“Who Thinks Abstractly?”: Quantum Theory and the Architecture of Physical Concepts

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Abstract. Beginning with its introduction by W. Heisenberg, quantum mechanics was often seen as an overly abstract theory, mathematically and physically, vis-à-vis classical physics or relativity. This perception was amplified by the fact that, while the quantum-mechanical formalism provided effective predictive algorithms for the probabilistic predictions concerning quantum experiments, it appeared unable to describe, even by way idealization, quantum processes themselves in space and time, in the way classical mechanics or relativity did. The aim of the present paper is to reconsider the nature of mathematical and physical abstraction in modern physics by offering an analysis of the concept of “physical fact” and of the concept of “physical concept,” in part by following G. W. F. Hegel’s and G. Deleuze’s arguments concerning the nature of conceptual thinking. In classical physics, relativity, and quantum physics alike, I argue, physical concepts are defined by the following main features—1) their multi-component multiplicity; 2) their essential relations to problems; 3) and the interactions between physical, mathematical, and philosophical components within each concept. It is the particular character of these interactions in quantum mechanics, as defined by its essentially predictive (rather than descriptive) nature, that distinguishes it from classical physics and relativity.

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1. INTRODUCTION

Beginning with its introduction in Werner Heisenberg’s famous paper, “Quantum-Theoretical Re-Interpretation of Kinematical and Mechanical Relations” ([1]), the mathematical formalism of quantum mechanics and then of higher-level quantum theories was noted, sometimes disapprovingly, for its unduly abstract and specifically algebraic character. Albert Einstein spoke of Heisenberg’s “purely algebraic method,” as against those of classical physics and relativity, primarily defined by differential equations, Einstein’s preferred way of pursuing the mathematical description of nature ([2], p. 378). This juxtaposition is not unqualified. Quantum mechanics does contain differential equations, in particular its most famous equation, Schrödinger’s equation. Einstein was of course aware of this, especially given that the mathematical equivalence of both versions was long established by the time of his comment. This is why he spoke of Heisenberg’s “*method*,” which was radically different from those of classical physics and relativity, or that of Erwin Schrödinger in his invention of his wave mechanics, accordingly much preferred by Einstein to Heisenberg’s method as well. The main mathematical difference between classical mechanics or relativity and quantum mechanics concerns the variables to which the equations of quantum mechanics apply. In classical mechanics and relativity, these variables are (differential) functions of real variables. In quantum mechanics they are matrix variables with complex, rather than real, coefficients (essentially Hilbert-space operators), whose multiplication Heisenberg, in his reinvention of matrix algebra, famously found to be noncommutative, as against the variables used in

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classical mechanics.[1] The nature of these variables may be seen as reflecting the more essentially algebraic and more abstract character of quantum mechanics and then higher-level quantum theories.

The abstractness of quantum mechanics was amplified and for some, again, most famously, Einstein, exacerbated by its apparent, and in certain interpretations actual, inability to describe, *even by way idealized models*, quantum processes in space and time, similarly to the way the formalism of classical mechanics or relativity describes the behavior of its objects. (I shall explain the underlined qualification presently.) Niels Bohr, who was arguably the first to offer a sustained interpretation of this type, known as "complementarity," spoke in this connection of the "symbolic," rather than "abstract," character of quantum mechanics (e.g., [3], v. 1, p. 48). The quantum-mechanical formalism does provide effective algorithms, and in this view only such algorithms, for, in general, probabilistic predictions concerning the outcomes of individual quantum events. The essentially probabilistic nature of these predictions corresponds to the experimental evidence, since the identically prepared quantum experiments (in terms of the physical state of the apparatus used) in general lead to different outcomes. This fact in itself does not exclude the possibility of descriptive or, as they are also called, realist (or causal) theories describing quantum objects and their behavior. However, it appears, at least, difficult to offer such theories.[2]

The above qualification concerning the role of "idealized models" in classical physics or relativity is important. These theories only offer idealized descriptive models for the actual physical processes that they consider, models providing excellent predictions in practical cases, such as, say, that of planets moving around the Sun, *the* paradigmatic early example of the situation that obtains in classical mechanics. In other words, these theories are *realist* only in the sense of being able to offer such models, rather than describe the actual objects and processes of nature. The debate concerning the epistemology of quantum theory, from the Bohr-Einstein confrontation on, may be seen as that concerning the ability or inability of quantum mechanics to offer idealized descriptive, and specifically causal, models for the behavior of individual quantum systems of the type classical physics or relativity provides (although relativity already poses certain complexities in this regard). The question is whether this lack of realism and causality, even at the level of idealized models, is due to the inadequacy of quantum mechanics as a physical theory, as Einstein thought, or, as Bohr thought, is due to the fact that nature itself may not allow us to do better than quantum mechanics does. Einstein's view would imply that quantum mechanics is deficient, or at least incomplete, analogously to the way classical statistical physics is incomplete by virtue of not providing the description of the individual classical systems, which, however, could be *described* by classical mechanics, at least ideally or in principle. In Bohr's view, although quantum mechanics does not offer a description of, or, correlatively, ideally exact predictions concerning, the behavior of individual quantum systems, it is, nevertheless, complete, that is, as complete as nature allows our theories to be in the case of quantum *phenomena*. By quantum *phenomena*, I mean, following Bohr, what we observe in measuring instruments, in contradistinction to quantum *objects*. In Bohr's view, nothing could be said about quantum *objects* themselves, apart from the effect of their interactions with measuring instruments upon the latter, which circumstance also implies or is correlative to the fact that an act of observation ineluctably interferes with the quantum system involved, to the point, again, of preventing us from saying anything about its independent behavior. Accordingly, the difference between quantum *objects* and observed quantum *phenomena* is irreducible. Quantum objects are unobservable as

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[1] Tensors of the second rank and higher used in relativity are matrices, and, accordingly, in general they do not commute. This noncommutativity, however, does not play a significant physical role in relativity, and it did not attract attention. It is only with quantum mechanics that noncommutative mathematics enters physics in an essential way.

[2] Bohmian mechanics is a descriptive and causal theory of that type in all of its versions thus far, at the expense of nonlocality (the existence of instantaneous physical connections between distant events, forbidden by relativity). Bohmian mechanics is a different theory of quantum phenomena, rather than merely a different interpretation of the standard quantum mechanics, which, and only which, I shall discuss in this article. There are also realist interpretations of the standard quantum mechanics. I shall put aside these interpretations, since these alternatives are not different from classical physics or relativity in their use of physical concepts, and hence, my argument concerning this use would apply to these interpretations (or to Bohmian theories). On the other hand, while both classical physics and relativity are commonly interpreted as realist, as they will be here, nonrealist interpretations of them, especially of relativity, have been offered (e.g. [4]). In these interpretations, the use of physical concepts in these theories would be analogous to that in nonrealist interpretations of quantum mechanics, as considered here.

such, especially elementary individual quantum objects ("elementary particles"), such as electrons or photons, and even in the case of macroscopic quantum objects, such as Josephson junctures, we can only observe them as quantum in the sense of observing their quantum effects upon measuring instruments. By contrast, in classical physics, in view of the fact that our observational interferences can be neglected or suitably compensated for, observed phenomena could be identified with objects and idealized accordingly. The quantum-mechanical situation was perfectly acceptable to Bohr and his followers (not that many in the long run), who also welcomed quantum mechanics as an effective response to the peculiar, strange nature of quantum phenomena, and as a major scientific achievement. By contrast, the situation was unpalatable to Einstein (who acknowledged the effectiveness and partial truth of quantum mechanics) and those who followed him, as, it appears, the majority of physicists and philosophers do. The debate has continued with undiminished intensity ever since and is still with us.

I shall, however, not be concerned with this debate here, although my argument cannot avoid relating to it either.[3] My aim is to consider the nature and structure, architecture of physical concepts, classical, relativistic, and, most especially, quantum-mechanical, the architecture that, I argue, is always physical, mathematical, and philosophical. Mathematical abstraction plays an equally crucial role in this architecture in classical physics, relativity, and quantum mechanics (or higher-level quantum theories). It also plays a crucial role in changing this architecture from classical physics and relativity to quantum mechanics, insofar as the mathematics of quantum mechanics no longer provides idealized descriptive models of quantum objects and their behavior. This change is, perhaps not coincidentally, accompanied by the use of more abstract mathematical theories, such as and in particular, that of Hilbert spaces over complex numbers, and in the case of continuous variables, even infinite-dimensional Hilbert spaces.

Somewhat earlier, roughly from the 1860s on, mathematics had undergone a parallel development toward a greater abstraction by separating itself from describing the physical world and by breaking its connections to physics, which connections defined much of the preceding history of mathematics (e.g., [6]). This separation defines "modern mathematics" arguably most decisively (even if not altogether uniquely), especially in the case of geometry and analysis. Algebra has always been a more abstract discipline, although it underwent a corresponding transformation as well, by introducing concepts, in particular those (such as "group") that were not derived from numbers, and by extending the concept of number itself. One finds echoes of algebraic modernism in Dirac's language of *q-numbers* (which were defined as abstract noncommuting entities, generalizing Heisenberg's matrix variables).

Quantum mechanics reestablished the connections between mathematics and physics on these more abstract and specifically algebraic grounds. It was took advantage of mathematical modernity, via such modernist theories as linear algebra and Hilbert spaces. The mathematics of Hilbert spaces may be seen as an extension of linear algebra, developed to solve certain problems of analysis, such as those dealing with integral equations. David Hilbert was one of the key figures in the rise of modernist mathematics, and the leading figure of the Göttingen school, the university where Heisenberg completed his Habilitation under Max Born, one of the founders of quantum mechanics, who was greatly influenced by Hilbert. Hilbert made major contributions to physics as well, most famously to general relativity. He is credited as a co-discoverer of the main equations of the theory, along with Einstein, who was, however, singularly responsible for the physics of the theory. A number of Hilbert's fellow mathematical modernists at Göttingen also contributed to physics. Hermann Minkowski introduced the concept of spacetime, Hermann Weyl made major contributions to relativity and quantum theory, and Amalie Emmy Noether, one of the founding figures of the twentieth-century abstract algebra, proved Noether's theorem, which relates symmetry and conservation laws. Göttingen was a place, even *the* place, where mathematical and physical modernism came together and defined the twentieth-century mathematics and physics.

[3] I have considered this debate in detail in ([5]), including as concerns the question of locality (put aside in this article as well), which was raised by Einstein early on, but which became especially prominent in the wake of Bell's theorem, from the 1960s on.

2. WHAT IS A PHYSICAL FACT, AND WHAT IS A PHYSICAL CONCEPT?

I borrow the title of this article from G. W. F. Hegel's essay "Who Thinks Abstractly?," which argues that it is not a philosopher or an educated person, to begin with, who thinks abstractly but a person in the street. Hegel illustrates his point by a striking example: "Who thinks abstractly? The uneducated, not the educated. … A murderer is led to the place of execution. For the common populace he is nothing but a murderer. …. One who knows men traces the development of the criminal's mind: he finds in his history, in his education, a bad family relationship between his father and mother, some tremendous harshness after this human being had done some minor wrong, so he became embittered against the social order—a first reaction to this that in effect expelled him and henceforth did not make it possible for him to preserve himself except through crime. … This is abstract thinking: to see nothing in the murderer except the *abstract* fact that he is a murderer, and to annul all other human essence in him with this simple quality" ([7], pp. 115-116). A person in the street thinks abstractly because such a person often (one should not over-generalize, be too *abstract* as concerns "persons in the street" either) uses facts and concepts, such as "murder" or "murderer," as fixed entities, and sees the world accordingly. A philosopher, by contrast, thinks concretely by going beyond the limits of crudely abstracted facts and uncritical uses of concepts, such as "murderer" or, again, "person in the street." A philosopher also goes beyond the view or concept of concept as something that is merely a generalization from particulars, say, the way we use the concept of "tree." Or more accurately, since concepts in this sense are unavoidable in and are essential to our thinking, a philosopher must carefully balance abstract and concrete thinking, and bring both to bear on philosophical problems, for example, the problem of justice, suggested by Hegel's example.

Hegel radically rethought both the concept of fact and the concept of concept. He argued that facts or propositions expressing them, wherever they occur (in daily life, philosophy, mathematics, or science) are always conceptually and historically *mediated*, although this mediation is often ignored or suppressed not only by people in the street but also by philosophers, or mathematicians and scientists. Mediation is at work beginning with our use of even the simplest propositions expressing "facts," say, the statement "this is a falling body," and we cannot refer to any fact apart from some propositions expressing them. While our observation or perception of "facts," say, again, that of a falling body, which may appear immediate and may sometimes be *more* immediate, it is still a subject of conceptual mediation, and in any event our *thinking* about it always is. Thus, even if we "merely" observe and think about a *falling* body, it involves, at the very least, some concept of "falling."

The concept of "falling body" is crucial for classical physics, to which it came from an already long history of its use in philosophy, theology, and elsewhere, and where it acquired new meaning or indeed became a new concept. On the one hand, all modern, post-Galilean, physics is based on a reduction, *abstraction* of most non-mathematizable properties of actual objects in nature (a procedure sometimes known as Galilean reduction), which allows one to develop idealized models of physical bodies and motions. On the other hand, this abstraction gave new concrete, especially mathematical, features to the concepts of physical body and motion. For example, one uses the (idealized) concept of a massive material point, moving along a straight or curved line in accordance with the corresponding differential equation. In other words, classical physics involves a complex interplay of the abstract and the concrete, and so do relativity and quantum theory. Einstein was a Hegelian when he told Heisenberg, in responding to Heisenberg's aspiration in his first paper on quantum mechanics to deal only with "quantities which in principle are observable," that theory decides what can be observed, and hence, one might add, what is abstract and what is concrete ([8], p. 63). Heisenberg's reflection on Einstein's point helped him to discover the uncertainty relations, which imply that the concepts of body and motion may no longer be applicable in quantum mechanics, and that, one might accordingly need new concepts of quantum objects.

I might also mention (although I cannot consider the subject in detail) the Hegelian character of, as they are sometimes called, constructivist arguments, beginning with those offered by such authors as Thomas Kuhn and Imre Lakatos in the 1960s, of the role of cultural factors in shaping scientific theories and, via these theories, scientific facts. While rarely invoked in these arguments (although both Kuhn and especially Lakatos were influenced by Hegel), Hegel thought deeply about this role, more deeply than many recent philosophers, historians, and sociologists of science who address the subject. In particular, he understood that the complexity of "facts" or "concepts," exceed their cultural determination, which, accordingly, requires critical analysis as well and should not be assumed, uncritically, as the only or even as always the dominant factor in determining scientific practice. Facts may be *always constructed*, but they are never *only socially* constructed, a view that more recently came to define the constructivist arguments in question as well (e.g., [9]).

Equally significant, especially for the present article, is that Hegel introduced a new *concept* of philosophical concept, which he introduced by the time of "Who Thinks Abstractly?" in *The Phenomenology of Spirit* ([10]). Hegel's concept of (philosophical) concept was recently developed in the work of Gilles Deleuze, especially in his and Félix Guattari's *What is Philosophy?*, which defines philosophy itself as the creation of new concepts ([11]). I shall follow this extension of Hegel's thought, in part because of its emphasis on the creation of new concepts. This concept of philosophical concepts gives them both concreteness and complexity. Two aspects of this concept of concept are especially crucial. First, a concept in this sense is never only an entity established by a generalization from particulars or any merely general or abstract idea, although a concept may involve such generalizations from particulars or general or abstract ideas. Indeed, a concept in this sense is always multi-component. "There are no simple concepts. Every concept has components and is defined by them. … It is a multiplicity. ... There is no concept with only one component" ([11], p. 15). Each concept is a multi-component conglomeration of concepts in their conventional senses, figures, ideas, images, metaphors, and so forth. Secondly, each concept is also fundamentally linked to a problem: "All concepts [in this sense] are connected to problems without which they would have no meaning and which can themselves be isolated or understood as their solution emerges" ([11], p. 16).

I shall adapt this concept of philosophical concept and the view of philosophy as the invention of new concepts to physical concepts and to theoretical physics, while, however, retaining their scientific and specifically mathematical specificity. I might note, in passing, that this concept of concept can also apply to mathematical concepts, such as that of Euclidean space (used in classical physics), Riemannian space (used in relativity), or Hilbert space (used in quantum mechanics), each of which, at the time of their creation, was a response to a problem. The concepts of even most abstract mathematics are as concrete and multi-component in the present sense as are those of physics or philosophy, and creative mathematics always involves the invention of new concepts. According to Frank Wilczek, "the primary goal of fundamental physics is to discover [create?] profound concepts that illuminate our understanding of nature" ([12], p. 239). It is equally important, however, to consider what is the nature and structure, architecture, of these concepts, such as, for example, those mentioned by Wilczek—gauge invariance, symmetry breaking, or the Higgs field. In particular, I shall argue that abstract mathematical concepts, such as that of Hilbert space or symmetry groups, used in quantum mechanics (similarly abstract mathematics is part of the concepts invoked by Wilczek), lead to concrete concepts in an analogous sense in physics, and they, again, have a similar conceptual concreteness in mathematics itself. While such concepts do involve generalization and abstraction, they, for example, the concept of "elementary particle," never amount to merely generalization or abstraction. This is true in classical physics as well. Just as new philosophical concepts, new physical concepts also respond to and aim to solve problems. In addition, as I shall argue here, these concepts have not only mathematical and experimental but also philosophical components. Galileo's *concept* of physics as a *mathematical* science of nature, which is itself a new philosophical concept, has shaped the disciplinary nature of modern physics. It would be next to impossible to speak of physical concepts, apart from the relationships, actual or potential, between their mathematical and experimental components, in contradistinction to philosophical concepts, although both types of concepts are equally multiple in their constitution. However, philosophical components of physical concepts are significant and often unavoidable, especially in the case of foundational concepts. A physical concept is a multi-layered conglomerate of physical, mathematical, and philosophical components in complex interactions with each other. It is true that some concepts that are used in physics, even foundational concepts, may be more strictly mathematical, more strictly physical, or more strictly philosophical. More often than not, however, we see physical concepts more narrowly because we miss their other dimensions, especially by not taking into account how these concepts actually function. Thus, while Bohr's concepts of phenomenon, quantum objects, and complementarity contain physical, mathematical, and philosophical strata, their mathematical aspects are less pronounced than in Heisenberg (where mathematics is dominant) and are easy to miss, which is not to say that they are not significant.[4]

Now, in confronting quantum phenomena, quantum theory, beginning with quantum mechanics, encountered and had to respond to a radically new situation, a new problem or a new configuration of problems. Both concepts, that of physical fact and that of physical concept, had to be reassessed, as against classical physics and even relativity (which, as I said, already posed some complexities in this respect). Just as those of classical physics and relativity, quantum-theoretical concepts are not only mathematical and experimental but are also philosophical, and, just as in classical physics and relativity, this architecture helps quantum theory to establish specific connections to nature. As noted from the outset, however, the nature of these connections is different in quantum theory, at least in the interpretation of the type adopted here. (Otherwise a more classical-like architecture of concepts would apply, and the case would not merit a special consideration.) Most especially, in this interpretation, the connections between concepts and nature are no longer those of idealized descriptions of nature, enabling, in the case of classical mechanics, (ideally) exact predictions concerning idealized objects of these theories, and, as a result, excellent practical predictions for the corresponding actual objects in nature. In quantum mechanics these connections only amount to (equally excellent) predictions, enabled by the formalism, predictions that are in general irreducibly probabilistic in character, even ideally, which, however, is, again, in accord with what is observed in quantum experiments. The situation is reflected in Bohr's concepts of "quantum phenomenon" (referring strictly to what is observed in measuring instruments) and "quantum object" (which cannot be described or even conceived of), mentioned above, which are *new* physical concepts in this sense. The *difference* between quantum phenomena and quantum objects (which radicalizes Kant's scheme of phenomena vs. things-in-themselves) is part of their architecture and makes both concepts irreducibly linked or even part of each other.[5]

The circumstance that quantum objects cannot be described or even *conceived* of, that is, they cannot be assigned any conceivable properties, does not stop the corresponding concept, the concept of inconceivable entities, from being a concept in the present sense, a new concept. Besides, this concept of is linked to other concepts, mathematical (say, symmetry groups, defining elementary particles) and experimental (those defined by particular effects observed in measuring instruments). Indeed, the concept of quantum objects may be seen as having a more complex architecture defined by these connections. From this perspective, the concept of quantum objects, at least of elementary constituents of nature, is still a problem in need of a new concept, which may or may not ultimately involve their inconceivable nature. As Heisenberg argued (in an article under this title), the question "What is an Elementary Particle?" was posed anew by Dirac's discovery of the positron, "perhaps the biggest change of all the big changes in physics of our century … because it changed our whole picture of matter" ([13], pp. 31-32). This question is still unanswered. Indeed it has become a different question more than once since then, and is now sometimes asked as "What is an elementary string?"

[4] I have discussed this point in ([5], pp. 128-133).

[5] The present argument may, thus, appear to, and in some respects does, depart from Bohr's insistence, as against Einstein's and Schrödinger's views, that quantum mechanics does not need new physical concepts. However, as I discussed in ([5]), the situation is more complex ([5], pp. 32-39). At stake in this exchange were new concepts that would describe the behavior of quantum objects, analogously to the way classical physical concepts describe classical objects in classical physics. Einstein and Schrödinger appear to have had in mind new concepts of this type. Bohr did not believe that such concepts were necessary or even possible. Bohr's concepts of quantum phenomena, quantum objects, and complementarity, are *new* concepts in the present sense, as Bohr, I think, would have agreed, although he did not consider this concept of concept itself.

3. AN INVENTION OF A NEW PHYSICAL CONCEPT: HEISENBERG'S "NEW KINEMATICS" OF QUANTUM THEORY

At some point in his work on his new quantum mechanics, Heisenberg realized that he did not need to have a space-time description of quantum objects to be able to predict the outcomes of the experiments that he was working with, which concerned atomic spectra. Heisenberg's strategy to abandon the search for a descriptive theory of these phenomena was not only unprecedented but also nontrivial because he *formally* retained the classical equations of motion. However, the variables to which these equations applied were mathematically different from those of classical physics and, correlatively, no longer related to the motion of quantum objects but only to the probabilities of predictions of the outcomes of quantum experiments. Heisenberg, thus, introduced a new concept of physical variable—mathematically (because they were no longer space-time variables or their functions as in classical physics or relativity), physically (because they no longer described physical objects and their behavior), and philosophically (because, given its predictive rather than descriptive character, it establishes a new form of epistemology).

According to Heisenberg, a new problem that physics had to solve in confronting quantum phenomena required a "new kinematics." In classical mechanics, "kinematics" refers to a representation, usually by means of differential functions, of the attributes of motion, such as spatial and temporal coordinates or velocities of a body. The representations of dynamic properties, such as momentum and energy, are dependent on and are functions of kinematical properties. By contrast, Heisenberg's "new kinematics" related its elements to what is observable in measuring instruments under the impact of quantum objects, rather than representing the attributes of the motion of these objects. The kinematical or, correspondingly, dynamical elements of Heisenberg's theory were no longer functions of variables defined by the properties of quantum objects or their behavior. These elements were conceived as infinite-dimensional square-tables, matrices, of complex variables, eventually rethought in terms of operators in a Hilbert space, or still more abstract entities, such as elements of C^*-algebras. They had, again, no classical-like specifiable relation to the attributes of motion of quantum objects themselves, as kinematical elements would in classical physics or (finite-dimensional) tensors of general relativity. They were designed only to relate, probabilistically, to the impact of quantum objects upon measuring instruments (in this case, the impact of light quanta, emitted by electrons, upon photographic plates registering spectra). These matrices had to be infinite to treat these data consistently, and could not even be "bounded" infinite matrices. It also became clear shortly thereafter that they needed to be infinite to be consistent with the uncertainty relations for the continuous variables. The classical (Newtonian, as in Heisenberg's paper, or Hamiltonian, in a full-fledged version of the theory) equations of motion are, again, formally retained, but are applied to matrix variables and no longer to variables describing the motion of quantum objects. They are no longer *equations of motion*. The variables in question are the "amplitudes" of probabilities of transitions between different stationary states of an atom. One borrows the term "amplitude" from the classical theory, but reinterprets it so as to deprive it of a reference to the physical (e.g., oscillating) motion of physical objects, which defines the use of the term in classical physics, and instead to relate it to the probabilities in question. These amplitudes are now known as "probabilities amplitudes." They are *complex vectors*, while the rule for moving from "amplitudes" to probabilities and hence to *real numbers* (between zero and one) is a special case of Born's rule, which applies more generally to Schrödinger's function, considered as the probability density.

Heisenberg's discovery became especially famous for the (in general) noncommutative nature of multiplication of his new kinematic elements. This noncommutativity is physically correlative to the fact that any two such elements X and Y in the product XY have to be taken as representing different stationary "states," as against classical theory, where such variables refer to the same physical state of the system in question. (This is true even for $XX=X^2$, although this multiplication is commutative.) This circumstance also entails, correlatively, both the uncertainty relations and Bohr's complementarity, although it took some time and hard thinking for, respectively, Heisenberg and Bohr to realize this. For, as against classical mechanics, both such (conjugate) variables can never be defined for the same physical state;

their definition is complementary in Bohr's sense because we can also always assign either one such variable or the other and define the respective physical states accordingly. Changing the order of measurement defines two different states and generally leads to different values of the corresponding variables. Moreover, if we measure first the ("coordinate") variable q (associated with the amplitude X) and then the ("momentum") variable p (associated with the amplitude Y), only p can be assigned a known value after the second measurement, never q—and vice versa. Our predictions concerning any measurement on the same object subsequent to the second measurement only depend on and are defined by this measurement, rendering the first measurement irrelevant for these predictions.

Thus, the (general) noncommutativity of Heisenberg's new kinematic elements is indeed crucial. However, an arguably even more decisive mathematical move of Heisenberg was the discovery of the different nature of these new variables themselves and of the way to multiply them as variables used in equations, which proved to be those of matrix algebra, with which Heisenberg was famously unfamiliar at the time. In Heisenberg's logic this step *preceded* the general noncommutativity of his new quantum-theoretical variables. He noted that this general noncommutativity was a consequence of his multiplication rule, which he applied in the paper only, *as so he thought*, to commuting multiplications, actually degrees of the same variable, but he did not use it in the paper itself, since the particular physical case he considered, that of an aharmonic oscillator, did not require it.[6] I would argue that, as concerns Heisenberg's *invention* of quantum mechanics, discovering the matrix-like character of his new kinematical elements and how to *multiply* them formed the most crucial step (noncommutativity followed automatically). Finding these elements was itself a founding theoretical move. That is, this arrangement of the relationships between observable quantities in infinite matrices of complex, rather than real, numbers (numbers never observable as such) is already a form of *theory*, not of observation of nature, which does not arrange anything in this way. At least in this respect, Heisenberg's *theory* defined what is observed (effects of quantum objects upon one or another classical medium) or not observed (the behavior of quantum objects themselves) and how, in accordance with Einstein's argument that our concepts and theories decide what could be observed. But it did so with an outcome quite different from the one Einstein would have preferred, in view of the fact that it suspended the observation of quantum objects and their independent behavior. The character of Heisenberg's new elements and of their *relationships* reflected the invention of a new concept of "physical variable." This concept, I argue, was not only mathematical and physical, but also philosophical, and it was to define most crucially the mathematical, physical and philosophical character of Heisenberg's theory.

In sum, in responding to the problem of inventing quantum mechanics as a theory of individual quantum processes, Heisenberg abandoned the project that defined classical physics: that of describing the behavior of physical objects by means of idealized mathematical models. He replaced it with the project in which a given mathematical formalism is used to predict the outcome of possible experiments on the basis of previously performed experiments. In a proper correspondence with the experimental evidence available at the time (and still in place now), such predictions were probabilistic. The scheme, Heisenberg stressed, "contains a complete determination not only of frequencies and energy values, but also of quantum theoretical transition probabilities" ([1], p. 268). Heisenberg accomplished this project by introducing a radically new concept of variables and a new *calculus* of these variables by re-inventing matrices and the rules for their multiplication, which also led to the introduction of noncommutativity into physics. This concept brought physics, mathematics, and philosophy into a new type of relationship.

I would like to trace Heisenberg's key steps in his invention of this concept by following his paper that introduced quantum mechanics. The key opening move was, again, that Heisenberg formally, but only formally, adopts the classical equations of motion in Fourier's representation. This step is prompted by what may be seen as a mathematical form of Bohr's correspondence principle. The principle more or less tells us that in the regions where classical physics can be used the predictions of quantum and classical theory should coincide, and the classical equations, with classical physical variables, work well

[6] I qualify because, technically, this is not true. While the equation itself used by Heisenberg did not involve noncommuting variables, hia integration of this equation in fact used noncommutative matrix multiplication, as Dirac was the first to notice.

for the large quantum numbers (when electrons are far from atomic nuclei). As Bohr often stressed and as Heisenberg explains in his uncertainty-relations paper, this does not mean that the physical processes themselves in these regions may be seen as classical and hence that classical equations describe them, but only that one can use these equations to make correct predictions in these regions ([14], pp. 72-76). In any event, classical equations (when using classical variables) do not work for small quantum numbers because they do not satisfy Bohr's rules for frequencies and the Rydberg-Ritz combination rules.

After announcing his intention to found his new mechanics on "relationships between quantities which in principle are observable," Heisenberg states what has not been possible in quantum theory so far and what is possible. He says: "[I]n quantum theory it has not been possible to associate the electron with a point in space, considered as a function of time, by means of observable quantities. However, even in quantum theory it is possible to ascribe to an electron the emission of radiation" ([1], p. 263). He then says: "In order to characterize this radiation we first need the frequencies which appear as functions of two variables. In quantum theory these functions are in the form [Bohr's rule]:

$$v(n, n-\alpha) = 1/h\ \{W(n) - W(n-\alpha)\} \quad (1)$$

and in classical theory in the form

$$v(n,\alpha) = \alpha v(n) = \alpha/h(dW/dn)\text{" ([1], p. 263).}^{7}$$

This difference leads to a difference "between classical and quantum theories as concerns the combination relations for frequencies, which correspond to Rydberg-Ritz combination rules." However, "in order to complete the description of radiation [in accordance with the Fourier representation] it is necessary to have not only frequencies but also the amplitudes" ([1], p. 263). As noted earlier, these "amplitudes," which are no longer amplitudes of physical, such as orbital, motion, are to be linked to the probabilities of transitions between stationary states; in other words, they are probability amplitudes. "The amplitudes may be treated as complex vectors, each determined by six independent components, and they determine both the polarization and the phase. As the amplitudes are also functions of the two variables *n* and α, the corresponding part of the radiation is given by the following expressions:

Quantum-theoretical:

$$\mathrm{Re}\{A(n, n-\alpha)e^{i\omega(n, n-\alpha)t}\}$$

Classical:

$$\mathrm{Re}\{A_{\alpha}(n)e^{i\omega(n)_{\alpha}t}\}\text{"}$$

The problem—a difficult and, "at first sight," even insurmountable problem—is now apparent: "[T]he phase contained in *A* would seem to be devoid of physical significance in quantum theory, since in this theory frequencies are in general not commensurable with their harmonics" ([1], pp. 263-264). This is the problem that Heisenberg, who starts this sentence with "at first sight," is about to solve in a radical way by changing the perspective completely, which involves the mathematical, physics, and philosophical moves here considered. He adds: "However, we shall see presently that also in quantum theory the phase had a definitive significance which is *analogous* to its significance in quantum theory" ([1], p. 264; emphasis added). "Analogous" could only mean here that the way it functions mathematically is analogous to the way the classical phase functions mathematically in classical theory; physically there is no analogy. If one considers "a given quantity $x(t)$ [a coordinate as a function of time] in classical theory, this can be regarded as represented by a set of quantities of the form

[7] Some of Heisenberg's notations are modified without affecting his actual formulas.

$$A_\alpha(n)e^{i\omega(n)\alpha t},$$

which, depending upon whether the motion is periodic or not, can be combined into a sum or integral which represents $x(t)$:

$$x(n, t) = \sum_{\alpha=-\infty}^{+\infty} A_\alpha(n)\, e^{i\omega(n)\alpha t}$$

or

$$x(n, t) = \int_{-\infty}^{+\infty} A_\alpha(n)\, e^{i\omega(n)\alpha t}\, d\alpha\text{” ([1], p. 264).}$$

Heisenberg is now ready to make his most decisive and most extraordinary move. He first notes that “a similar combination of the corresponding quantum-theoretical quantities seems to be impossible in a unique manner and therefore not meaningful, in view of the equal weight of the variables n and $n - \alpha$” ([1], p. 264). “However,” he says, “one might readily regard the ensemble of quantities $A(n, n - \alpha)e^{i\omega(n, n-\alpha)t}$ [an infinite square matrix] as a representation of the quantity $x(t)$” ([1], p. 264). The arrangement of the data into square tables is a brilliant and—in retrospect but, again, only in retrospect (since it also changed our view of what is natural in quantum physics)—natural way to connect the relationship (transitions) between two stationary states, and it is already a great concept. However, it does not by itself establish an *algebra* of these arrangements, for which one needs to find the rigorous rules for adding and multiplying these elements—rules without which Heisenberg cannot use his new variables in the equations of the new mechanics. To produce a quantum-theoretical interpretation of the equation of motion, as applied to these new variables, Heisenberg needs to be able to construct the powers of such quantities, beginning with $x(t)^2$. The answer in classical theory is of course obvious and, for the reasons just explained, obviously unworkable in quantum theory. “In quantum theory,” Heisenberg proposes, “it seems that the simplest and most natural assumption would be to replace classical [Fourier] equations (3) and (4) by

$$B(n, n - \beta)e^{i\omega(n, n-\beta)t} = \sum_{\alpha=-\infty}^{+\infty} A(n, n-\alpha)A(n-\alpha, n - \beta)e^{i\omega(n, n-\beta)t}$$

or

$$= \int_{-\infty}^{+\infty} A(n, n-\alpha)A(n-\alpha, n - \beta)e^{i\omega(n, n-\beta)t}\, d\alpha\text{” ([1], p. 265).}$$

This is the main postulate, the (matrix) multiplication postulate, of Heisenberg’s theory, “and in fact this type of combination is an *almost* necessary consequence of the frequency combination rules” [equation (1) above] ([1], p. 265; emphasis added). “Almost” is an important word here. While Heisenberg, to some degree, arrives at this postulate in order to get the combination rules right through a complex process of “guessing” (not the best word here)—by manipulating the correspondence principle and the data—the justification or derivation is not strictly mathematical. The rule is ultimately justified by an appeal to experiment. This combination of the particular arrangement of the data and the (re)invention through physics of an algebra of multiplying his new variables is Heisenberg’s great invention.

This multiplication is in general noncommutative, and the scheme essentially amounts to the Hilbert-space formalism, with Heisenberg’s matrices serving as operators. As noted earlier, this noncommutativity is often seen as Heisenberg's greatest discovery, especially in the context of the full-fledged matrix formalism developed by Born, Jordan, and Heisenberg himself, and Dirac's and von Neumann's versions of quantum mechanics. However, it is the introduction of the concept of matrix variables that appears to me to be more crucial, from both perspectives, that of Heisenberg's process of discovering quantum mechanics and that of the view of theoretical physics as the invention of new concepts. Noncommutativity is just an automatic consequence. This concept is not only physical and mathematical but also philosophical. It is philosophical because the arrangement of quantities available from the experimental data was, first, a new phenomenological and theoretical procedure and, secondly, it was correlative to a radically new epistemology, based only on probabilistic predictions concerning future events and not on a description of quantum objects and their behavior in space and time.

Two years later, Heisenberg made the following remark in his uncertainty-relations paper, which also aimed to provide, via the uncertainty relations, an intuitive [*anschaulich*] physical meaning to quantum mechanics (the mathematical equivalence of matrix and wave mechanics was established by then). He said in a revealing footnote: "Schrödinger describes [matrix] quantum mechanics as a formal theory of frightening, indeed repulsive, abstractness and lack of [intuitive] visualizability [*Anschaulichkeit*]. Certainty one cannot overestimate the mathematical (and in that sense also intuitive) mastery of the quantum-mechanical law that Schrödinger's theory has made possible. However, as regards questions of physical interpretation and principle, the popular view of wave mechanics, as I see it, has actually deflected us from exactly those roads which were pointed out by the papers of Einstein and de Broglie, on the one hand and by the papers of Bohr and by [matrix] quantum mechanics on the other hand" ([14], p. 82, note; translation modified). Heisenberg is right to credit the mathematics of Schrödinger's wave theory and its physical contributions, and this theory, too, was based on the invention of new concepts, concepts that were mathematical, physical, and philosophical, albeit closer to the (descriptive) ideal of classical physics or relativity. He was also correct as concerns Bohr and matrix mechanics. On the other hand, both Einstein and de Broglie had, as we know, rather different views of the road to be taken, views philosophically much closer to Schrödinger's argumentation. Heisenberg's footnote is appended to the following statement: "as we can think through qualitatively the experimental consequences of the theory in all simple cases, we will no longer have to look at quantum mechanics as unphysical or abstract" ([14], p. 82). We do, however, need to rethink what are physical facts and what are physical concepts in quantum physics, in other words, the relationships between the formalism and experiments (which establish quantitative laws of the theory), and thus between mathematics and physics, and physics and nature. Indeed, we need to *create* new concepts that respond to this new situation. This is, I argue, what Heisenberg had essentially accomplished beginning with his first paper on quantum mechanics.[8]

We can describe the type of thinking used by Heisenberg as follows. Consider a trace on a silver-bromide screen, found, say, in the double-slit experiments. It would seem "natural" to assume that a classical-like object hits the screen—"natural," that is, given the preceding history of physics as classical physics, since quantum physics also changes what is and is not natural for us to think. This assumption, however, poses major difficulties, as would the assumption, prompted by the data (an interference-like pattern) obtained in repeating the individual runs of the double-slit experiment many times in one of its set-ups (with both slits opened and no counters installed) or in other quantum experiments. (It became clear earlier that one could not assign an electron dimensions even classically, since the forces of its negative electricity would tear it apart.) Accordingly, it is thinking in terms of classical physics that becomes abstract now insofar as this thinking abstracts observed facts from what in fact define them. Heisenberg proceeded altogether differently in order to avoid these difficulties, while enabling the theory to properly predict the outcomes of the relevant quantum experiments, initially those concerning spectra. (It is easily shown that the situation that obtains in the double-slit experiment is correlative to both the uncertainty relations and the probabilistic nature of quantum predictions, and that both of the latter are correlative in turn.) Reversing the way of thinking defining classical physics, Heisenberg *abstracted*

[8] It may be argued that, although a tremendous achievement, in some respects the uncertainty-relations paper is a partial retreat in this respect ([5], pp. 192-194, 216-217).

quantum phenomena from classical physics, or classical physics from quantum phenomena. "Facts" of quantum physics now referred only to what is observed in our measuring instruments under the impact of quantum objects, in other words, to what Bohr later defined as quantum *phenomena*, in juxtaposition to quantum *objects*. It is of course essential that quantum objects exist as something different from classical physical objects, and hence cannot be idealized in the manner of classical physics. The "abstract" mathematics of quantum mechanics becomes part of a multi-component conceptual architecture enabling one to predict, again, in probabilistic terms, the data observed in measuring instruments, rather than to describe the behavior of quantum objects on the model of classical physics.

That does not mean that a classical-like, descriptive approach might not again work in future physics, as Einstein hoped (and some, again, believe it works even now). In this case, however, new multi-component concepts, vis-à-vis those of Heisenberg and Bohr, will be necessary, as Einstein and Schrödinger, again, thought they would be. For now, given that our philosophical views powerfully shape our thinking in physics and that our fundamental physical theories are incomplete (although quantum mechanics, within its scope, is intact thus far) one could only make our Bayesian bets on this future.

4. CONCLUSION

Heisenberg's new concept of quantum variables, thus, exhibits all three main aspects of physical concepts, as understood here—their multi-component multiplicity; their essential relations to problems; and the role of and interactions between physical, mathematical, and philosophical components in their architecture. This view of physical concepts, I argue, applies to all of modern, post-Galilean, physics, rather than only to quantum theory; and while the epistemological differences, established by Heisenberg's concept, between quantum mechanics and classical physics or relativity are crucial, it is this general architecture of physical concepts and its role in physics that I would like to stress in closing. A concept is akin to a crystal or a lens through which a physicist or a philosopher sees the world and our interactions with it. One could think, with Deleuze, of Spinoza (whose professional occupation was polishing lenses), polishing the lenses of his philosophical vision by inventing new philosophical concepts ([15], p. 14). One could also think of Newton, whose investigation of light in his optics was accompanied and shaped by the sharpening of his scientific and philosophical vision, or one can return to Heisenberg's conceptual optics, which, too, was physically concerned with light (spectra). The "molecular" constitution, as it were, of each such conceptual crystal in the case of physical concepts is defined by a complex configuration of its physical, mathematical, and philosophical "atoms." This complexity, however, also helps us to improve our optics and better focus it.

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