinsky and Piet Mondrian, abstract painting were dramatic examples of modernist art, contemporary, and in some of their aspects, parallel to abstract mathematics and quantum mechanics. Quantum mechanics brings together the art of experiment and the art of mathematics; and by so doing it also tells that, while the effectiveness of mathematics in quantum theory may be enigmatic, it is extraordinary, and that it might be extraordinary not so much in spite of being enigmatic but because of it.

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