Bernhard Riemann's Conceptual Mathematics and the Idea of Space

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Abstract:

This essay explores the power and fertility of mathematical imagination, as they are manifest in the thought of a nineteenth-century German mathematician Bernhard Riemann (1826–1866), one of the greatest and most imaginative mathematicians who ever lived. Riemann introduced radically new ideas in every main field of modern mathematics: algebra, analysis, geometry, and topology. These ideas transformed each of these fields and played major roles in making mathematics into what it is now. The essay considers in particular two interrelated aspects of Riemann's work: the first is his concept of "manifold(ness)," which transformed our mathematical, physical, and philosophical understanding of spatiality; and the second is the *conceptual* character of Riemann's mathematical thinking as responsible for the radical nature of his ideas, such as those concerning spatiality and/as manifoldness. The essay also addresses, in closing, some of the implications of Riemann's ideas for modern physics, most especially for Albert Einstein's general relativity—his non-Newtonian theory of gravitation.

Introduction

Richard Feynman once said that "a new idea is extremely difficult to think of. It takes a fantastic imagination."[1] It is, accordingly, all

1. Richard Feynman, *The Character of Physical Law* (Cambridge, MA: MIT Press, 1995), p. 172.

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the more remarkable that Bernhard Riemann (1826–1866), a nineteenth-century German mathematician—and one of the greatest and most imaginative mathematicians who ever lived—thought of radically new ideas in every main field of modern mathematics: algebra, analysis, geometry, and topology. His ideas transformed each of these fields and, in part through establishing new connections among them (one of the hallmarks of Riemann's thought), played major roles in making each field and mathematics as a whole into what they are now. As Detlef Laugwitz observes in his biography of Riemann, *Bernhard Riemann: Turnings Points in the Conception of Mathematics*: "It is an amazing fact that fundamental parts of modern mathematics have their origins in Riemann's analysis."[2] Laugwitz's subtitle is worth noting for its plural of "turning *points*," and for its suggestion that these were the points at which not only mathematics itself, but also our conception of it changed. Given my limits here, I shall focus primarily on two interrelated aspects of Riemann's work. The first is the power and fertility of his thought and imagination as manifest in his concept of "manifold," or "manifoldness," one of his great inventions, which transformed our mathematical, physical, and philosophical understanding of spatiality. The second is the *conceptual* character of Riemann's mathematical thinking in general as responsible for the radical nature of his ideas, especially those concerning spatiality and manifoldness.

Following Gilles Deleuze and Félix Guattari's *What Is Philosophy?*, I understand by "thought" a confrontation of the human mind with chaos, and in particular, the way this confrontation takes place in art, philosophy, and science, including mathematics.[3] These modes of thought are, for Deleuze and Guattari, the primary ways in which the mind confronts chaos. This confrontation itself, they argue, is necessary for thought to be *true* thought—the dynamic engagement of mental activity, especially against the static dogmatism of opinion. Deleuze and Guattari see chaos as a grand enemy, but also as the greatest friend of thought and its best ally in its yet greater struggle—that against opinion, *doxa*, always an enemy, "like a sort of 'umbrella' that protects us from chaos." They argue that

2. Detlef Laugwitz, *Bernhard Riemann: Turning Points in the Conception of Mathematics*, trans. Abe Shenitzer (Boston: Birkhäuser, 1999), p. 130. Although Laugwitz specifically invokes analysis, where Riemann made most of his contributions, his ideas in geometry and topology were just as revolutionary and significant, and Riemann combines different fields even while working (overtly) in a particular field.

3. Gilles Deleuze and Félix Guattari, *What Is Philosophy?* trans. Hugh Tomlinson and Graham Burchell (New York: Columbia University Press, 1994).

> the *struggle against chaos* does not take place without an affinity with the enemy, because another struggle develops and takes on more importance: the struggle *against* opinion, which claims to protect us from chaos itself. . . . [T] he struggle with chaos is only the instrument in a more profound struggle against opinion, for the misfortune of people comes from opinion. . . . And what would *thinking* be if it did not confront chaos? (202, 208)

Imagination, then, may be understood as the capacity of thought to invent new forms of this confrontation, including new *forms* of combining old forms, assuming that any other new "forms" are even possible (since no thought starts from nothing). To the degree that they are, as might be in Riemann's case, they are indeed exceptionally rare and require a truly fantastic imagination, as Feynman asserted.

Riemann's mathematics, I also argue, is *conceptual* mathematics, a form of mathematics that gives primacy to thinking in mathematical concepts and their properties and structure, rather than to thinking primarily in terms of (by manipulating) formulas, which dominated mathematics before Riemann, or in terms of sets, as is most common in present-day mathematics. I shall suspend the well-known difficulties and paradoxes of the concept of "set" (which are not germane to my argument here), and shall use what is called in mathematics a "naïve definition" of set, which is sufficient for my purposes. In Pierre Cartier's formulation, via Bourbaki, "a *set* is composed of *elements* capable of having certain *properties* and certain *relations* among themselves or with elements of other sets."[4] Thus natural numbers (1, 2, 3, . . .) form a set, or a sphere is a set, with each point as an element of this set. A two-dimensional sphere (in the three-dimensional Euclidean space), with center (x_0, y_0, z_0) and radius r is defined in analytic geometry as a locus of all points (x, y, z) satisfying the equation $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$, a formula that was de facto used even before Descartes, the inventor of analytic geometry. This is, however, not the way in which Riemann would have primarily viewed either object; instead, he considered natural numbers or the sphere first as a certain specifically determined concept—in the case of the sphere, that of (continuous) manifold.

4. Pierre Cartier, "A Mad Day's Work: From Grothendieck to Connes and Kontsevitch, the Evolution of Concepts of Space and Symmetry," *Bulletin of American Mathematical Society* 38:4 (2001): 393. Nicolas Bourbaki is the pseudonym of a group of mostly French mathematicians, formed during the 1930s and still in existence. One of the group's aims was to formulate and present in a series of books, with the maximal possible rigor, all of mathematics as founded on set theory—a program never fulfilled and currently more or less suspended.

The above formula for the sphere remains important and plays its role, but it is secondary to the manifoldness of the sphere, which, as I shall explain, tells us more about it as a mathematical object.

Riemann's concept of manifold or manifoldness (*Mannigfaltigkeit*) is a product of this way of thinking, coupled with the extraordinary power of his mathematical and philosophical imagination. Riemann radically reimagines the nature of spatiality: phenomenal, philosophical, mathematical, and, as shall be seen, physical. Riemann defines a space, such as a plane or a sphere, as a conglomerate of local spaces and networks of relationships among them, rather than as merely a conglomerate (set) of points, as it would be from the set-theoretical perspective. More specifically, a space as a manifold is a conglomerate of (local) spaces, each of which can be mapped by a (flat) Euclidean, or Cartesian, coordinate map and treated accordingly (and thus also given geometry), without allowing for a global Euclidean structure for the whole, except in the limited case of a Euclidean homogeneous (flat) space itself. Every point has a small neighborhood that can be treated as Euclidean, while the manifold as a whole, in general, cannot. In the case of the sphere, one can imagine small circles on the surface around each point, and project each such circle onto the tangent plane to this point to a regular circle on this plane. If the first circle is very small, the difference between two circles becomes very small as well and can be neglected, allowing one to treat both circles as Euclidean. What is most crucial, however, is not how a given space is composed of points, but how it is composed of other spaces, some of which are considered as local subspaces, specifically those, called "neighborhoods," that surround each point of this space, as the small circles just described do in the case of the sphere.

The concept suggests a kind of "sociological" definition of space, as Yuri Manin called it in the context of the category and topos theories, which arguably extend Riemann's idea to its ultimate limits, at least for now.[5] The significance of Riemann's concept, however, extends not only to other areas of mathematics and physics, but also to philosophy. Indeed, I shall first consider the relationships between mathematical and philosophical thinking, which are central for my discussion here, since it is philosophical rather than (technically) mathematical and explores primarily the philosophical content of Riemann's concept of manifold. Riemann's concept is, I shall argue, best understood by viewing it as having a conglomerative,

5. Yuri I. Manin, "Georg Cantor and His Heritage." 2002. http://arxiv1.library.cornell.edu/abs/math/0209244.

multi-component architecture (mathematical, philosophical, and other), rather than by segregating its components into separate concepts or clusters among or within each field.[6] It is true that, insofar as this kind of philosophical discussion bypasses technical aspects of mathematics, certain aspects of mathematical thinking may be lost. Perhaps, however, these losses are not as severe as they might appear, and something in our understanding of the *mathematical aspects* of Riemann's concept is gained as well.

Mathematics, Philosophy, and Conceptual Thinking

The relationships between mathematical and philosophical thought, and between both and ordinary thought, beginning with our general phenomenal intuition (defined here simply as what appears to our consciousness immediately rather than through philosophical mediation), is a delicate and difficult matter, and my reflections on it are unavoidably limited. I begin with an observation made by Hermann Weyl, a follower of Riemann and a great twentieth-century mathematician in his own right, who also made major contributions to physics—specifically, to relativity and quantum theory. In his 1917 book *The Continuum*, Weyl says:

> The conceptual world of mathematics is so foreign to what the intuitive continuum presents to us that the demand for coincidence between the two must be dismissed as absurd. Nevertheless, those abstract schemata supplied us by

6. This *concept* of concept corresponds more to Deleuze and Guattari's view of philosophical thought, defined by them as an invention of new concepts in this multi-component sense (*What Is Philosophy?* [above, n. 3], pp. 1–23). They view mathematical or scientific thinking as proceeding (confronting chaos) by means of functions, propositions, and frames of reference. In this respect, the present argument, or Riemann's thinking, challenges, or rather *complements*, Deleuze and Guattari's argument in *What Is Philosophy?* The latter distinguishes between mathematical or scientific and philosophical thought, and sees only the latter as primarily conceptual in this sense. Ultimately, however, they suggest the interactive, if still heterogeneous (this qualification also applies to the present argument), relationships between mathematical and philosophical thinking, or between both and artistic thinking. I have to put art aside here, even though it can be linked to Riemann's thinking via the question of composition, which defines creative activity in art in Deleuze and Guattari. Riemann defined a given space as composed by other spaces, rather than merely by points (one could think of Jackson Pollack's paintings as suitable images here). Not coincidentally, Riemann, primarily via his concept of manifold, was one of the inspirations for Deleuze's and Deleuze and Guattari's philosophy. I suspect that they would have agreed that Riemann's concept of manifold conforms to their view of philosophical concepts, and they say nearly as much in their appeal to Riemann's idea on manifold in their works. I have discussed the relationships between Riemann and Deleuze, along the lines of the present analysis, in "Manifolds: On the Concept of Space in Riemann and Deleuze," in *Virtual Mathematics*, ed. Simon Duffy (Manchester, UK: Clinamen Press, 2005).

> mathematics must underlie the exact sciences [e.g., physics] of domains of objects in which continua play a role.[7]

As Weyl was undoubtedly aware, this assessment could be extended to other conceptual junctures between mathematics and our phenomenal intuition. The question of continuity, however, appears to pose particularly thorny problems in this respect. It is also especially pertinent to Riemann's work, including to his thinking concerning space, although Georg Cantor's set-theoretical problematic, against which Weyl positions his own concept of the continuum, is Weyl's primary context here. Riemann's thought was a major inspiration for Weyl in *The Continuum* and to the critique of Cantor that Weyl undertook there. Weyl expressly addressed Riemann's ideas in his classic *Space-Time-Matter*, to which I shall return below, and elsewhere in his works, particularly in *The Idea of Riemannian Surface* (1913).

Weyl's assessment of the problem of the continuum requires qualification, as Weyl himself must have been aware. The *coincidence* between the conceptual world of mathematics and our phenomenal intuition may be dismissed, but the *interactions* between them are unavoidable. The significance of these interactions may be obscured if one assumes that mathematics comes to us from some mathematical reality existent independently of human thought (a view often defined as "Platonist," without always a due regard for Plato's own thinking). It becomes more transparent if, instead, one sees mathematics as, in Niels Bohr's words, a "refinement" of our ordinary thought, "supplement[ed] . . . with appropriate tools to represent relations for which ordinary verbal expression is imprecise and cumbersome."[8] This refinement does sometimes reach the point, pondered by Weyl, of divorcing our mathematical thinking from our ordinary phenomenal intuition *nearly* altogether, which could, it is true, make the Platonist's view especially tempting—as it was, for example, to Cantor (Weyl's and Riemann's positions on the subject are more complex). Nearly, but, I would contend, not completely!

In addition, mathematical intuition and thought are further coupled to the philosophical modes of intuition and thought, which

7. Hermann Weyl, *The Continuum: A Critical Examination of the Foundation of Analysis*, trans. Stephen Pollard and Thomas Bole (1918; reprint, New York: Dover, 1994), p. 108.

8. Niels Bohr, *The Philosophical Writing of Niels Bohr*, 3 vols. (Woodbridge, CT: Ox Bow Press, 1987), p. 2:68. The point is given further subtlety by Edmund Husserl in "The Origins of Geometry," in Husserl, *The Crisis in European Sciences and Transcendental Phenomenology*, trans. David Carr (Evanston, IL: Northwestern University Press, 1970), pp. 375–377.

cannot (any more than mathematical ones) be fully dissociated from our general phenomenal intuition and thought. One finds the relationships among all three even within each one; that is, there is some philosophical and general thinking in mathematics, some mathematical and ordinary thinking in philosophy, and some philosophy and mathematics in our ordinary thinking. In making his point concerning the difference between the conceptual world of mathematics and our general phenomenal intuition, such as that of spatiality or continuity (which are interrelated), Weyl refers to Bergson's *Creative Evolution*. He says that "[i]t is to the credit of Bergson's philosophy to have pointed out forcefully this deep division between the world of mathematical concepts and the immediate experience of continuity of phenomenal time (*la durée*)."[9] The point itself concerning the division in question is philosophical, rather than mathematical, and, accordingly, its affinity with or origin in Bergson's philosophy does not complicate Weyl's point (or Bergson's own point) concerning the split between the conceptual world of mathematical and our general phenomenal intuition, or between either or both and philosophical thinking. The point is, however, complicated by the fact that Bergson's philosophical ideas concerning continuity and spatiality have a Riemannian genealogy, in part via Einstein's work, itself based in Riemann's geometry and his concept of manifold. Bergson thinks of space or time in terms of manifoldness in the sense derived from, even if not identical to, that of Riemann. I add "time," because Bergson's concept of the continuum of time is phenomenally spatial and manifold-like as well. Weyl did not need Bergson (though he sometimes needed Einstein) to mediate Riemann's ideas, which Weyl knew much better than Bergson as concerns their *technical* mathematical aspects. On the other hand, Bergson's philosophy, along with that of Franz Brentano, Edmund Husserl, Immanuel Kant, Johann Gottlib Fichte, and others, might well have helped Weyl to appreciate more deeply the philosophical dimensions and underpinning of Riemann's mathematics.

The case is, however, merely one especially telling instance of the fact that our thinking and concepts traffic among different domains, here between Riemann's mathematics and Bergson's philosophy or the philosophical thinking of others such as Deleuze, who were influenced by Riemann. In other words, if mathematics provides a powerful source for the ideas that can be developed in other domains of thought by virtue of its conceptual richness, the reverse is equally true, even though mathematicians tend to shun philosophical

9. Weyl, *Continuum* (above, n. 7), p. 90.

aspects or genealogies for their concepts. Weyl notes both of these tendencies in *The Continuum*, as he traces the philosophical genealogy of the idea of continuum in the phenomenology of Brentano, Bergson, and Husserl, as well as in the thought of such earlier figures as Fichte. This Kantian philosophical tradition is especially significant to intuitionist mathematics, with which Weyl's argument has certain affinities and that he came to embrace a bit later (albeit only temporarily).[10] Weyl's philosophical acumen was extraordinary and is perhaps unmatched in twentieth-century mathematics or science, and he makes use of this tradition throughout his work, including in his discussion of Riemann in *Space-Time-Matter*. On the other hand, as he observes:

> We cannot set out here in search of a definitive elucidation of what is to be a state of affairs, a judgment, an object, or a property. This task leads into metaphysical depths. And concerning it one must consult men, such as Fichte, whose names may not be mentioned among mathematicians without eliciting an indulgent smile.[11]

The concepts themselves, however, of "state of affairs," "judgment," "object," or "property" are unavoidable in mathematics.

These are general philosophical concepts, shared by mathematics and other scientific fields and, of course, by our everyday thinking. Philosophical genealogies for specific mathematical concepts, however, such as that of continuum, are hardly less important for mathematical thinking, as Riemann's work makes particularly evident. Riemann was significantly influenced by philosophy, which shaped his mathematics and his concepts, above all his concept of manifold or manifoldness as *Mannigfaltigkeit*—that is, as certain multi-foldedness within a unity (the German *Falt* means "fold") in the case of space, defining a given space as composed by other (local) spaces and the relationships among them. The standard English mathematical term is "manifold," used as a noun. Riemann's concept may owe a debt to Kant, who appears to be the first to consistently use the term in philosophy, and possibly also to certain theological

10. Briefly, "intuitionism," founded by the Dutch mathematician Lucius E. J. Brouwer, is defined by the following argument. To claim that an object with certain properties exists means that a mathematician can construct this object as available to our phenomenal intuition, rather than merely deduce this existence through the set of axioms and propositions free of contradiction. It is, accordingly, not surprising that Cantor's work (essentially dealing with objects that are beyond the grasp of our phenomenal intuition) was unacceptable to intuitionists.

11. Weyl, *Continuum* (above, n. 7), p. 7.

ideas; Riemann was originally trained in theology and the German for "trinity" is *Dreifaltigkeit* ("three folded into one," as it were).[12]

Riemann's concept thus results from a complex superposition of mathematical, philosophical, and theological thinking. The disciplinary specificity of mathematics remains important and should be rigorously respected, since otherwise it would be difficult to understand or even to speak of mathematics without a disregard for both its history and the way it actually works as mathematics—as against, for example, philosophy. In particular, mathematics and all mathematical natural sciences, such as physics, are defined by their capacity to give its objects, such as spaces, mathematically exact and specifically numerical features, which capacity became a defining part of the disciplinary practice of mathematics. This need not be the case in philosophy, which, when it adopts mathematical concepts and makes them into philosophical ones, tends to leave this numerical component of mathematics behind. The disciplinary specificity of mathematics or that of philosophy (however difficult to establish) does not, however, prevent the interactions between philosophical and mathematical thinking, or between each and other forms of thinking: it inevitably inflects these interactions, especially socially and institutionally. What is at stake, in other words, is an inevitably and irreducibly complex—heterogeneous yet interactive—field of determinations, both phenomenal and cultural.

As noted at the outset, Riemann's conceptual mathematics is different from set-theoretical mathematics, which is foundationally the dominant form of mathematics since Cantor, defining mathematical objects in terms of sets. I am not saying that they are simply opposed: although not without some losses in translation, they could be mathematically translated into each other, as was most of Riemann's mathematics, including his concept of manifold. Besides, set-theoretical mathematics inevitably involves mathematical (or philosophical) concepts, *beginning* with that of set (which is both mathematical and philosophical). The special position of this concept, however, as the primary, all-comprehending concept, defines the main difference between set-theoretical mathematics and Riemann's conceptual mathematics. The latter proceeds by working with specific concepts, such as "space," "manifold," "function," "series," and so on, and with their respective architectures in their relationships with one another without pre-comprehending them by a single concept, such as that of set. Each concept has a particular

12. According to Laugwitz, the ideas of Johann F. Herbart appear to be especially significant for Riemann; see *Bernhard Riemann* (above, n. 2), pp. 277–292.

mode of determination, such as discrete versus continuous manifold, and the constitutive elements or aspects of these concepts, such as points, are related through this determination, rather than considered in terms of formulas on the one hand and sets on the other.

Both the general ideology just described (Derrida would speak here of a "transcendental signified," which is also a "transcendental concept") and the set-theoretical specificity are important in establishing the difference between Riemann's and set-theoretical mathematics.[13] It is true that Riemann's work precedes set theory, which was, accordingly, not available to him, and it is difficult to guess how he would respond to it if he lived to see it (as he could have, given that he died at age 40, just a few years before Cantor introduced his set theory). My point at the moment is, however, philosophical rather than historical. It is difficult, although not impossible, to follow now Riemann's way of doing mathematics. It was nearly unique even in his time, and has been seen as controversial on account of its perceived (not always accurately) lack of rigor, especially if rigor is understood (as it often is) in set-theoretical terms—that is, as defined by proper manipulations of sets and their elements, and the relationships among these elements or among different sets. It is possible, however, to practice set-theoretical or so-called categorical mathematics along the lines of Riemann's general philosophy of working with multiple concepts, whereby the concept of set no longer occupies a unique transcendental position and does not govern, even in principle, all of the concepts involved. One could see these interactive relationships among concepts and different fields of research without an underlying transcendental, all-pre-comprehending master concept or set of concepts as non-Euclidean mathematics (which is not the same as non-Euclidean geometry, which merely deals with the mathematics of spaces different from Euclidean ones). This heterogeneity appears to shape most mathematics (apart, perhaps, from Euclidean two-dimensional geometry), even though many practitioners of mathematics, especially those of the Platonist persuasion, believe otherwise.[14] This understanding of the multiple interactive relationships among concepts that are no lon-

13. The term "transcendental" is taken here in the Kantian sense of the condition of the possibility of, in this case, all other signifiers and concepts, pre-comprehended by such a transcendental signified or concept, even though it may be hidden or never appear as such.

14. I have discussed the idea of non-Euclidean mathematics in *The Knowable and the Unknowable: Modern Science, Nonclassical Thought, and the "Two Cultures"* (Ann Arbor: University of Michigan Press, 2002), pp. 126–132.

ger governed by a single concept has appeared in philosophy at least since Friedrich Nietzsche; more recently, it has been made prominent by Derrida's work, especially his early work (which is closer to Nietzsche).[15] According to Derrida, there could be no single concept uniquely defining thought, and hence no single form of thought. Riemann's work offers a powerful example of the practice of this plurality in mathematics, which, it follows, is never only mathematics.

Geometry, Topology, and the "Sociology" of Space

While both geometry and topology are concerned with the mathematics of space, each is distinguished by its different provenance: *geometry* (geo-*metry*) has to do with measurement (initially that of the earth), while *topology* disregards measurement or scale and only deals with the structure of space qua space and with the *essential* shapes of figures. Topological figures are themselves generally seen as continuous spaces (possibly embedded in other spaces), although one can sometimes give a kind of (mathematical) continuity even to (phenomenally) discrete objects. Insofar as one deforms a given figure continuously (i.e., insofar as one does not separate points previously connected and, conversely, does not connect points previously separated), the resulting figure is considered the same; this sameness is difficult and perhaps impossible to think of by means of our general phenomenal intuition and, therefore, is mathematical. Indeed, the proper mathematical term is "topological equivalence." Thus all spheres, of whatever size and however deformed, are topologically equivalent, despite the fact that some of the resulting objects are no longer spheres, geometrically speaking. Such figures are, however, topologically distinct from tori, since spheres and tori cannot be converted into each other without disjoining their connected points or joining the disconnected ones, the holes in tori making this impossible. Such topological properties can be related to certain algebraic and numerical properties, most especially through the so-called cohomology and homotopy theories, which are among the great achievements of modern mathematics, extending in part from Riemann's work.

Discovering these relationships between topological (rather than only geometrical) properties and algebraic and numerical properties was crucial for making topology a mathematical discipline, since, as noted above, mathematics is disciplinarily defined by its capacity

15. See, in particular, Derrida's discussion, first via Nietzsche and then via Leibniz, in *Of Grammatology*, trans. Gayatri C. Spivak (Baltimore: Johns Hopkins University Press, 1974), pp. 19–20, 50, 75–80.

to give its objects mathematically exact and specifically numerical features. One might argue that the ancient Greeks had *philosophical* topology, as is suggested by Plato's concept of *khora* in *Timaeus*, which may even be seen as already questioning the very concept of spatiality. But they did not have a mathematical discipline of topology; their only mathematical (exact and quantifiable) science of space was geometry. Anticipated by Leibniz's conception of "analysis situs" (the term used by Riemann and for a while after him), topological ideas were gradually developed by Riemann and others, especially Henri Poincaré, whose work was uniquely responsible for establishing topology as a mathematical discipline.[16]

The concept of set—understood, again, as a multiplicity composed of elements having certain properties and certain relations among themselves or with elements of other sets—was introduced by Cantor shortly after Riemann's death in 1866. The concept has shaped foundational thinking in mathematics, including topology, and the very understanding, mathematical and philosophical, of what defined foundational thinking in mathematics from then on. As concerns space, perhaps the most crucial question is whether, *in approaching space*, one considers space itself as a primary, grounding concept or whether, with Cantor and most subsequent mathematics, one considers it as derived from the concept of set by, say, considering a given space as a particular set of points. I am referring to the mathematics of *spatiality* rather than mathematics in general (hence my emphasis), since, as explained earlier, there is no general, all-pre-comprehending transcendental concept governing all of mathematics in Riemann.

More recently, roughly from the 1950s onward, the so-called category theory offered an approach to mathematics that is closer to that of Riemann, especially through its connections to topology and algebraic geometry, a field developed in part following Riemann's work.[17] In this approach, spatiality—or at least a certain conception of a spatial type called "topos," developed in the so-called topos theory of Alexandre Grothendieck—replaces set as the primary concept, or forms the ultimate transcendental concept as it did for Grothendieck himself, whose philosophical orientation was arguably Platonist. One need not, however, see topos (or indeed set) as

16. The term "topology" was introduced by Johannes B. Listing, a contemporary of Riemann and, like Riemann, a student of Gauss.

17. One could also see category theory or especially (given its Platonist orientation) Grothendieck's work as an extension of, rather than in juxtaposition to, Cantor's mathematics, as Manin suggests; see "Georg Cantor" (above, n. 5), p. 8.

the grounding concept of all mathematics, and one can use the concept of topos even more effectively than that of set to question the very possibility of a transcendental mathematical concept. Topos theory also allows one to avoid the well-known paradoxes of set theory, such as that of the concept of the set of all sets. This concept, according to traditional set theory, cannot be consistently defined, since such a set can be immediately shown to be a member of itself if and only if it is *not* a member of itself. Topos theory allows for esoteric constructions such as spaces consisting of a single point and yet having an architecture that defines them, rather than structureless entities (classical points) or spaces without points, sometimes slyly referred to by mathematicians as "pointless topology." These constructions are far from pointless, however, insofar as each suggests that "space," or something space-like in character, is a more primary concept than that of "point" or "set of points" (as Kant already suggested at the phenomenal level). Below, I shall return to topos theory as an extension of Riemann's "sociological" understanding of space. Before I do so, however, and in order to make these connections clearer, I shall first discuss Riemann's understanding of space in more detail.

According to Riemann in his habilitation lecture, "On the Hypotheses Which Lie at the Bases of Geometry," which introduced the idea of manifold and Riemannian geometry,

> [t]he concepts of magnitude are only possible where there is an antecedent general concept which admits of different specialisations. According as there exists among these specialisations a continuous path from one to another or not, they form a *continuous* or *discrete* manifoldness [*Mannigfaltigkeit*]; the individual specialisations are called in the first case points, in the second case elements, of the manifoldness. Concepts whose specialisations form a *discrete* manifoldness are so common that at least in the cultivated languages any things being given it is always possible to find a concept in which they are included. (Hence mathematicians might unhesitatingly found the theory of discrete magnitudes upon the postulate that certain given things are to be regarded as equivalent.) On the other hand, so few and far between are the occasions for forming concepts whose specialisations make up a *continuous* manifoldness, that the only simple concepts whose specialisations form a multiply extended manifoldness are the positions of perceived objects and colours. More frequent occasions for the creation and development of these concepts occur first in the higher mathematics.[18]

18. Bernhard Riemann, "On the Hypotheses Which Lie at the Bases of Geometry," trans. William Kingdon Clifford, *Nature*, 8:183–84 (1873): 14–17, 36, 37 (hereafter HBG), available at http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Geom/WKCGeom.html, which I cite throughout this article. The passage just cited occurs in

As this description suggests and as Riemann's overall discussion in the lecture makes clear, he defines mathematical objects, both as a class (e.g., manifolds, or continuous and discrete manifolds) and each specifically (e.g., a given manifold), not in terms of ontologically pre-given assemblies ("sets") of points, which have or are then endowed with a certain set of relations between them; instead, he defines them in terms of concepts—specifically, in the case of continuous manifolds (most crucial here), as that of a space defined by a conglomerate of local spaces and the interactions between them. Hence, he speaks here of both general concepts and specializations (and subconcepts corresponding to such specializations) through which magnitudes could be defined. Each concept has a particular mode of determination, such as discrete versus continuous manifolds or those modes that further specify a given subclass of manifolds or even a single manifold, whose properties are defined by this determination rather than only or primarily by formulas. Continuous and discrete manifolds are thus *different* concepts, subspecies of the concept of manifold.

This view implies a redefinition of the very concept of "concept" in mathematics and beyond—specifically, according to Deleuze and Guattari, in philosophy (keeping in mind the disciplinary specificity of each field). Like a philosophical concept, a mathematical concept, even when it is a general one, is never merely a generalization from particulars or an abstract idea, but is always defined by a specific architecture, which has multiple components. Such a concept may and usually does include concepts in the conventional sense (of generalizing from particulars), but only as components, as part of this more complex architecture. Thus Riemann's conceptual mathematics goes beyond merely giving an essential priority to thinking in terms of mathematical objects as concepts rather than to calculational approaches, defined as thinking by manipulating formulas or equations; while his habilitation lecture contains only one formula(!), his argument is nevertheless unlikely to be accessible to a lay reader, certainly not without a major effort.[19]

section 1. Riemann presented his habilitation lecture in order to obtain his position as Privatdozent at Göttingen (the rough equivalent of a tenured associate professor). Habilitation, which is still necessary in Germany and other countries to gain a permanent university position, requires a second dissertation, which must be defended before an academic committee (chaired by Gauss in Riemann's case) (translation modified by author).

19. Laugwitz's discussion of Riemann's conceptual mathematics takes a more conventional view of mathematical concepts and, in part as a result, appears to me to miss the radical aspects of Riemann's thought discussed here; see *Bernhard Riemann* (above, n. 2), pp. 303–307.

Riemann's conceptual approach allows him to get hold of the properties of mathematical objects and of relationships that would be difficult and perhaps impossible to handle or establish otherwise than by means of the concepts involved. No conglomerate of formulas as such may allow us to say as much about, say, a sphere (which can, of course, be defined analytically by an equation), as seeing it as a manifold and by exploring its properties as those of a manifold, even if in conjunction with formulas and equations. For example, the topological difference between spheres and tori is a rich source of information concerning these objects, and Riemann was the first to realize and to take advantage of this fact. The flow of a liquid on a sphere, for instance, where turbulence is unavoidable, is very different from flow on a torus, where it can be free of turbulence. This fact has major mathematical as well as physical significance and implications. Riemann also pursues (as effectively as anyone) an analysis of and work with formulas, but he does so primarily as auxiliary to establishing and analyzing conceptual determinations, which, he shows, tell us more about mathematical objects than formulas. Riemann uses the same approach in his analysis of functions of complex variables, or his work on the so-called ζ-function and the distribution of prime numbers. The ζ-function can be given by a formula, but Riemann derived his remarkable results by considering certain deeper properties reflected in this formula—in other words, by treating it as a mathematical concept in the sense I am describing.

It is worth noting that Riemann speaks of "points" only in the case of continuous manifolds, and uses the term "elements" for the simplest constitutive entities comprising discrete manifolds. This is astute, since points *qua* points only appear in relation to some continuous space, such as a line or a plane, although the situation involves considerable mathematical complexities, especially when considered set-theoretically. Riemann appears to have sensed these complexities, although he allows for the possibility that discrete manifolds may function mathematically as spaces, or that space in nature may be a discrete manifold. Mathematics has subsequently developed ways to speak of discrete *spaces*, or even spaces with a finite number of points or, again, single-point spaces. Riemann, however, primarily pursues a conception of space as a continuous (three-dimensional) manifold.

The key conceptual component of his approach is to define a topologically and geometrically complex space, particularly a curved space such as a sphere, not as a constitution of points, but as a space that could be covered by maps whereby it can be treated as locally Euclidean, even though globally the space itself may not be Euclidean.

The idea of mapping a given space with local spaces is significant for our understanding of Euclidean spaces as well, in contrast to the set-theoretical view. Topology describes a space not so much by its points, but by the class of its so-called *open* sets—the concept that underlies Riemann's concept of manifold, but that allows for a more general mathematical definition of topological spaces, which need not be locally Euclidean and hence need not be manifolds. One can conceive of such sets on the model of open intervals of the line: say, all points between 1/4 and 3/4, except for these two points themselves, which are its boundaries. The standard mathematical notation is]1/4, 3/4[. A closed interval, usually designated as [1/4, 3/4], includes its boundaries. Open or closed intervals can be thought of alternatively as spaces or sets, or both. The *problem* of the continuum is to determine how a given continuum is constituted (as a set) by its points, and whether we can exhaust the straight line by a set of real numbers, the problem correlative to Cantor's continuum hypothesis.[20]

Brouwer questioned, on intuitionist grounds, the legitimacy of the set-theoretical concept of the continuum of the straight line as constituted by real numbers or even by points, which in principle he regarded as inaccessible to human intuition, general or mathematical (which he was more reluctant to dissociate than Weyl). A continuous space such as a straight line (or what appears to us intuitively as such) may, however, be described or defined differently (which was acceptable to Brouwer): not by the set of its points, but by a class of its open subspaces *covering* it. Such subspaces may overlap, as, for example,]1/4, 3/4[and]1/2, 1[, generating new open subspaces (in this case,]1/2, 3/4[) in the overall covering atlas. Any open interval containing a given point is called a "neighborhood" of this point. Thus both]1/4, 3/4[and]1/4, 1/2[would be neighborhoods of

20. The statement of the hypothesis admits a number of different formulations, and the mathematical or philosophical equivalence of these formulations is, in turn, a complex matter. For the present purposes, the hypothesis concerns the set of real numbers as an infinite cardinal (a concept of number introduced by Cantor to compare the number of elements of different sets, especially different infinite sets). The question is whether there exists a set whose power (a number of elements) is less than that of the set of real numbers, but larger than the power of the set of natural numbers. The answer is, again, complex, although the problem is generally considered to be solved in mathematics. Cantor's statement is viewed as "undecidable"—that is, unprovable to be either true or false within certain systems of axioms. Kurt Gödel, who introduced the idea of undecidable propositions, also contributed to the solution of the continuum problem, finally reached by Paul J. Cohen in 1963. Cohen, however, argued later that Cantor's statement could be seen as false, even "obviously false"; see Cohen, *Set Theory and the Continuum Hypothesis* (1966; reprint, New York: Dover, 2008), p. 151.

1/3, and the first of these neighborhoods will contain the second, or will overlap with a neighborhood such as]1/6, 2/5[. Topologically, all such intervals are equivalent, and]0, 1[represents any of them.[21]

It is true that if one appeals to open intervals as sets, as is usual, the concept of the line sketched here retains the concept of set as a primitive concept. The approach itself, however, offers a more general definition of space as comprised or, again, covered by other spaces, which allows for using this structure as a primitive one by replacing the concept of covering a space by "open sets" with the concept of covering it by "open spaces." A topological space is defined as (covered by) a collection of such open spaces as subspaces of the initial spaces by providing certain (algebraic) rules for the relationships among these subsets.

These ideas arguably find their most radical incarnation in Grothendieck's topos theory. The theory is prohibitively difficult in view of its mathematical abstractness and rigor; however, the essential philosophical ideas involved may be sketched here. In this approach, a given space, X, may, at least initially, be left unspecified as concerns its internal constitution. What would be specified instead are the relationships between this space and other spaces of the same type, which are seen as "mapping" or "covering" X. The internal constitution of X is then defined through these external relationships among spaces. We could call the structure defining these relationships the "arrow structure": $Y \rightarrow X$, where the arrow designates the relationship in question, also known as a "morphism," between two spaces. (The notation $Y \rightarrow X$ is used in mathematics.) An especially important example of the arrow structure is what is called a "fiber bundle" or "sheaf" (the latter is, technically, a somewhat different concept, but this difference is not essential here): Y is "fibered" into a conglomerate of subspaces F (fibers), each of which is "projected" onto ("covers") each point of X. Grothendieck's "topos" is a still more complex object, defined as a conglomerate ("topos") of sheafs over a given space.

21. The procedure sketched above can also be used to define the topology of curves or higher-dimensional spaces, flat or curved. It enabled Riemann to define manifolds of any dimensions, even infinite-dimensional ones, as collections of covering maps. As will be seen in the next section, the approach also allows one to define a curved surface or a manifold of dimension three and higher in terms of its inner properties, rather than in terms of its relation to its ambient Euclidean space, where such a manifold might be placed. The infinitesimal flatness of such spaces does not prevent them from having a curvature at any point.

The procedure just outlined thus enables one to specify and give a space-like structure to a given object of a spatial, or even, conventionally, not spatial, type not in terms of its *intrinsic* structure (e.g., a set of points with relations among them), but, to return to Manin's language, "sociologically," through its external relationships with other objects of the same type—or, in mathematically rigorous terms, of the same "category," such as the category of Riemannian spaces as manifolds.[22] Space, accordingly, is always defined by a multiplicity ("society") of other spaces, rather than by a multiplicity of its points. A space may be without points, but it cannot be without (other) spaces to which it must relate. One does not have to give a uniquely privileged position to a Euclidean space, whether seen in terms of sets of points or otherwise; instead, a Euclidean space is just one specifiable object of a large categorical multiplicity—say the category of Riemannian spaces—and is merely marked by a particularly simple way by which we can measure the distance between any two points. Again, the most crucial feature of his approach is that any given space (it may be a point, for example, which gives a point a conceptual complexity and specificity) is defined in terms of its relations to other spaces. These spaces may be, but need not necessarily be, subspaces of a given space, or, as in the case of Riemann's manifolds, spaces mapping subspaces of a given space. One can also generalize the notion of neighborhood in a similar fashion by defining it as a relation between a given point and the spaces associated with it in this way—for example, via fiber bundles or sheafs, as just explained.

One of the starting points of Riemann's reflection on space was the possibility of non-Euclidean geometry, which led him to a new type of the non-Euclidean geometry, that of positive curvature. Positive curvature means that there are no parallel shortest or geodesic lines crossing any point external to a given geodesic. In Euclidean

22. Manin, "Georg Cantor" (above, n. 5), p. 7." "Categories" are defined by category theory as multiplicities of mathematical objects endowed with given structures and the relationships among them—in particularmorphisms or arrows, which are, as just explained, the mappings between these objects that preserve this structure. Studying such morphisms allows us to learn about the individual objects involved, often to learn more than we would by considering them only or primarily individually. Categories themselves may be viewed as such objects, and in this case, one speaks of "functors" rather than "morphisms." Because topology relates topological or geometrical objects, such as manifolds, to algebraic ones, especially the so-called groups (multiplicities defined by a single algebraic operation, such as multiplication, and certain properties of this operation), by its very nature it deals with functors between categories of topological or geometrical objects, such as manifolds, and categories of algebraic objects, such as groups.

geometry, where geodesics are straight lines, there is only one such parallel line, and in non-Euclidean geometry of negative curvature or the hyperbolic geometry of Johann Bolyai and Nikolai Lobachevsky (the first non-Euclidean geometry discovered), there are infinitely many such lines. Riemannian geometry encompasses all of these as special cases. Significant as the discovery of non-Euclidean geometry was for the history of mathematics and intellectual history, it was only a small part and, as Weyl says, in retrospect "a somewhat accidental point of departure" for Riemann's radical rethinking of the nature of spatiality and related developments.[23] Eventually, this rethinking had a nearly equal impact on the history of mathematics and, more indirectly, on intellectual history.

Riemann's concept of manifold is the most crucial part of this rethinking, and Riemannian geometry would most properly refer to the study and the very definition of space in terms of manifolds, especially continuous manifolds. Riemann also considers manifolds of higher dimensions and even of infinite dimensions; as noted above, he also considers discrete manifolds (which mathematically have the dimension zero), formed by isolated rather than continuously connected elements. (The concept of discrete manifold becomes important for Riemann's view of space in physics.) In modern usage, the term "manifold" is reserved primarily for continuous manifolds, Riemann's most significant contribution to modern geometry, and to our understanding of space in general.[24] A manifold is a conglomerate of (local) spaces, each of which can be mapped by a (flat) Euclidean, or Cartesian, coordinate map, without allowing for a global Euclidean structure of the whole, except in the limited case of a Euclidean space itself. Weyl speaks of Riemann's mathematics of manifolds as "a true *geometry*, . . . a doctrine of *space itself* and not merely like Euclid, and almost everything else that has been done under the name of geometry, a doctrine of the configurations that are possible in space."[25]

I shall now discuss in more detail the key geometrical principles behind the concept of manifold, beginning with Riemann's extension of Gauss's ideas concerning the *internal* geometry of curved surfaces—that is, the geometry independent of the ambient (three-dimensional) Euclidean space where curved spaces could be placed.

23. Hermann Weyl, *Space-Time-Matter*, trans. Henry L. Brose (1918; reprint, New York: Dover, 1952), p. 92.

24. Riemann mainly considered the so-called differential manifolds, which means that one can define differential calculus on such objects.

25. Weyl, *Space-Time-Matter* (above, n. 23), p. 102.

Riemann's main contribution in this respect was his discovery that Gauss's concept of (internal) curvature could be extended, via the so-called tensor calculus, to measurement in curved spaces of dimension three and higher, which Einstein used in his theory of general relativity (his non-Newtonian theory of gravitation). It is worth stressing that Riemann is concerned not with curves in a flat space, but with the curvature of space itself, which is why Riemann's concept is so important, given that the space in which we live is of three or possibly even higher dimensions. One can also treat spaces of lower dimensions as independent rather than embedded. The curvature of space could also be assumed to vary from point to point, from a neighborhood to a neighborhood. One of Riemann's contributions (again, crucial to Einstein's work) was his understanding that the concept of manifold is general enough to allow for such variations. Earlier non-Euclidean geometries retained some Euclideanism by conceiving of the corresponding spaces as globally homogenous curved spaces with the same constant curvature, positive or negative. In Riemann's geometry, spaces of variable curvature are allowed as well.

A related though separate feature of this new spatiality, again crucial to Riemann and to Einstein, extends Leibniz's view concerning the relational nature of all spatiality. In this view, actual space is no longer seen as a given, ambient (flat) Euclidean or as Newtonian absolute space or, in Weyl's words, as an (infinite) "residential flat" (*flat* is a fitting pun here) where geometrical figures or material things are put;[26] instead, it emerges as a (continuous) manifold whose structure, such as curvature, would, again, be determined *internally*, mathematically or physically (for example, by gravity, as in Einstein's general relativity), rather than in relation to an ambient space, Euclidean or not. From this point of view, the notion of empty space might be entertained mathematically or philosophically, and perhaps experienced phenomenally, but, as Leibniz grasped, it is difficult to apply to the physical world. In any event, all spaces, mathematical or physical, become subject to investigation in their own terms and on equal footing and in multiple sociological relations to other spaces, rather than in relation to an ambient or otherwise uniquely primary space. The internal, constant or variable structure of a given space may again be determined sociologically by the relations among this space and other spaces, whatever such spaces may be. As indicated earlier, in the case of Riemann's manifolds, this relation is defined

26. Ibid., p. 98. The concept of absolute space, or that of absolute time, introduced in his *Principia*, ultimately troubled Newton as well.

by the locally Euclidean structure of neighborhoods, by Euclidean maps locally covering a given manifold, without a global Euclidean map or even, in general, a single global non-Euclidean map, except in special cases of spaces of constant curvature (zero, positive, or negative).

Riemann's approach clearly required a very different way of thinking about and indeed imagining space from the one that defined mathematical and philosophical thinking concerning spatiality, which still defines most of our common Euclidean thinking concerning space. I would like in closing to consider the significance and implications of Riemann's ideas concerning space in physics, where, in particular, they were crucial for Einstein's general relativity theory, which, along with quantum mechanics, defines our present understanding of the ultimate constitution of nature.

Spatiality, Phenomenality, and Materiality from Riemann to Einstein

Thinking along the lines of Leibniz and against Newton, Einstein gave a rigorous physical meaning to the ideas concerning spatiality considered here and extended them by arguing that space and time are not given, but arise as the effects of our instruments, such as rods and clocks, and, one might add, of our perceptual and conceptual interactions with those instruments. Space is thus possible as a phenomenon (or as a concept) by virtue of two factors, which form, in Kant's language, the condition of the possibility of spatiality as well as of temporality: the first is the presence of matter and technology, such as instruments (or natural objects that assume their roles); and the second is our perceptual phenomenal machinery, whose role might be primary, as Kant argued, but whose very existence may only be possible by virtue of the materiality of our bodies. Einstein's theory itself concerns the role of materiality (gravity) in defining physical space. While the theory does tell us something about how nature works, rigorously speaking, it still only describes certain phenomena rather than nature itself, as Weyl argues, via Brentano and Husserl (*Space-Time-Matter* 1–10). As Weyl also argues, however, no theory can do more.

Riemann offers remarkable intimations of Einstein's theory. In the final section of his lecture "Application to Space," where he refers to Archimedes, Galileo, and Newton, Riemann proceeds from the contrast between discrete and continuous manifolds, saying:

> The question of the validity of the hypotheses of geometry in the infinitely small is bound up with the question of the ground of the metric relations of

> space. In this last question, which we may still regard as belonging to the doctrine of space, is found the application of the remark made above; that in a discrete manifoldness, the ground of its metric relations is given in the concept of it, while in a continuous manifoldness, this ground must come from outside. Either therefore the reality which underlies space must form a discrete manifoldness, or we must seek the ground of its metric relations outside it, in binding forces which act upon it.[27]

Riemann, however, goes on to say:

> The answer to these questions can only be found by starting from the conception of phenomena which has hitherto been justified by experience, and which Newton assumed as a foundation, and by making in this conception the successive changes required by facts which it cannot explain. Researches starting from general concepts, like the investigation we have just made, can only be useful in preventing this work from being hampered by too narrow views, and progress in knowledge of the interdependence of things from being checked by traditional prejudices.[28]

Riemann thus allows for the possibility that spatial reality may correspond to a discrete manifold and, in this case, its nature or structure could be established from within—that is, given by a (specific) mathematical concept that defines such a space. This may ultimately prove to be the case, although in most physical theories thus far—quantum theories included—space has been viewed as a continuous manifold in Riemann's sense. There have been exceptions, and more recently, the idea of the underlying discreteness of physical space acquired new prominence and gained new conceptual grounds in the context of quantum gravity, which would bring together general relativity and quantum theory. While equally essential for our understanding of nature, these two theories are thus far are irreconcilable, and no workable theory that would resolve this difficulty exists as yet. The main argument in question, however, concerns continuous manifolds and space as continuous phenomena, especially in the context of Leibniz, who anticipated Riemann's view, or Einstein, who gave this view a rigorous physical content. Einstein was able to do so by replacing Newton's assumption about the instantaneous action of gravity with the idea of curved space, where the speed of all processes is finite (i.e., below the speed of light in a vacuum).

According to Weyl, "Riemann rejects the opinion that has prevailed up to his own time, namely, that the metrical structure of space is fixed and [is] inherently independent of the physical phe-

27. HBG (above, n. 18), section 3, no. 4 (translation modified by author).

28. Ibid.

nomena for which it serves as a background, and that the real content takes possession of it as of residential flats" (98). It is usually a *flat* (Euclidean) residential flat, although flatness is less crucial here than the fact that the flatness or curvature of a given space is determined by matter. The space of our universe may prove to be flat on average, as recent cosmological theories seem to suggest, although (and this is crucial here) it is curved, and variably so in smaller regions, such as a solar system or a galaxy. Weyl then says, "*[Riemann] asserts, on the contrary, that space in itself is nothing more than a three-dimensional manifold devoid of any form; it acquires a definite form only through the advent of the material content filling it and determining its metric relations*" (emphasis in original).[29] It would perhaps be more accurate to say that space may be given phenomenally at most as a three-dimensional manifold. Physically, a space may be (in Riemann's view, as well as that of Einstein and Leibniz, could *only* be) co-extensive with matter: actual bodies (as in Leibniz) or propagating fields such as electromagnetism or gravity, or both (as in Riemann and Einstein). But the phenomenal component remains irreducible. Weyl adds that "[l]ooking back from the stage to which Einstein brought us, we now recognize that these ideas can give rise to a valid [physical] theory only after *time* had been added as a fourth dimension to the three-space dimensions."[30] He concludes:

> [W]e see that "the inner ground of metric relations" [of the continuous manifold that forms the reality that underlies physical space] must indeed be sought elsewhere. Einstein affirms that it is to be found in the "binding forces" of *Gravitation*. . . . The laws according to which space-filling matter determines the metrical structure are laws of gravitation. The gravitational field affects light rays and "rigid" bodies used as measuring rods in such a way that when we use these rods and rays in the usual manner to take measurements of objects, a geometry of measurement is found to hold which deviates very little from that of Euclid in the regions accessible to observation. These metric relations are not the outcome of space being a form of phenomena, but of the physical behavior of measuring rods and light rays as determined by the gravitational field.[31]

In sum, the gravitational field determines the manifold in question (and to some degree our phenomenal perception of it) and its curvature: the gravitational field shapes space and, moreover, shapes

29. Weyl, *Space-Time-Matter* (above, n. 23), p. 98.

30. Ibid., p. 101.

31. Ibid., pp. 101–102.

it as a Riemannian manifold. This fact radically transforms our philosophy of space and matter and of their relationships, first and most obviously in mathematics and physics, or in the philosophy of mathematics and physics, but also elsewhere. For instance, Einstein's techno-material efficacy of space and time, and of space-time, is not unlike the efficacy of Derrida's *différance*, which produces, as effects, multiple differences, proximities, and interactions between and among entities that in an un-deconstructed regime would be seen as unconditionally separate or opposite. Derrida sees *différance* as the material efficacy of both spatiality and temporality, of the spatiality of space and the temporality of time, or of the spatiality of time and the temporality of space.[32] Materiality is conceived by him so as to include the materiality of writing, using the term "writing" in Derrida's extended sense, reciprocal with a certain radical idea of materiality, coupled to the idea of technology via *différance* and other Derridean "neither terms nor concepts," such as trace, supplement, dissemination, and so forth. This broader view of materiality allows one to extend Einstein's technological argument concerning space and time to all our cultural production: all cultural artifacts, scientific theories included, become effects or products of this material "*différantial*" dynamics and thus are written in Derrida's sense by means of technologies of culture (beginning with pens and pencils, but hardly ending with them).[33] An analogous type of argument was developed in the constructivist social studies of science, where, more recently, an uncritical view of social constructivism as a single determining "technology" of such productions has been reexamined, bringing the resulting constructivist argument closer to the position offered in this essay.[34]

It also follows that the Riemannian or any other *phenomenality* of space is only possible by virtue of *materiality*.[35] This materiality is that of the material constitution of the *bodies* we possess and their material history, as well as that of technology. This technology is still

32. Jacques Derrida, *Margins of Philosophy*, trans. Alan Bass (Chicago: University of Chicago Press, 1980), p. 13.

33. Given my limits here, for a proper argument, I must refer to my previous discussion of this aspect of Derrida's thought in *The Knowable and the Unknowable* (above, n. 14), pp. 184–199.

34. See, in particular, Bruno Latour, *Pandora's Hope: Essays on the Reality of Science Studies* (Cambridge, MA: Harvard University Press, 2002).

35. I borrow this language from Paul de Man's discussion of Kant in his "Phenomenality and Materiality in Kant," in *Aesthetic Ideology* (Minneapolis: University of Minnesota Press, 1986), pp. 70–90.

enabled by our bodies and the universe that gives rise to our bodies and the perception, cognition, and thinking they enable (often helped by technology). One can here take advantage of the diverse concepts designated by the term "body," from the ultimate (quantum) constituents of nature, to human bodies, to bodies of stars and galaxies, to the body of the universe itself, or to political, cultural, and textual bodies (which have their own forms of materiality). The ultimate nature of this materiality may be unavailable not only in practice, but more crucially in principle to our phenomenality or conceptual capacities, to our thinking and imagination. Indeed, it may well be unavailable to us even beyond the way that Kantian things-in-themselves are unavailable, with the implication that terms such as materiality and matter are finally inapplicable. This unavailability is, however, quite different from an unavailability in practice, or even the unavailability in principle defined by theological or quasi-theological arguments, in which the ultimate constitution and architecture of the world is only available to God. The existence of such unavailable entities could, however, only be established by virtue of their capacity to affect what is available to us and produce available effects. The material objects considered by quantum theory are a good example, since here our thinking, our imagination, and our bodies are capable of arriving at a new conception of unavailability, and (with the help of the universe) to build technologies that establish the existence of unavailable material objects.[36]

Quantum theory—physically, an essentially discrete theory—takes us beyond Riemann's concept of continuous manifold, although Riemann's ideas concerning discrete manifolds may yet prove their significance for quantum theory. There have been some investigations along these lines since the early days of quantum theory, and quite a few physicists think that, given the ultimately quantum nature of physics, the ultimate constitution of space (or time) is unlikely to be continuous and would need to be quantized and considered as discrete in turn. On the other hand, even though quantum theory is discrete insofar as it deals with irreducibly discrete phenomena, most of its mathematics is continuous. Some of Riemann's ideas in continuous mathematics, and the developments that owe to them, are as significant for quantum theory as they are for relativity.

As I stressed at the outset, Riemann's revolutionary contributions are many and diverse, all the more so because there are also connec-

36. I discussed quantum mechanics in this context in *Reading Bohr: Physics and Philosophy* (Berlin: Springer-Verlag, 2006).

tions among them, and some of these connections are still not fully understood. One such contribution is to number theory: it concerns the distribution of prime numbers among natural numbers. Riemann was led to this contribution by his investigation of the ζ-function, known as the "Riemann ζ-function" (or "Riemann-Euler ζ-function"), and his famous hypothesis concerning the distribution of the zeros of this function, often considered the greatest unsolved problem of modern mathematics (after the Fermat theorem was proven by Andrew Wiles a few years ago). At first glance, this number-theoretical problematic appears to have nothing to do with physics and seems particularly remote from quantum theory; as has become apparent during the last decade or so, however, these ideas may in fact have deep connections with the highest and most complex developments of quantum theory. It is not possible to address here these developments (since new ideas and findings continue to proliferate), except to note that it is the product of extraordinary thought and imagination, approaching even those of Riemann, who is difficult to match and perhaps impossible to surpass.[37]

Perhaps the greatest achievement of Riemann's thought is this continuous impact of his ideas, both separately and in their interactions, through which they have shaped modern mathematical thought and imagination and will undoubtedly continue to do so. If there could be a single name for this mathematics, "Riemannian" might well be the best candidate, although, as explained earlier, there are also good reasons to call it "non-Euclidean" or "non-Cantorian." That no such single name is possible is, however, entirely in the spirit of Riemann's mathematics and the spirit of his thought, which, as I have argued here, cannot—nobody's thought can!—be only mathematical either

37. See Cartier, "A Mad Day's Work" (above, n. 4), pp. 406–407.