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The visualizable, the representable and the inconceivable: realist and non-realist mathematical models in physics and beyond

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The project of this article is twofold. First, it aims to offer a new perspective on, and a new argument concerning, realist and non-realist mathematical models, and differences and affinities between them, using physics as a paradigmatic field of mathematical modelling in science. Most of the article is devoted to this topic. Second, the article aims to explore the implications of this argument for mathematical modelling in other fields, in particular in cognitive psychology and economics.

1. Introduction

The project of this article is twofold. First, it aims to offer a new perspective on, and a new argument concerning, realist and non-realist mathematical models (defined by their different relations to the objects or phenomena these models consider), and differences and affinities between them, using physics as a paradigmatic field of mathematical modelling in science. Most of the article is devoted to this topic. Second, the article aims to explore the implications of this argument for mathematical modelling beyond physics, in particular in cognitive psychology and economics. In general, even in considering physics, the aims of this article are more explorative than directed towards offering definitive claims, although it will make several, sometimes strong, claims as well.

The difference between realist and non-realist models has been a major concern in the history of mathematical modelling in physics following the introduction of quantum mechanics. Quantum mechanics was the

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first scientific *theory* that used a mathematical model that resisted realist interpretation and that was given non-realist interpretations by N. Bohr and others who followed him or ‘the spirit of Copenhagen’, as W. Heisenberg called it, interpretations making the mathematical formalism of quantum mechanics the first non-realist mathematical *model* [1, p. ix].[1] As this formulation indicates, ‘theory’ (such as quantum mechanics) and ‘model’ (such as the mathematical formalism of quantum mechanics) are different concepts, at least as they will be understood here. I shall properly explain them in §2 and will only offer brief summary comments here. By a ‘mathematical model’ (henceforth, ‘model’), I refer to a mathematical formalism of a given theory cum an interpretation of this model within this theory. A theory will be understood as a broader conceptual structure, an organized assemblage of concepts. It would, for example, establish how the model defined by this theory relates to experimentally observed phenomena, which entails an interpretation of this model. A physical theory always involves at least some interpretation of the model it uses by virtue of giving a physical meaning to this model, for example, again, by establishing the way in which it relates to the experimentally observed phenomena considered by the theory. ‘Realist’ and ‘non-realist’ are interpretive conceptions, which may be adopted by different theories using the same model. For simplicity, however, I shall also speak of the corresponding interpretation of the theory containing a given model, interpreted by the theory, such as quantum mechanics, although, rigorously speaking, a different interpretation defines a different theory, as well as a different model. The mathematical formalism of classical mechanics is, or is commonly interpreted, as a realist (and causal) model, although, as I shall argue, not always of the same type, in part depending of which version of the formalism one uses. The mathematical formalism of quantum mechanics is a mathematical model that (in any version of the formalism, which are essentially equivalent mathematically) allows for either realist or non-realist interpretations, which may be different within each type. In contrast to classical mechanics where realist and causal interpretations are common and effective, and are easy to uphold, realist or causal interpretations of quantum mechanics have proved difficult to develop and to convincingly argue for.

Instead, however, of unconditionally juxtaposing realist and non-realist models in physics, this article, while acknowledging and accentuating their differences, argues for certain significant affinities between them, specifically between *non-visualizable realist* models and *non-realist* models (which are always non-visualizable). Some realist models are visualizable: they are available to a pictorial representation by our phenomenal intuition, and thus may also be called, more gracefully, *pictorial* models. I shall, nevertheless, use the more awkward ‘non-visualizable’ for the sake of uniformity. (These concepts will be given more proper definitions in §2.) Visualizable models were the first models developed in physics, with Descartes, Galileo and Newton. Non-visualizable realist models arrived over a century later, with J.-L. Lagrange’s and W. R. Hamilton’s analytical mechanics, and then were used in classical statistical physics and relativity, and more recently in chaos and complexity theory. Non-realist models, as defined here, entered physics only with quantum mechanics.

This article, thus, will consider three types of models: *visualizable realist* models (hereafter visualizable models), *non-visualizable realist* models, and *non-realist* models, which are non-representational and, hence, non-visualizable. Visualizable and non-visializable realist models are both representational, but in the case of visualizable models in a stronger, more descriptive and geometrical, sense. Non-visualizable realist models are essentially algebraic, as are non-realist models, which circumstance reflects the affinity between them argued for in this article. The distinction between visualizable models and non-realist models, as non-visualizable models, was often used by Bohr in referring to quantum objects and processes, in his interpretation. Bohr sometimes used German ‘*Anschaulichkeit*’ (for intuition) to address visualizability, although he also spoke of pictorial visualization or pictorial representation, or the absence thereof in

[1]This characterization is preferable to a more common rubric of the Copenhagen interpretation, because it is not possible to speak of a single Copenhagen interpretation, even in Bohr’s own case. Bohr changed his views a few times. Here, I shall be primarily concerned with Bohr’s ultimate view, reached by him in the late 1930s in the course of his decade-long debate with A. Einstein. I have considered the development of Bohr’s views in [2].

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the case of quantum objects and their behaviour. Bohr did not consider, at least not expressly, non-visualizable realist models, perhaps because, as explained below, such models are always underlain by visualizable realist models. His interpretation of quantum phenomena and quantum mechanics was, however, non-realist, rather than only non-visualizable.

Visualizable models have continued to be operative in classical physics, especially in considering elemental individual physical objects and processes, usually represented as motions of dimensionless points endowed with mass. This representation may be seen as an idealization, an idealized mathematical refinement, of our daily phenomenal intuition, specifically of motion of material bodies that we observe in the world. Such models are, generally, presumed, as they were by Lagrange and Hamilton, to apply at the ultimate underlying physical stratum defining the systems considered by theories using non-visualizable realist models, in contrast to theories using non-realist models, such as quantum mechanics, at least, again, in certain interpretations, such as those in the spirit of Copenhagen. This difference is correlative to another key difference. Non-visualizable realist models of classical mechanics and specifically analytical mechanics offer (ideally) exact predictions concerning the elemental individual processes they consider. The predictions offered by the formalism of quantum mechanics, *however interpreted*, are probabilistic or statistical even in the case of elemental individual quantum processes and the corresponding events. This has to be the case even in realist interpretations of quantum mechanics or alternative realist theories (as in the case in Bohmian mechanics), because the probabilistic or statistical nature of quantum predictions corresponds to what is observed in quantum experiments. Identically prepared quantum experiments, even those dealing with elemental (unsubdividable) quantum objects, such as electrons or photons, and processes in general lead to different outcomes. This difference obtains regardless of the capacity of our measuring instruments, unlike in the case of classical physics where we can improve our predictions to the point of them being ideally exact by improving this capacity. According to the interpretation to be ultimately adopted here, 'the statistical Copenhagen interpretation', it may not even be possible to assign probabilities to the outcomes of individual quantum experiments. It is only possible to ascertain statistical regularities pertaining to multiple repeated experiments.

The remainder of the article proceeds as follows. Section 2 outlines the main concepts used here, and an analysis of the concept of mathematical model, and of the functioning of mathematical models of all three types considered here (visualizable, non-visualizable realist and non-realist) in physics.[2] Section 3 discusses the models of the Lagrangian and Hamiltonian analytical mechanics as non-visualizable realist models, and their differences from visualizable models of classical mechanics and, conversely, their affinities with non-realist models of quantum mechanics. It also comments on the non-visualizable models of classical statistical physics and relativity. Section 4 is devoted to quantum mechanics, addressed, in part historically, via Heisenberg's discovery of quantum mechanics and Bohr's interpretation of it. It is true that neither Bohr nor Heisenberg expressly considered the concept of model, as, for example, J. von Neumann or D. Hilbert did. However, Heisenberg's and Bohr's thinking is crucial for understanding the nature of non-realist mathematical models in physics and the reasons for their emergence in quantum theory. This thinking provided most of the necessary ingredients or, one might say, a 'model' for the concept of a non-realist probabilistic model in physics and beyond that is the article's primary concern. Von Neumann's interpretation of quantum mechanics and his concept of model are realist. Section 5 considers the nature of quantum probability and offers an argument for the statistical rather than only probabilistic nature of quantum theory, even in considering elemental individual quantum processes and events. It also offers a statistical (non-realist) interpretation of quantum mechanics in the spirit of Copenhagen, 'the statistical Copenhagen interpretation'. Section 6 explores some among the implications of the analysis given in the preceding sections for mathematical modelling beyond physics, especially in

[2]Parts of §§2 and 5 are adopted from [3] and part of §6 from [4], but are significantly modified in view of the present discussion of models, a subject not addressed in these articles. There are also substantive conceptual differences between them and the present article.

cognitive psychology and economics. Finally, by way of conclusion, §7 reflects on 'the spirit of experimentation' enabled by non-realist theories and models in physics and beyond.

2. Concepts, theories and models

Modern, post-Galilean, physical theories are typically defined by mathematical models by means of which they relate to material objects, such as bodies or fields, and their behaviour, usually in order to predict this behaviour as manifested in the experiments considered by these theories. Such models may be descriptive or (a more general concept) representational or (a still more general concept) realist, and derive their predictive capacity from their realist and especially representational nature, as in the case of models used in classical mechanics. They may also be strictly predictive, without being representational or realist, as in quantum mechanics, at least, again, in certain interpretations. The concepts just invoked—theory, model, description, representation, and realism or reality—require proper definitions, which this section is designed to offer. These definitions are not exhaustive, and they cannot be, given the diverse and sometimes diverging understanding of these concepts, and the controversies surrounding them. They are, however, sufficiently general for the purposes of this article and for avoiding misunderstandings concerning its use of these concepts.[3]

I begin with the concept of concept itself, often used without further explanation in physical or even philosophical literature. I shall define a theory as an organized conglomerate of concepts in the present sense, which are, in physics, associated with certain physical objects or (they are not always the same) phenomena, defined by experiments. My definition of concept follows G. W. G. Hegel and G. Deleuze and F. Guattari, who take their inspiration from Hegel. They apply their definition to philosophical concepts, in juxtaposition to scientific or mathematical ones [11, pp. 24–25]. I extend this definition to physical and scientific as well as mathematical concepts, which may involve philosophical concepts as components.

A concept in this sense is not merely a generalization from particulars (which is commonly assumed to define concepts) or a general or abstract idea, although a concept may contain such generalizations and abstract ideas, specifically abstract mathematical ideas. A concept is a multi-component entity, defined by the *organization* of its components, which may be general or particular, and some of these components are concepts in turn. It is the relational organization of these components that is most crucial in defining a concept. Consider the concept of 'tree', even as it is used in our daily life. On the one hand, it is a single generalization of all (or most) particular trees. On the other hand, what makes this concept that of 'tree' is the implied presence of further elements, components or sub-concepts, such as 'branch', 'root', 'leaf', and so forth, and most crucially the relationships between them. The concept of tree acquires further features and components, indeed becomes a different concept, in botany. This is characteristic of scientific extension of daily terms and concepts, arguably especially in quantum mechanics, where most daily terms and concepts are used, in Bohr's words, 'in a way hardly compatible with common language and practical definition' [12, vol. 2, pp. 63–64]. Single-component, 'simple', concepts are rare, if possible at all in rigorous terms, as opposed to provisionally disregarding their multi-component architecture. In practice, there is always a cut-off in delineating a concept, which results from assuming some of the components of this concept to be primitive entities whose structure is not specified. These primitive concepts could, however, be specified by an alternative delineation, which would lead to a new overall concept, containing a new set of primitive unspecified components. The history of a given concept, and every concept has a history, is a history of such successive specifications.

[3]These concepts have a long history of debates in the history and philosophy of science, and recent decades have witnessed a renewed critique and reconsideration of them in the work of T. Kuhn, I. Lakatos, P. Feyerabend and their followers, especially in the so-called constructivist studies of science. The literature on these subjects (both more traditional, such as that in the analytic philosophy of physics, and more revisionist, such as representing the work just mentioned) is massive, and it cannot be addressed here. To give a few pertinent references, for more restrained post-Kuhnian philosophical approaches, see [5, 6], and in the context of the relationships between classical and quantum physics [7, pp. 177–233], and for more radically constructivist treatments, see Galison [8], Latour [9] and Hacking [10]. These works also contain extensive further references.

The same type of process defines the history of a given theory, *as an organized assemblage of concepts*, modified or transformed in the course of this history, for example, from Galileo to Newton and then to Lagrange and Hamilton in classical mechanics, or from Heisenberg and Schrödinger to Dirac and then to von Neumann in quantum mechanics. The history of a given theory is also defined by the history of its interpretation. The history of quantum mechanics was that of persistent and seemingly uncontainable proliferation of its interpretations, multiplying even within each type ('Copenhagen', 'many worlds', 'consistent histories', etc.). 'An organized assemblage of concepts' will serve here as the definition of a theory, using the term concept in the sense just defined. Theories relate to certain manifolds of phenomena or (they are, again, not always the same) objects, which may define the 'reality' considered by a given theory. I define a mathematical model, which is, again, the only kind of models (and correspondingly theories) I consider here, as a mathematical structure that enables a theory to establish a mathematical relation, representational or predictive, to the observed phenomena or objects, which need not be observable (quantum objects are not) that the theory considers. Thus, by a 'quantum theory', I refer to any theory accounting for quantum phenomena, among them the 'standard quantum mechanics' (introduced by W. Heisenberg and E. Schrödinger in 1925–1926), with which and the model comprised by its mathematical formalism I shall be primarily concerned here, and which is henceforth designated as 'quantum mechanics', as against, for example, 'Bohmian mechanics'. The latter (in any of its versions) is a theory defined by a mathematically different model of reality, rather than a different interpretation of the standard quantum-mechanical formalism, and it will only be mentioned in passing here. By quantum phenomena, I refer to those physical phenomena in considering which Planck's constant h must be taken into account, and by quantum objects those entities in nature that, through their interactions with measuring instruments (or what functions as such), are responsible for the emergence of quantum phenomena. By 'quantum physics', I refer to the totality of quantum phenomena and experiments concerning them (experimental quantum physics) and quantum theories (theoretical quantum physics). The terms 'classical phenomena' and 'classical objects' (the difference between them, while still present, could be neglected in classical physics, unlike in quantum physics), 'classical mechanics', 'classical theory' and 'classical physics' will be used in parallel. In order to define the three main types of models considered here—visualizable, non-visualizable realist and non-realist—I need to define the concepts of reality and realism. Before I do so, however, I would like to discuss several examples of physical concepts, especially pertinent to this article.

My first example is the concept of a moving body in classical mechanics. It has multiple components (physical, mathematical and philosophical), beginning with the concept of motion, defined by such component concepts as position and velocity or momentum, mathematized by means of differential functions of real variables. This concept has its history, extending even to the pre-Socratics, but especially to Aristotle's concept of motion, some features of which are found in classical physics and relativity, although not in quantum mechanics. Aristotle's concept, of course, lacked the mathematical architecture found in the concept of motion as defined by modern classical physics and its mathematical model(s) from Descartes, Galileo and Newton on, which architecture defined a new concept of motion. However, both of these concepts of motion, that of Aristotle's physics and that of modern physics, retained their connections to the daily-life concept of motion, which they refined, in the second case, by giving it a mathematical architecture [12, vol. 2, p. 72; 1, p. 11]. As will be seen, this point is important for defining visualizable mathematical models, such as those of Descartes, Galileo and Newton, which are also essentially geometrical, although they involve algebraic components as well.

Einstein's concept of relativistic kinematics in special relativity was a new, revolutionary physical concept, which implied a radical departure from the classical physics concept of motion, because of a new law for the addition of velocities, $s = (v + u)/(1 + (vu/c)^2)$, which becomes relevant when these velocities are close to c. The physical motion defined by this law (again, when the velocity is close to c or is c, as in the case of photons) has no counterpart in our phenomenal intuition of velocity. As such, it also reflects a radical change in our physical, as well as philosophical, understanding of space and time and leads to a fundamentally different physics.

This concept of motion is no longer a (mathematical) refinement of a daily concept of motion in the way the classical concept of motion is. The nature of motion defined by this law, and hence the behaviour of photons, is not visualizable. (Photons are ultimately quantum objects and, hence, in the present view are beyond representation or even conception.) As I argue here, the Lagrangian and Hamiltonian versions of classical mechanics already move beyond this refinement by virtue of their algebraically abstract character. Both versions, however, presupposed that the motion of individual classical objects is physically classical and is visualizable. Accordingly, unlike relativity, analytic mechanics does not reflect any change of the classical physical concept of motion, but only in the mathematization of this concept.

Einstein's concept of gravity in general relativity, mathematically represented by Riemannian manifolds of, in general, *variable curvature*, defined by the presence of matter (including that of fields), was another major new concept introduced by Einstein. The architecture of this concept is complex in its physical, mathematical and philosophical aspects. Its history extends to both Galileo (the equivalence principle, which postulates the equivalence of the inertial and gravitational mass), and the Leibniz–Newton debate concerning the nature of space and time, and to the history of non-Euclidean geometry, to which B. Riemann gave its proper mathematical foundations. The formalism of the theory, its mathematical model, could be seen as yet another form of the Lagrangian formalism, first given to general relativity by Hilbert's derivation of Einstein's equations of general relativity. The theory could also be given a Hamiltonian formulation.

Heisenberg's concept of quantum variables, as infinite unbounded matrices with complex elements (in effect, operators in a Hilbert space over complex numbers, in a more rigorous formalism established by von Neumann shortly thereafter [13]) is fundamentally different from the descriptive or representational concepts of classical physics or relativity. It was the first physical concept of this kind, and it was expressly developed as such. It was defined by making each such variable a mathematical entity enabling only the probabilistic predictions (essential via a Born-type rule) concerning *quantum phenomen*a, observed in measuring instruments, without providing a mathematically idealized description or representation of the behaviour of *quantum objects* responsible for the appearance of these phenomena. This is in accord with Bohr's concept of quantum phenomena, defined as what is observed in measuring instruments and thus as irreducibly different from quantum objects, which are never observable as such. Nobody has ever seen, at least thus far, a moving electron or photon. It is only possible to register traces of this 'movement' (assuming even this concept applies) left in measuring instruments, traces that do not allow us to reconstitute this movement in the way it is possible in classical physics or relativity. Mathematically, an especially novel feature of Heisenberg's formalism was that his new variables, in general, did not commute, that is, the product of PQ was not, in general, equal to QP: $PQ - QP \neq 0$. This feature eventually came to represent Heisenberg's uncertainty relations constraining certain simultaneous measurements, most notable, those of the momentum (P) and the coordinate (Q), associated with a given quantum object in the mathematical formalism of quantum mechanics and (correlatively) the complementary nature of such measurements in Bohr's sense. Heisenberg referred to these variables and their predictive functioning as 'new kinematics', which was not the best term to use under the circumstances. As its etymology suggests, kinematics conventionally refers to variables representing the motion of physical objects, as it does, again, differently in both classical physics and relativity. As Heisenberg commented even before his paper introducing quantum mechanics was published: 'What I really like in this scheme is that one can really reduce *all interactions* between atoms and the external world … to transition probabilities' [14] (letter to Kronig, 5 June 1925; cited in [15, vol. 2, p. 242]; emphasis added).

This epistemology, including the difference, thus irreducible, between quantum objects and quantum phenomena, became the core of Bohr's interpretation of quantum phenomena and quantum mechanics as complementary, an interpretation that stemmed from Heisenberg's thinking just described but that was developed under the impact of Bohr's subsequent exchanges with Einstein. Bohr's concept of a quantum object defined it as unavailable to any description or representation, or possibly even conception, although it is, again, not clear whether Bohr would have been willing to go that far. This definition gives this concept a special status insofar as this concept has no conceptual architecture that could be determinately ascertained, or in the second eventuality possibly in principle be applicable to it in any way. Phenomena, by contrast, are defined by what is actually observed, as effects of the interactions between quantum objects and measuring instruments, and thus are subject to a classical description, under certain constraints, such as those defined by the uncertainty relations. Considered, in their own right, the observable parts of measuring instruments are described by classical physics and its concepts, although these concepts cannot describe how such observed effects come about, nor predict them. (Measuring instruments also have quantum and, hence, unobservable parts, through which they interact with quantum objects.) In classical physics, specifically in classical mechanics, this difference between objects and phenomena, while still, technically, valid, as defined, for example, on Kantian lines, can be disregarded, and objects and phenomena can be treated as the same. Correlatively, the behaviour of objects could be treated as independent of observation, because the interference of measuring devices, beginning with our bodies, with the process of observation or measurement can be neglected or compensated for.

As in Heisenberg's original scheme, in Bohr's interpretation, the mathematical concepts comprising the formalism of quantum mechanics only relate, in terms of probabilistic predictions, to what is observed in measuring instruments, even, in contrast to classical mechanics, in dealing with primitive individual quantum processes and events. This situation appears to be correlative to the epistemology just described, as defined by the fact that, unlike in classical mechanics, one cannot obtain any information about quantum objects, even the primitive individual ones, without interfering appreciably in their behaviour. This makes all such information defined by the effects of the interaction between quantum objects and measuring instruments. In this regard, as Bohr emphasized, the recourse to probability or statistics in quantum theory is essentially different from the use of probability and statistics in classical statistical physics, or in dealing with such events as a coin toss. There this use is necessitated by the mechanical complexities of the systems considered, which cannot be tracked in practice, but which emerge from the mechanical behaviour of the ultimate individual constituents that is causal and, in the first place, amenable to a realist representation by means of classical mechanics. In quantum mechanics, at least if interpreted in the spirit of Copenhagen, the behaviour of the ultimate constituents of matter is not amenable to a realist representation and, as a consequence, is never causal, which makes the recourse to probability unavoidable even in this case, again, in accord with what is observed in quantum experiments.

Finally, I would like, by way of a contrast with Heisenberg and Bohr, to mention Schrödinger's concept of wave function, initially developed by Schrödinger as a representational or realist concept, in a deliberate juxtaposition to Heisenberg's scheme. While the key mathematical features of this concept have survived and become indispensable in quantum mechanics, the representations or realist nature of the concept and Schrödinger's wave mechanics as a whole, as a realist theory, has proved to be difficult to sustain. Schrödinger abandoned his initial project of wave mechanics shortly thereafter, although he eventually tried to return to it. The wave function was converted by others into a predictive concept of the Heisenbergian type, via Born's rule.

I shall now define the concepts of reality and realism, in the case of realism, two sets of concepts. These concepts could and have been defined in a great variety of ways, and they have been discussed and debated from the pre-Socratics on. The present sets of definitions will not be able to capture all concepts thus designated, which is, as I said, impossible. Nevertheless, the concepts of reality and realism defined here are, I believe, sufficiently general to encompass a large spectrum of both concepts currently used in physics and the philosophy of physics.

By 'reality', I refer, very generally, to that which exists or is assumed to exist, without making any claim concerning the character of this existence. In the case of physics, it is nature or matter, which generally, but not always (although exceptions are rare), assumed to exist independently of our interaction with it, and to have existed when we did not exist and to continue to exist when we will no longer exist. This assumption also holds in Bohr's and other interpretations in the

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spirit of Copenhagen, in the absence of any representation or even conception of the character of this existence, which defines realism.

I define realism, then, as, in each corresponding (realist) theory, a specific set of claims concerning what exists and, especially, *how* it exists, provided by the corresponding theory, again, meaning by a theory an organized conglomerate of concepts, accounting, either in a realist way or not, for the phenomena they consider. In this definition, any form of realism is more than only a claim concerning the existence, *reality*, of something, such as physical objects, which we can describe, at least ideally, in classical physics, or quantum objects, about which, at least in certain interpretations, nothing else could be said or even thought. Instead, realism is defined most essentially by claims concerning the *character* of this existence.[4] The extent of such claims or degree of their specificity may vary. Nevertheless, realist theories, as they are understood here, may be seen as belonging to the following two types.

According to *the first type of realism*, a realist theory would offer a description or, more generally, representation, in modern physics typically an idealized and mathematized representation of the objects or systems considered by the theory and their behaviour, or sometimes, as in so-called structural realism, of the structures defining such systems and their behaviour. Such an idealized representation retains some of the features of the actual objects and processes considered and disregard others, in particular those that cannot be mathematized. The mathematical formalism of a given realist theory comprises a *representational* mathematical model of reality, according to the present definition of a model. Such a model is sometimes identified with this theory, although in this article a theory is understood as having a larger conceptual architecture than does a model, typically limited to the mathematical aspects of the theory. Quantum mechanics, or classical physics or relativity, is hardly limited to its mathematical formalism, for one thing, by virtue of an interpretation this formalism requires. All modern, post-Galilean, physical theories, again, proceed by way of idealized mathematical models, even if these models are not realist or representational. Realist models are representational models—idealized mathematical representations of physical objects and processes considered, again, as part of a given theory, as a conglomerate of concepts, which also gives an interpretation to the model it uses. As I said, for simplicity, I also speak of this interpretation as an interpretation of this theory itself.

All realism in physics (or elsewhere) is *conceptual* realism, as was acutely realized by Einstein, who saw the mediation of mathematical concepts as irreducible and essential, and who saw the practice of theoretical physics as that of the invention of new concepts through which one can approach reality.[5] He argued that a viable realist representation of physical reality could only be achieved by means of conceptual construction, 'the free choice of [mathematical] concepts', rather than by means of observable facts themselves, which Einstein sees as the empiricist 'philosophical prejudice', found, for example, in E. Mach's philosophy [16, p. 47]. A theory, however, always adds dimensions that are other than mathematical to the mathematical concepts of its mathematical models, and concepts that are other than mathematical. Such a choice of concepts is never entirely free and may perhaps be better seen in terms of experimenting with concepts.

The first type of realism allows for different degrees to which our models 'match' reality. As noted above, these questions could be and, beginning at least with Kant, have been posed concerning classical mechanics, where our descriptive (visualizable) idealizations are more in accord with our phenomenal experience. The question has also been posed concerning the degree, if any, to which the mathematical architecture of relativity (more removed from this experience, as noted above) corresponds to the architecture of nature, as opposed to serving as a mathematical model for correct predictions, in this case, ideally exact, concerning relativistic phenomena [17]. I shall return to these questions in the next section.

[4]Realist theories are sometimes also called ontological theories. The term 'ontology' has additional philosophical connotations that require qualifications. These qualifications, however, would still allow one to see ontological theories as realist in the present sense, which makes these terms interchangeable as far as this article is concerned.

[5]Mathematical realism, sometimes also known as mathematical Platonism, is something else, in part by virtue of the fact that it deals with mental rather than physical reality, and it will not be addressed here.

Visualizable models comprise a particular type of representational realist models. As other key concepts under discussion here, the concept of visualization has various and sometimes divergent meanings. Feynman diagrams are sometimes seen as visualizations of quantum processes of high energy governed by quantum field theory. Helpful as they are, however, Feynman diagrams are merely *diagrams* (heuristic devices): they do not really represent the quantum processes to which they refer, even if one does not hold that these processes are *not visualizable* or otherwise representable, or even conceivable, as they are according to the interpretation adopted by the article. The role of Feynman diagrams is to help one to work more easily with the formalism of quantum electrodynamics or quantum field theory in order to make probabilistic predictions concerning the outcomes of relevant experiments, in the same way it is done in quantum mechanics. Bohr's concept of complementarity, too, is sometimes seen as 'regaining' the lost 'visualization' in quantum mechanics [18,19, pp. 68–69]. I would argue that this is a misconception, at least as concerns the behaviour of quantum objects, and Bohr himself never spoke of and never saw complementarity in this way. Complementarity does, it is true, relate to visualizable complementary phenomena observed in measuring instruments, but never to any visualization or other representation of quantum objects. There are other helpful heuristic tools of visualization used in quantum physics. This is, however, not the kind of visualization that I have in mind.

By visualization, I mean the possibility of a (phenomenally) visually configurable geometrical representation, a picture, of the behaviour of an individual physical object in classical mechanics, paradigmatically represented by a motion of a material body mathematically idealized as a dimensionless massive point. As noted above, Bohr sometimes also speaks of a pictorial visualization or pictorial representation, with German *Anschaulichkeit* (intuition) in mind. As I shall discuss in more detail in the next section, this idealization is a (mathematized) refinement of our daily perception and our (visualizing) intuition of motion, although visualizable realist models are still conceptual models. It is true that there are classical systems the (idealized) behaviour and hence the corresponding models of which cannot be thus visualized, for example, multi-component systems of great mechanical complexity, such as, again, paradigmatically, those of classical statistical physics. The corresponding models are statistically predictive, rather than representational. Models of analytical mechanics, or of chaos and complexity theories are representational, but they are not, in general, visualizable, although some computer models provide partial visualizations of such models, as in the case of chaos theory, famous for these computer images. However, in all of these cases, the individual behaviour of the ultimate constitutive objects from which the behaviour of these systems emerges is represented by the model of classical mechanics and, as such, is at least locally or infinitesimally visualizable, even though this visualization may be approximate.

In classical mechanics, this visualization is essentially connected to the possibility of ideally representing the motion of classical objects by means of differential calculus, which reflects this geometrical visualization of the continuous or, more accurately, differential motion, along a straight or curved trajectory, of an object. In fact, this motion, again, at the level of this idealized model, can be considered as infinitesimally linear, which circumstance led Newton to the invention of calculus, although he was compelled to recast his argument, initially developed via calculus, in terms of Euclidean geometry in *Principia* [20]. One can, literally, draw a picture of such motion (as a small curve with a tangent vector as a given point on it) on a blackboard, actual or that in our mind, which makes these models more descriptive. From this perspective, visualization is in effect equivalent to the applicability of the concept of classical motion as infinitesimally represented by differential functions of time and coordinates.

This representation implies that position and the velocity (or momentum) are both defined at any moment of time and, hence, such a representation is impossible in the case of quantum objects, due to the uncertainty relations. In the present view, no representation of their behaviour, either as continuous (or differential) or discrete, by any means is possible, even in the case of primitive or elemental individual quantum objects and processes. This makes quantum mechanics or its mathematical model non-realist. Quantum discreteness only occurs at the level of quantum phenomena observed in measuring instruments, the change of which behaviour under the impact of quantum objects is not classically controllable, even though the independent behaviour of these instruments is described by the idealized (differential) models of classical physics, as just outlined. There is no physical continuity or differentiability in quantum mechanics or quantum field theory as concerns the behaviour of quantum objects. The quantum-mechanical mathematical formalism does involve differential calculus in Hilbert spaces over complex numbers, without, in the present interpretation, representing any physical reality. This formalism only relates (via certain rules, such as Born's rule) to the probabilities or statistics of the outcomes of possible future quantum experiments on the basis of the data obtained in previously performed quantum experiments. These circumstances compelled Bohr to speak of the impossibility of pictorial visualization of quantum objects and processes. As Bohr ultimately maintained as well, they are beyond any other representation and possibly beyond conception, although Bohr, again, might not have been willing to go as far.

It is, I should qualify, possible to see a claim concerning the existence of something to which a theory relates without representing it as a form of realism. Such appeals to realism are found in advocating interpretations of quantum mechanics that are not realist in the present sense: that is, these interpretations do not assume that the formalism of quantum mechanics represents, even ideally, the behaviour of quantum objects, but only serves to predict the outcome of quantum experiments. However, placing this view of reality outside realism is consistent with a more common use of the term realism and is advantageous in the context of interpretations in the spirit of Copenhagen. Conversely, as I explain below, certain realist theories of the second type about to be defined are sometimes termed 'non-realist'.

The second type of realism would presuppose an independent architecture (which may be temporal) of reality governing the behaviour of the ultimate objects considered, even if this architecture cannot be represented, even ideally, either at a given moment in history or perhaps ever, but if so, only due to practical limitations. In the first of these two eventualities, a theory that is merely predictive may be accepted for the lack of a realist alternative, but under the assumption or with a hope that a future theory will do better, for example and in particular by virtue of being a realist theory of the first type, just considered. Einstein adopted this attitude towards quantum mechanics, which he expected to be eventually replaced by a realist theory, essentially, a field theory of a classical type, on the model of Maxwell's electrodynamics, followed by Einstein's general relativity. In general, even in the second eventuality, the ultimate constitution of nature is often deemed conceivable following models of classical physics, possibly adjusted to accommodate new phenomena, such as electromagnetic ones, and new concepts, such as field, classical, or even quantum, or more recently automata. Relativity still follows classical physics in its model of reality as concerns realism and causality, although, as explained earlier, ultimately not visualizability. However, generally, the second type of realism assumes that a proper specific theory or model of reality is not and may not be available. If it were, the situation would amount to the first type of realism.

What, then, unites both conceptions of realism and thus defines realism most generally is the assumption that an *architecture* of reality, rather than only reality itself, exists independently of our interactions with it, or at least that the concept of architecture or structure would apply to reality. This presupposition, again, defines structural realism, although the latter, usually makes more specific claims concerning this structure (e.g. [21]). In other words, realism is defined by the assumption that the ultimate constitution of nature possesses attributes, 'elements of reality', and the structured relationships among them that may be either (i) known in one degree or another and, hence, represented, at least ideally, by a theory or model or (ii) unknown or even unknowable. This difference between (i) and (ii) separates two types of realism defined here. Both allow for and the second implies the difference between 'objects', defined as what ultimately exists in nature, especially in its ultimate constitution, and 'phenomena', defined as what appears to our mind. This difference, as the difference between noumena (things-in-themselves) and phenomena (appearances or representations constructed by our mind), grounds Kant's philosophy. The same type of difference also grounds Bohr's distinction between 'quantum phenomena' (defined as what is observed in measuring instruments) and 'quantum objects', which are unobservable and unknowable or unrepresentable, and possibly inconceivable, and thus are beyond thought itself, rather than only knowledge or representation. This distinction follows Kant but, for the reasons explained below, it takes a more radical form in Bohr's epistemology. Kant's position may be seen as realist or allowing for the possibility of realism *in the present definition*, realism of the second kind, although, as will be seen presently, Kant's argument involves further epistemological complexities and its realist or non-realist character may be a matter of interpretation or choice of terminology.

As I indicated above, sometimes theories (including models they contained) defined here as realist theories of the second type are seen as 'non-realist'. Such theories are, however, different from non-realist theories in the present sense, because this view still at least allows that objects or reality, while unknown and possibly even unknowable, is still conceivable, thinkable, in certain organizational, architectural terms, the difference I shall further explain presently, via Kant. As I said, however, sometimes non-realist theories in the present sense are, conversely, seen as realist, because they still assume the existence of reality, however much beyond representation or even thought this reality might be, although this is relatively uncommon.[6] It may be added that one also could, following G. Berkeley, understand non-realism as the denial of the existence of external reality or matter altogether. While this concept, used by Berkeley against Newton, is not without relevance to physics and specifically quantum theory, it is rarely adopted and will be not considered here. As defined here, realism at the very least assumes that the *concepts* of independent properties and organization can in principle apply to reality, no matter how much off the mark at a given point or even ever our conceptions of this constitution may be. However, the hope of most proponents of realism is that our theories can eventually capture something of this architecture. In addition, more commonly than not, this architecture is conceived on the ideal of classical physics and is assumed to be approachable along the lines of the first type of realism.

Now, interpretations of quantum phenomena and quantum mechanics in the spirit of Copenhagen, beginning with that of Bohr (at least in its ultimate version, developed in the late 1930s), not only do not make any of the realist assumptions just considered but also disallow all of them. In order to appreciate the degree of this break with realism, it may be helpful to revisit Kant's epistemology, arguably the weakest form of realism available, even bordering on non-realism. Kant's epistemology places the ultimate (noumenal) reality beyond our knowledge or understanding based on this knowledge, both of which are associated strictly with phenomena or appearance to our mind. However, while *beyond knowledge*, Kant's noumena are not *beyond thought*. They are, according to Kant, thinkable, insofar as this thinking is logical. While a rigorous scientific theory in physics or elsewhere, even a non-realist one, must of course be logical, here, as in all realist theories, this requirement gives the possible architecture of noumena and thus nature a logical character, a conception that is a product of human thought [22, p. 115]. It is true that we are capable of thinking in a logically consistent way beyond things that we know or that are phenomenally available to our understanding. Kant associates this type of thinking with what he calls reason (*Vernunft*), which is a different and higher faculty than understanding (*Verstand*), dealing with phenomena only, and which could in principle reach the ultimate nature of things. Kant recognizes that our thinking concerning the character of noumena or things-in-themselves may be wrong, even if it works in practice. However, his view of the situation still logically implies that this thinking may also be correct, even though it may not be possible to definitively verify its correctness. Indeed, Kant appears to suggest that some claims of reason are in fact determinately correct. I shall, however, put this part of his argument aside, because it represents a form realism of the first type considered above and is, hence, of less interest here, as against more sceptical dimensions of his thought that only allow that such claims may be true but are never certain to be true. (In this regard, his thinking comes close to that of D. Hume, whose philosophy Kant,

[6]Thus, Hacking's concept of 'entity realism' (versus 'theory realism') and some of its avatars, developed during recent decades in the philosophy of science and debates concerning the question of reality there, as mentioned in footnote 3, may be argued to be examples of this use of the term realism [6].

in general, aims to supersede.) As Kant also argued, under this assumption of only possible rather than determinate truth of our conception of ultimate reality, this conception need not be justified in theoretical terms: a practical justification, defined by the workability of such a conception, may suffice [22, p. 115]. This is why this is a borderline position between realism and non-realism. However, because such a conception might still be true, this view implies at least a possibility of a realist representation, albeit never guaranteed to be correct or be verifiable, in the first place.

On the other hand, it is also possible that, in the case of certain phenomena, no conception of the corresponding noumenal reality that could be even practically justified could be found, while, conversely, it is the absence of such a conception that could be at least practically justified. Although the latter possibility does not appear to have been considered by Kant, this is precisely what happens in the case of quantum objects and processes, as inferred from the data manifested in quantum phenomena, according to Bohr's and other interpretations in the spirit of Copenhagen. If one applies Kant's reasoning just outlined to quantum objects and processes, a conception that we may form of them (and nothing *a priori* prevents us from doing so) may be incorrect or it may not be possible to be certain whether it is correct or not. However, if this conception allowed us to predict correctly the outcomes of quantum experiments, it would be practically justified. Indeed, at least in physics or elsewhere in science, any realism, even the most direct one, is only meaningful if it is justified in practice, and in particular if, even when it is representational, it enables effective predictions, of either probabilistic or (ideally) determinate nature, concerning future physical phenomena. Quantum objects qua quantum objects, in particular the elemental ones (such as electrons or photons, or other elementary particles) and their independent behaviour are experimentally unobservable, and hence any conception we form of them could only be theoretically verified or justified indirectly. Bohr's argument is that, given the character of quantum *phenomena* and data observed in them, in particular statistical or, as in the EPR-type experiments, statistically correlated data, no such realist conception concerning quantum phenomena appears to be justified either theoretically or practically, at least as things stand now and if, in addition, one assumes locality. Although it took a while and some 'help' from Einstein to realize its significance, the latter qualification and the question of locality became crucial to our understanding of quantum phenomena and quantum mechanics from the time of the EPR thought experiment [23], and especially in the wake of such more recent developments as the Bell and the Kochen–Speker theorems.

Bohr's and related non-realist interpretations are, then, at least practically justified. If, however, no representation of quantum objects and processes is possible consistently with the nature of quantum phenomena, it is also *possible* that no such conception is possible at all, even if Bohr, again, was unwilling to go so far. Either assumption appears to be in principle consistent with Bohr's, arguably, strongest statement on the subject to the effect that an '*analysis*' of quantum phenomena beyond a certain point and, hence, of the ultimate nature of quantum objects and their behaviour, is '*in principle* excluded'—an *analysis*, but not necessarily a conception [12, vol. 2, p. 62]. Bohr might, however, be understood as having adopted this stronger view, because of his emphasis of the lack of causality in quantum processes, if one agrees (as Bohr might have) with Wittgenstein's contention in the *Tractatus* that we cannot conceive of processes that are not causal [24, p. 175]. I am not saying that this stronger assumption is logically or physically necessary, but only that it is *interpretively possible* (hence my emphasis on 'possible' above). There do not appear to be experimental data to prefer either assumption, or for that matter to definitively claim either beyond their interpretative consistency or effectiveness. For all practical purposes, these two assumptions are pretty much equivalent as far as physics is concerned. They are, however, different philosophically.

It remains important that, in either case, we still deal with interpretations of quantum phenomena and quantum mechanics, rather than 'the truths of nature'. These truths, to the degree one can speak of truth at all, are only represented by the data observed in quantum phenomena and the probabilistically or statistically correct predictions provided by quantum mechanics itself, or locality, again, as things stand now. Either interpretation is, however, logically consistent and is in accord with these truths, which justifies, at least practically, either assumption just discussed. Either interpretation could still be challenged, and that of Bohr has been even on the basis of the currently available data (how successfully is another matter). Either could also be eventually refuted in view of new data.

That said, the non-realist reasoning based on either assumption is not simply 'an arbitrary renunciation' of an analysis or the possibility of realist representation of the ultimate constitution of nature underlying quantum phenomena [12, vol. 2, p. 62]. It is an *argument* that, on the basis of a rigorous analysis of quantum phenomena and quantum mechanics, one is compelled to conclude that this constitution *might be* beyond the reach of representation by means of quantum mechanics or otherwise, or even of thought itself, at least as things stand now, and offer a corresponding (logically consistent) interpretation of both. It is not, accordingly, that this constitution is something that is merely unknown or unthought at the moment and that could in principle become known or thought one day, which way of thinking of reality would, in the present definition, qualify as realism. The reality defined here is ineluctably beyond representation or possibly thought, is reality without realism, or reality beyond realism, which, importantly, remains an interpretation that could be abandoned in favour of a more realist one. Quantum objects are real and, as such, have effects on the world we observe, and yet prevent us from representing them, by the formalism of quantum mechanics or possibly otherwise, or even from forming any conception of them. As this stands now, that is, insofar as the present experimental data hold, in this interpretation of quantum phenomena, defined by this data, this reality is not part of our knowledge and possibly not thought itself. Each encounter with this reality, say, in any given quantum experiment, is a singular, unique encounter, without implying that there is some single overall, all encompassing, total reality which we always confront. While each time unknowable or unthinkable, this reality is each time different. It is, again, possible that nature one day will allow for a fundamental theory that no longer requires this concept of reality and hence will violate and will compel us to abandon non-realism, as Einstein and many others following him have hoped and continue to hope. Most physicists and philosophers have always assumed that it is unreasonable to go that far in renouncing realism that has been and remains so successful in physics apart from quantum mechanics. This resistance was well realized by Bohr [12, vol. 2, p. 63]. At stake is a confrontation, dramatically enacted in the Bohr–Einstein debate, between two radically different ways of thought. The question, which is still open and which might remain open for years to come, is whether nature will ultimately allow for the first, realist architecture. Einstein hoped and even believed that it will, while Bohr thought that it might not, which is not the same as to say that it never will.

In non-realist interpretations of quantum phenomena and quantum mechanics, only predictions, moreover, in general, only of a probabilistic or statistical nature, concerning quantum phenomena, even those associated with primitive individual quantum processes, are possible. This is one of the reasons (but not the only reason) for such interpretations, because this character of quantum predictions is an experimental fact. As noted from the outset, the repetition of identically prepared experiments, in general, leads to different outcomes. Unlike in classical physics, this difference cannot be improved beyond a certain limit (defined by Planck's constant, h) by improving the conditions of measurement, a fact also reflected in the uncertainty relations, which are equivalent or correlative to the statistical nature of quantum predictions. This character of quantum predictions must, accordingly, be equally maintained, as must also, and correlatively, be the uncertainty relations, by realist interpretations of these theories or alternative theories, such as Bohmian theories. In Bohmian theories, a given quantum object is assumed to possess both position and the momentum, defined exactly at any moment of time, which allows for realism and causality. However, these theories both, and again correlatively, reproduce the statistical predictions of quantum mechanics and retain the uncertainty relations, because a given measurement always disturbs, *actually disturbs*, the object and displaces the value of one of these properties. This type of disturbance of independent causal quantum behaviour is in conflict with Bohr's interpretation and other interpretations in the spirit of Copenhagen [12, vol. 3, pp. 63–64].

Most representational or realist models in physics are also predictive and derive their predictive capacity from their representational or realist capacity. Furthermore, some mathematical models used in classical physics are strictly predictive, without offering a description of the objects and processes considered by the corresponding theories. The models used in classical statistical physics are of that type, because they only offer statistical predictions concerning (large) classical systems. These models are, however, underlined by the representational, even visualizable, model of classical mechanics, which is assumed to describe (causally) the behaviour of individual constituents of these systems. The situation is somewhat more complex but is not essentially different in chaos and complexity theories, which are probabilistically predictive as well, due to the mechanical complexity of the systems considered and their sensitivity to the initial conditions. By contrast, mathematical models used in quantum mechanics and quantum field theory are only predictive and, moreover, only probabilistically or statistically predictive, even as concerns elemental individual quantum objects and processes, at least in interpretations that follow the spirit of Copenhagen. It is an intriguing question whether one can have strictly predictive *non-representational* or *non-realist* (rather than only non-visualizable realist) models that are not probabilistic or statistical: we do not appear to have examples of such models thus far. I mean here *predictive models* that, similarly to those of quantum mechanics or quantum field theory, would not allow for or would resist a realist interpretation. Otherwise, classical or relativistic realist models, which make ideally exact predictions, could be interpreted, on Kantian lines, as only predictive rather than realist.

Finally, as indicated above, there is still a form of realism associated with non-realist interpretations of quantum phenomena and quantum mechanics. This realism is defined by the interpretation of the physics of measuring instruments in which the outcomes of quantum experiments are registered as the effects of the interaction between quantum objects and these instruments. (These are objective and objectively communicable truths mentioned above.) These instruments, rather *their observable parts*, are assumed to be described by classical physics. On the other hand, the interaction between quantum objects and measuring instruments is quantum, and hence it is not amenable to a realist treatment, but in each case, this interaction leaves, as its effect, a trace, a mark, in a measuring instrument. The origin of this 'trace' is beyond the reach of experiment or theory, or, again, of thought itself, and in this sense the very concept of trace may be inapplicable, insofar as it cannot refer to that which could in principle be traced. The mark itself, however, can be treated, by means of a realist account, and made part of a permanent record, which can be unambiguously discussed and communicated, and in this sense may be seen as realist and objective. These are, again, these effects that compel the introduction of non-realist interpretations of quantum phenomena and quantum mechanics.

The lack of causality is an automatic consequence, especially if one places the reality of quantum objects and processes beyond thought altogether, because causality would imply an at least partial conception of this reality. Conversely, as noted above, any conception of this reality may have to be causal, because, as Wittgenstein contended, we cannot conceive of processes that are not causal [24, p. 175]. However, even if one adopts a weaker view, which only precludes a representation of this reality, causality is difficult and ultimately impossible to maintain because in order to do so one requires a sufficient representation, analogous to that which obtains in classical physics [25]. Schrödinger expressed this difficulty, albeit by way of a very different assessment of the type of argumentation just outlined, which he saw as 'a doctrine born of distress', in his cat-paradox paper. He said: 'if a classical state [defined by the ideally definite position and the definite momentum of an object at any moment of time] of an object does not exist at any moment, it can hardly change causally' [26, p. 154]. I need, however, to define causality.

Causality as understood in this article (and by Schrödinger here) is an ontological category—part of reality. It relates to the behaviour of physical systems whose evolution is defined by the fact that the state of a given system is, at least as idealized by a given theory and model, determined at all moments of time by their state at a particular moment of time, indeed at any given moment of time. This concept is in accord with Kant's principle of causality, which states (this statement of the principle has been commonly used since) that if an event takes place, it has a [determinable] cause of which it is an effect [22, pp. 305, 308]. Quantum phenomena violate this principle, because no determinable event or process could ever be established as the necessary cause of a given event, and only statistical correlations between certain events could be ascertained. Accordingly, the concept of causality just defined may be more properly designated 'classical causality'. However, given that most of my discussion of causality concerns classical causality, and I shall refer by 'causality' to classical causality, which is also in accord with most standard uses of the term, but not all of them. In particular, the term 'causality' is sometimes used in accordance with the requirements of relativity, which further restricts causes to those occurring in the backward (past) light cone of the event that is seen as an effect of this cause, while no event can be a cause of any event outside the forward (future) light cone of that event. In other words, no physical causes can propagate faster that the speed of light in a vacuum, *c*, which requirement also implies temporal locality. Thus, one can also speak of relativistic causality, although relativity theory itself, special or general, is a classically causal theory, while quantum mechanics is not, at least in the present view. In speaking of the lack of causality in quantum mechanics, I only refer to the inapplicability of classical causality, found in classical physics, and not about any incompatibility with relativity. The compatibility with relativistic requirements would be maintained, say, as concerns the way in which a given experiment already performed determines a possible outcome of a future experiment, a determination that is not classically causal, at least in the present interpretation.

I define 'determinism' as an epistemological category, part of our knowledge of reality. Just as in the case of causality. Determinism denotes our ability to predict the state of a system, at least, again, as defined by an idealized model, exactly, rather than probabilistically, at any and all points once we know its state at a given point. This category may, too, be called classical determinism, because one could define determinism in a more quantum-mechanical way. Schrödinger's equation is sometimes seen (in my view, not rigorously) as deterministic, and sometimes as causal. Once again, however, unless qualified otherwise, I shall by determinism refer to classical determinism. Determinism is sometimes used in the same sense as causality, and in the case of classical mechanics, which deals with single objects or a small number of objects, causality and determinism essentially coincide. Once the system is large enough, one needs a superhuman power to predict its behaviour exactly, as was famously noted by P. S. Laplace, albeit in order to argue for the causal nature of the world.

While, however, it follows automatically that non-causal behaviour, *considered at the level of a given model*, cannot be handled deterministically, the reverse is not true. The underlined qualification is necessary because we can have causal models of processes in nature that may not be ultimately causal. The fact that the causal models of classical physics apply and are effective within the proper limits of classical physics does not mean that the ultimate character of the actual processes that are responsible for classical phenomena is causal. They may not be, for example, by virtue of their ultimately quantum nature. Nor, conversely, does the non-causal character of a model, for example, that of quantum mechanics in the present interpretation, guarantee that the ultimate nature of reality is non-causal. It may ultimately prove to be causal, and, to begin with, amenable to a realist treatment, and many, again, hope it will be.

As all models, non-causal or, in the first place, non-realist, such as quantum-mechanical, models of the ultimate constitution of nature are (interpretive) idealizations, formed by theories that contain these models. In particular, rigorously it is only determinism that quantum phenomena preclude, because, as noted from the outset, identically prepared quantum experiments in general lead to different outcomes. As against classical physics, the difference between these outcomes cannot be improved beyond certain limits (reflected in uncertainty relations), for example, by improving the precision of our instruments, in the way we can, at least in principle, in classical physics. This makes individual experiments unrepeatable as concerns their outcomes, as against the statistics of multiply repeated experiments, which are repeatable.

It would be difficult, if not impossible, to have a scientific theory without being able to repeat at least the statistical data established and confirmed by experiments. The lack of causality or, to begin with, realism is *interpretive inferences* from this situation, which makes randomness or chance the fundamental element of quantum physics.[7]

By 'randomness' or 'chance', I refer to a manifestation of the unpredictable. Randomness and chance are not quite the same, but the difference between them is not germane for my argument, and I shall primarily speak of randomness. It may or may not be possible to estimate whether a random event would occur, or even to anticipate it as an event. A random event may or may not result from some underlying causal dynamics unavailable to us. Thus, as noted above, in classical statistical physics, randomness and the recourse to probability that result from it are due to insufficient information concerning systems that are at bottom causal but whose mechanical complexity prevents us from accessing their causal behaviour and making deterministic predictions concerning this behaviour. Indeed, in considering phenomena in question in classical statistical physics, when quantum events could be neglected, we could in certain circumstances even observe such individual behaviour as causal. Once quantum effects come into play, as in the case of the thermodynamics of black-body radiation, this assumption could no longer be made, as Einstein was the first to realize in the case of Planck's law, which he showed to be incompatible with this assumption [28]. The situation found in quantum physics is thus different from that of classical physics, given the difficulties of sustaining arguments for the causality of the independent behaviour of quantum objects, even elemental individual quantum objects, 'elementary particles' (a concept on which I shall comment below). If an interpretation is non-realist, the absence of causality is, again, automatic, and the recourse to probability or statistics is unavoidable, even in the case the elemental constitutive processes and events.

Probability and statistics deal with providing estimates of the occurrences of certain individual or collective events, which defy deterministic handling (whether there is or is not a hidden underlying causality determining these events), in physics or science usually in accordance with mathematical probability theories. The terms 'probabilistic' and 'statistical' are generally used differently. 'Probabilistic' refers to our estimates of the probabilities of either individual or collective events, such as that of a coin toss or of finding a quantum object in a given region of space. 'Statistical' refers to our estimates concerning the outcomes of identical or similar experiments, such as that of multiple coin tosses or repeated identically prepared experiments with quantum objects, or to the average behaviour of identical objects, or objects treated as identical.[8] A given definition of probability may already reflect this difference, as in the case of the Bayesian versus the frequentist understanding of probability. The Bayesian understanding of probability defines it as a degree of belief concerning the occurrence of possible individual events on the basis of the relevant information we possess and thus making probabilistic estimates involved, generally, subjective, although there may be agreements (possibly among a large number of individuals) concerning such estimates. The frequentist understanding, sometimes also referred to as 'frequentist *statistics*', defines probability in terms of sample data by the emphasis on the frequency or proportion of these data, which is often seen as more objective, although this can be debated. The Bayesian approach allows one to make estimates concerning individual and even unique events, say, betting on the outcome of a basketball game or even, as in Pascal's wager, on the existence of God and the salvation of the soul, rather than on frequently repeated events, such as repeated coin tosses. In the latter case, our estimations are defined by previous experience of the same or closely similar events. It is true that, technically, no two coin tosses are ever quite the same, the point used by Bayesian theorists against frequentist approaches

[7]This statement does not exclude the possibility of causal or realist interpretations of quantum mechanics, or the alternative causal or realist quantum theories, such as Bohmian mechanics, or theories defined by an assumption of a deeper underlying (pre-quantum) causal dynamics, which makes quantum mechanics an 'emergent', surface-level theory. Khrennikov's theory, invoked in §6, is a theory of this type [27].

[8]The standard use of the term 'quantum statistics' refers to the behaviour of large multiplicities of identical quantum objects, such as electrons and photons, which behave differently, in accordance with, respectively, the Fermi–Dirac and the Bose–Einstein statistics.

to probability (e.g. [29, pp. 317–320]). In the frequentist view, however, they are sufficiently similar to be treated as statistically identical.

This brief summary sidesteps some of the deeper aspects of probability, but it suffices for my purposes. I shall address some of these aspects below.[9] I conclude my discussion of probability by reiterating the following two points. First, probability has a special temporal structure by virtue of its, correlatively, irreducibly futural and irreducibly discrete character, because one can only estimate future discrete events, thus strictly corresponding to the ultimate character of all quantum events, about which, moreover, only probabilistic or statistical predictions are possible, even in dealing with primitive individual events. This qualification is crucial, for otherwise, this situation pertains to all probabilistic or statistical situations, such as those of classical statistical physics. Quantum mechanics and higher level quantum theories, if understood in the spirit of Copenhagen, only provide (very good) estimates in strictly probabilistic or statistical terms of the outcomes of certain future events and the basis of certain previous events, but say nothing about what happens between these events. There is no story to be told and, in the strongest form of non-realism assumed here, no concept to be formed as to how these outcomes come about, even these theories give us, with great exactitude, the probabilities or statistics of observed in quantum experiments.

The second aspect of probability I would like to note is as follows. Randomness or chance introduces an element of chaos into order, *unless they are seen as defining the world as a manifold of random events, perhaps underlying a perceived order*, and reveals that the world confronts us with this element, *even if the ultimate constitution of nature is assumed to be classically causal*. The two underlined qualifications describe two opposing forms of realism or ontology, the second of which is much more common. But the first one has been sometimes entertained as well, beginning with the pre-Socratics, and may indeed be called 'the Jocasta ontology', since it was assumed and was dramatically expressed by Jocasta, Oedipus's mother and wife, in Sophocles' *Oedipus the King*: 'Fear? What should a man fear? It's all chance, chance rules our lives. Not a man on earth can see a day ahead, groping through the dark. Better to live at random, best we can' [34, p. 146, ll. 1068–1072]. No appeal to probability is possible under these conditions: next to nothing can be estimated with any degree of belief. This view is proved illusory in the play, because the lives of the characters are ultimately ruled by fate and, thus, by classical causal ontology, or at least so the play is commonly interpreted, because there are dissenting readings. In any event, probability introduces an element of order into situations defined by the role of randomness in them and enables us to handle such situations better. Probability or statistics is, thus, about the interplay of randomness and order. This aspect of probability takes on a special, even unique, significance in quantum physics because of the presence of statistically ordered correlations (not found in classical physics) between certain data, such as those of the EPR-type experiments. There correlations are correctly predicted by the formalism of quantum mechanics and rules, such as Born's rule or various forms of the projection postulates, which are added to, rather than are inherent in, the formalism. This, again, does not mean that it is the only formalism that can predict these correlations. Bohmian mechanics, the predictions of which coincide with those of standard quantum mechanics, predicts them, but at the expense of non-locality.

I conclude this section by defining the concept locality. By locality, I refer to the prohibition on the instantaneous propagation or transmission of physical influences between spatially separated physical systems. Non-local theories, such as Bohmian mechanics (in all of its versions), allow for and even entail such connections, even though it may not be possible to actually trace or enact these connections by human means. Non-locality in this sense is usually, albeit not always, seen as undesirable. Standard quantum mechanics appears to avoid it, although the question of the locality of quantum mechanics or quantum phenomena is a matter of great subtlety and much controversy, especially in the wake of the Bell and Kochen–Specker theorems and related findings, which also led to alternative conceptions of locality, sometimes linked to other concepts, such as 'separability' or 'no signalling'. These alternative conceptions require a separate discussion,

[9]See [30,31] and references therein. On the Bayesian philosophy of probability, in two different versions of it, see [32,33].

which cannot be undertaken here and is not necessary for my argument. It is important to note, however, that these theorems and findings are sometimes interpreted as implying that quantum mechanics or even nature itself is non-local. This view was already suggested by Einstein, in view of the EPR-type (thought) experiments, and challenged by Bohr, in spite or even because of these experiments. It is true that quantum phenomena allow, specifically in EPR-type situations, for *predictions*, for example and in particular by means of quantum mechanics, at an, in principle, arbitrary distance on the basis of measurements performed at a given location. However, as Bohr argued in his reply to the EPR, it is possible to interpret quantum mechanics so as to preserve locality, and thus to avoid what Einstein famously called in this connection a 'spooky action at a distance' [35]. In this view, one could speak of spooky *predictions* at a distance, *spooky* because there is no physical explanation of how these predictions and quantum correlations come about [36, pp. 16, 271].

Locality is commonly associated with relativity. However, the concept of locality, including as implied by Einstein's arguments (insofar as they are based on prohibiting any instantaneous action at a distance) concerning the EPR-type experiments, is independent of other keys concepts with which it is linked in relativity, such as the Lorentz invariance of special relativity. Indeed, the latter is violated in general relativity, where it is only locally or infinitesimally valid, but where locality is strictly maintained. Technically, too, relativity prohibits a transmission of physical influences not only instantaneously but also faster than the (finite) speed of light in a vacuum, a requirement that could, in principle, be violated, while still allowing for locality, as different speed limits on physical actions may emerge. Einstein's 'spooky action at a distance', too, refers to an instantaneous action, in conflict with relativity as well, but not defined by it. The standard quantum mechanics is not relativistic, but it is, or may be interpreted as or required to be local. The locality of quantum mechanics also implies that it is in compliance with the requirements of relativity, but it may be a deeper fact, as the Bell and the Kochen–Specker theorems and related findings may indicate as well. This need not mean that locality is quantum in origin, but only that it might be a deeper aspect of nature than those, such as the Lorentz invariance, defining relativity theory, which, however, is local as well.

3. From Newtonian mechanics to analytical mechanics to quantum mechanics

As must be apparent from the preceding discussion, the difference between realist and non-realist models of reality is fundamental and irreducible, whatever features these models possess otherwise, in particular, whether realist models are visualizable or not. However, there are also important affinities between *non-visualizable realist* models, found in classical physics and relativity theory, and *non-realist* models, found in quantum mechanics and the higher level quantum theories.

As discussed above and as was often noted by Bohr and Heisenberg, both physics and even mathematics originate as forms of *refinement* of our general phenomenal perception, thinking and language, originally developed from the capacities of our neurological constitution to accommodate and to represent in a realist manner the experiences of daily life. Some aspects of this perception, thinking and language are proto-geometric (spatial) and proto-physical (related to motion). One may call such intuitive phenomenal representations daily-life realism, which may function as naive realism insofar as it is direct, rather than mediated by concepts or representations, mathematical or other, defined by concepts. Rigorously speaking, no realism, wherever it is found, including in daily life, is ever entirely free from mediation of concepts. This mediation may not be always apparent and is sometimes repressed, but it is never entirely absent. Certainly, the mediation by concepts in their conventional sense (of generalization from particulars) is unavoidable, at least when language is involved, for otherwise language would not be possible. Nevertheless, our more immediate phenomenal visualization of bodies and motions is an important part of daily-life realism. Visualizable mathematical models, which emerged in physics with Descartes, Galileo and Newton, may be seen as mathematically idealizing daily-life phenomenal representations of processes, such as and in particular motion.

However, beginning with ancient astronomy, physics has also used suitable technical and mathematical means in order, in Bohr's words, 'to represent relations for which ordinary verbal expression is imprecise or cumbersome' [12, vol. 2, p. 68]. In mathematics and physics alike, this mathematical refinement can over time lead and, by the nineteenth century, had led to a nearly complete break between mathematical thinking or intuition and general phenomenal thinking or intuition. In reflecting on the idea of continuum (a concept still under debate in the philosophy of mathematics and mathematical logic) in 1918, H. Weyl said: 'the conceptual world of mathematics is so foreign to what the intuitive continuum presents to us that the demand for coincidence between the two must be dismissed as absurd' [37, p. 108]. Weyl added, however: 'Nevertheless, those abstract schemata supplied us by mathematics must underlie the exact sciences of domains of objects in which continua play a role' [37, p. 108]. Weyl's point concerning the conceptual world of mathematics vis-à-vis our general phenomenal intuition obviously exceeds the question of the continuum and pertains to most of twentieth-century mathematics, certainly that used in relativity and quantum theory. Exact sciences and specifically physics could, however, use mathematics differently and even take advantage of this divorce between mathematics and our general phenomenal intuition, to the point, reached by quantum theory, of abandoning realism altogether, rather than only a visualizable one. Weyl was undoubtedly aware that, as concerns its relationships with our general phenomenal intuition, the situation had also changed in physics, with the help of mathematics, for nearly a century by the time of his comment. It is true that this divorce from our phenomenal intuition reached its radical form in quantum mechanics a few years after Weyl's statement. But it was nearly as radical already in relativity, to which Weyl made major mathematical contributions. As noted earlier, the famous relativistic law of addition of velocities (defined by the Lorentz transformation) in special relativity, $s = (v + u)/(1 + (vu/c)^2)$, for collinear motion ($c$ is the speed of light in a vacuum), runs contrary to any intuitive representation of motion that we can have. This divorce becomes even more pronounced in quantum mechanics, beginning with Heisenberg, who was also able to take advantage of this divorce in a new way and to a hitherto unprecedented degree, thus moving physics beyond realism.

Although it would be difficult to speak of a single event of origin here (Descartes' analytic geometry certainly contributed), this divorce could, I would argue, be especially traced to Lagrange's and Hamilton's analytical mechanics, as connoted by the term 'analytical' (echoing analytic geometry as well). As I argue here, one could only meaningfully speak of the proximity between physics and our phenomenal intuition in classical mechanics, the physics of visualizable models, before Lagrange, although, again, one cannot do so entirely without qualifications there either, or even in daily life. Still a geometrically idealized visualization is meaningful in classical mechanics and could be and has been, and still is, used for physical descriptions and correlative predictions, as both Galileo's *Dialogues* [38] and Newton's *Principia* [20] demonstrate. *Principia* recasts Newton's argument, initially developed via calculus, in terms of Euclidean geometry, which was, according to Newton, the true language of mathematical demonstration, also, in the sense of visualization. In addition to not being a rigorous mathematical theory at the time, calculus would have introduced major complications into the geometrical visualization of mechanics, a complication that Newton might have preferred to avoid, unlike Lagrange or Hamilton. On the other hand, as explained above, the visualizable model(s) of classical mechanics are linked and even defined by a form of visualization found in and leading to calculus, although calculus had moved far beyond any visualization by the end of the eighteenth century, in accord with Weyl's comment cited above, made in part with calculus in mind.

Both of these aspects of calculus, visualizable and non-visualiable, are found in Lagrange's and Hamilton's analytical mechanics. On the one hand, unlike those of Galileo and Newton, Lagrange's and then Hamilton's formalisms were no longer mathematically linked to a phenomenal or specifically geometrical representation of the physical systems considered. On the other hand, both formalisms were developed under the assumption that a visualizable model, considered in the preceding section, would apply to the behaviour of each individual constituent of any physical system considered. This representation, however, while essential as an underlying conception, played next to no role in establishing or using either formalism, which enabled excellent predictions, the same as in Newton's physics, but made more generally and efficiently, also in the situation where it would be next to impossible to use Newton's equations. Both formalisms were, thus, *conceptually* realist, but no longer visualizable. (All realism is, again, conceptual.) A break with the *geometrical* representation of even individual physical processes and ultimately with all realism, algebraic one included, came with quantum mechanics. However, based in the mathematics of Hilbert spaces and the form of calculus they contain, quantum mechanics was 'algebraic', compelling Einstein to speak of Heisenberg's 'purely algebraic method of description of nature', a method that offered no representation, for Einstein ultimately based in geometry, of the ultimate constitution of nature [39, p. 378]. Hilbert spaces over complex numbers, used in quantum mechanics, are not visualizable. Indeed, they are 'spaces' only metaphorically, by virtue of having certain algebraic properties possessed by phenomenally visualizable spaces, such as the three-dimensional Euclidean space (the only space we can phenomenally conceive of) or two-dimensional surfaces, flat or curved.

Nevertheless, Lagrange's and then Hamilton's recasting of classical mechanics as analytical mechanics was a major shift, which, as just indicated, vastly extended the range of application of mathematical modelling in physics and beyond, for example, eventually to economics. The Lagrangian or Hamiltonian phase-space representation of classical mechanical systems, even individual ones, was, in each case, a multidimensional (symplectic) manifold, as we understand it now, rather than a Euclidean space. As such, this representation no longer corresponded to our phenomenal intuition and was primarily a mechanism for correct predictions, in this case (ideally) exact, of the behaviour of classical physical systems. This departure from visualization was reinforced by the use of generalized coordinates and velocities, which helped to extend models based in analytical mechanics beyond physics.

Both Lagrange and Hamilton, and most subsequent practitioners of analytical mechanics, again, assumed an underlying mathematical architecture of physical reality and the possibility of a pictorial representation of the behaviour of individual classical objects. The formalism itself, however, not only made one question the degree to which our models 'match' reality, but also whether they match it all. In other words, the question becomes: to what degree does the Lagrangian or Hamiltonian mathematical model of classical physics, or the models of classical statistical physics, or those of relativity (generally presumed to be a realist theory) correspond, even as idealizations, to the architecture of nature, as opposed to merely enabling correct predictions concerning the phenomena considered? As, again, Kant realized, these questions could be posed even concerning the classical mechanics of individual systems. In such cases, however, the descriptive and specifically visualizable idealizations used, at least in relatively simple cases, are more in accord with our phenomenal experience, which is no longer the case in considering more complex classical systems, treated by Lagrangian or Hamiltonian models, or in relativity, and then, more radically, beyond realism, in quantum theory.

Nevertheless, the quantum-mechanical and then quantum-field-theoretical formalisms are extensions of the Lagrangian and Hamiltonian formalism, both mathematically and physically. The shared mathematical features are obvious. The shared physical features are subtler and more complex. Although quantum theory went much further than analytical mechanics on this road, ultimately to the point of renouncing realism altogether, the Lagrangian and Hamiltonian formalism, while realist, had features that could be transferred into quantum theory, or 'quantized', some literally, others more conceptually. Thus, both formalisms essentially track how the potential (V) and kinetic (T) energy vary in the course of a given physical process, while the sum of both, represented by the Hamiltonian, H, remains constant. This tracking is equally possible in quantum mechanics and then quantum field theory, now without, because of the uncertainty relations, a possibility of tracking or even defining the trajectories of quantum objects themselves. By the same token, both the Hamiltonian ($H = T + V$) and the Lagrangian ($L = T - V$) are defined, in the operator formalism, for a given quantum system in the absence of causality and realism, even conceptual, let alone visualizable, realism. As was realized even before quantum mechanics was discovered, the laws of energy and momentum conservation do not depend on causality even in classical physics. As Bohr noted, 'the great fruitfulness of [the laws of the conservation of energy and momentum], already in classical physics, depends upon the fact that one may extensively apply them without following the course of the phenomena [the course of the processes resulting in the appearance of phenomena?] in space and time' [12, vol. 1, p. 94]. The proof of energy conservation, again, shared with classical physics, was a major early achievement of matrix mechanics and a crucial testimony to its viability.

Also, the integral of the Lagrangian is the action, S, which, while of course continuous in classical mechanics, is measured in the same units as Planck's constant, h, fittingly (indeed for this very reason) called by Planck 'the quantum of action', now also made discrete, rather than continuous as in classical physics. Hamilton's principle of minimal action is fully retained in quantum theory as well and was used by Schrödinger in his derivation of his equation. He also used, in this derivation, via F. Klein, the mechanical-optical analogy in a reverse parallel with Hamilton's optical-mechanical analogy used by Hamilton [36, pp. 161–171].

Classical statistical physics and the thermodynamics of Bolzman, Maxwell, Planck, and Gibbs, Maxwell's electromagnetic theory, and relativity, special and general, were all influenced by and followed Lagrange's and Hamilton's methods. Maxwell's electrodynamics in particular was a theory or the Lagrangian–Hamiltonian type. It, too, presupposed causality and realism at the ultimate level, now understood, following both Faraday and Maxwell, in terms of fields, a concept that is not easily, if, rigorously, at all, visualizable, at least not in full measure. Attempts at developing visualizations of Maxwell's theory have not been lacking, of course, beginning famously with Maxwell's own. I would contend, however, that they have only been and worked as heuristic devices, *diagrams*, never quite able to capture the complexity of the concept, as Maxwell was arguably the first to realize as well [40]. Maxwell's electrodynamics was then extended, with radical implications, to special relativity, some of the concepts of which, such as, again, that of the relativistic addition of velocities, are even further removed from pictorial visualization or any human intuition. General relativity amplified these complexities and introduced new ones in the case of gravity, which can also be described, as it was by Hilbert, by a new form of Lagrangian formalism. These complexities, again, compel one to ask how realist the theory ultimately is, as opposed to being a mathematical model for predicting, in this case ideally exactly, the behaviour of relativistic systems [17].

It is not coincidental either that classical statistical physics was primarily an atomic theory, dealing with the ultimate microscopic constitution of matter, and that Planck's discovery of his radiation laws, defined, again, by the *discrete* quantum of action, which ushered in quantum theory, occurred at the intersection of electromagnetism and thermodynamics, and thus classical statistical physics. His law itself was, as I noted, proven, by Einstein, to be incompatible with classical statistical physics. Atomic physics was progressively driving its mathematical models away from our general phenomenal intuition and visualization, thus joining another trajectory in this same direction extending from the Lagrangian and Hamiltonian versions of mechanics to electromagnetism to relativity. While, physically, culminating in the first trajectory, mathematically, quantum mechanics and then quantum field theory were extensions of Lagrangian and Hamiltonian methods.

Admittedly, these methods were born from the spirit of classical physics and, thus, causality and realism, both retained in Lagrangian and Hamiltonian versions of classical mechanics, or in classical electrodynamics and then relativity. Nevertheless, the break from visualization and, more generally, phenomenal intuition entered physics with Lagrange, whose approach was, it is true, not motivated by new physics, but, which is my main point here, by a different type of mathematical modelling, no longer linked to our phenomenal intuition and its visualization. It was a product of experimenting with mathematics. Once atomic theory enters the picture, this link may no longer be possible, although relativity, again, breaks this link nearly altogether as well. According to Heisenberg:

> It is not surprising that our language [or conceptuality] should be incapable of describing processes occurring within atoms, for ... it was invented to describe the experiences of daily life, and these consist only of processes involving exceedingly large numbers of atoms.

> Furthermore, it is very difficult to modify our language so that it will be able to describe these atomic processes, for words can only describe things of which we can form mental pictures, and this ability, too, is a result of daily experience. Fortunately, mathematics is not subject to this limitation, and it has been possible to invent a mathematical scheme—[quantum mechanics]—which seems entirely adequate for the treatment of atomic processes.
>
> [1, p. 11]

Not everyone at the time, beginning with Einstein, or, often following Einstein, since then, saw this scheme as 'entirely adequate' for the treatment of atomic processes or, technically, quantum phenomena. This was because this scheme, in Heisenberg's hand and in interpretations, such as that of Bohr, did not describe or otherwise represent atomic (quantum) processes at all, which was its main inadequacy for Einstein and others, Schrödinger, again, among them.

Mathematics' freedom from this limitation is found in classical physics as well. The difference is that Heisenberg's scheme, if interpreted in the spirit of Copenhagen, not only bypasses a visualizable model of individual quantum objects and processes, but also renounces non-visualizable realist models. It remains crucial and was part of the background of Heisenberg's statement, that the model of classical statistical physics, underlain by the visualizable model of classical mechanics, cannot apply in quantum theory, if Planck's law is correct. Accordingly, that Heisenberg *found* a mathematical scheme that could predict the data in question was as fortunate as that mathematics is free of this limitation, for this is also the case in classical physics and in relativity, beginning, again, at least with Lagrange's and Hamilton's analytical mechanics. It is true that matrix algebra was introduced in mathematics before Heisenberg, who was famously unaware of it and had to reinvent it, although unbounded infinite matrices that he used were not previously studied in mathematics and were given a proper mathematical treatment by M. Born and P. Jordan after Heisenberg's discovery. But, even if Heisenberg had been familiar with it, his scheme would still needed to be invented as a mathematical model dealing with quantum phenomena, which, as Heisenberg realized, was possible to do in terms of probabilistic or statistical predictions in the absence of any representation of quantum objects and their behaviour.

4. From Heisenberg to Bohr: mathematics and measurement in quantum mechanics

Bohr immediately grasped the revolutionary nature of Heisenberg's discovery and approach, as is clear from his comments made in 1925, following Born and Jordan's work, which developed Heisenberg's initial insights, but before Schrödinger's introduction of wave mechanics:

> In this theory [matrix mechanics] the attempt is made to transcribe every use of mechanical concepts in a way suited to the nature of the quantum theory, and such that in every stage of the computation only directly observable quantities enter. In contrast to ordinary mechanics, the new mechanics does not deal with a space-time description of the motion of atomic particles. It operates with manifolds of quantities which replace the harmonic oscillating components of the motion and symbolize the possibilities of transitions between stationary states .... These quantities satisfy certain relations which take the place of the mechanical equations of motion and the quantization rules [of the old quantum theory].
>
> [12, vol. 1, p. 48]; also [12, vol. 1, pp. 70–71]

The correspondence principle, as previously used, more ad hoc, by Bohr and others, stated that the predictions of quantum theory must coincide with those of classical mechanics at the classical limit, say, in the region of large quantum numbers, when electrons are far away from atomic nuclei. (Their behaviour is still quantum there.) Heisenberg gave the principle a more rigorous mathematical form, which may be called 'the mathematical correspondence principle'. Thus understood, the principle required that the equations of quantum mechanics convert into those of classical mechanics at the classical limit. Heisenberg, again, realized that he did not

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need a space–time description or any realist representation of quantum objects and processes to be able to predict the outcomes of the experiments he considered. His decision to forgo this representation was a daring step, because throughout the history of classical mechanics from Galileo on, and then electromagnetism and relativity, such a representation appeared to be necessary for making successful predictions concerning the behaviour of the systems considered. As explained above, while designed only for making statistical predictions concerning large systems of great mechanical complexity, classical statistical physics, too, was only valid under this assumption, which, again, came into conflict with the data (statistical or otherwise) obtained in experiments with quantum theory, a conflict inherent in Planck's law.

Heisenberg made yet another, equally unexpected, move that proved to be crucial and in fact enabled him to give the correspondence principle its mathematical form. Quantum theorists before Heisenberg had tried to change classical equations while keeping the same physical variables, in part by thinking in representational terms of classical mechanics and in order to be able to continue to do so. They saw equations and not variables as the problem. Heisenberg reversed this thinking, in view of the mathematical correspondence principle. He regarded classical equations, now considered purely formally, and the lack of descriptive or representational capacity of the resulting theory as solutions. These were classical variables and it was their attempted realist and causal use that were problems. Heisenberg retained the classical equations of motion, while replacing classical variables, which are real differential functions, with new variables, which were infinite unbounded matrices with complex elements. These elements no longer related to the motion of quantum objects but only to the probabilities of predictions of the outcomes of quantum experiments. Formally, or symbolically, they corresponded to classical amplitudes of motion in the Fourier representation. In Heisenberg's scheme, however, and in quantum theory since then, these 'amplitudes' are no longer amplitudes of any physical motions, which makes the name 'amplitude' an artificial, *symbolically* used, term. These amplitudes were linked to the probabilities of transitions between stationary states: they are what we now call probability amplitudes. The corresponding probabilities themselves were derived by a form of Born's rule for this limited case (Born's rule is more general). One takes square moduli of the eigenvalues of these matrices, which gives one real numbers, corresponding, once suitably normalized, to the probabilities of observed events. (Technically, one needs the probability density functions, but this does not affect the essential point in question.) The standard rule for adding the probabilities of alternative outcomes is changed to adding the corresponding amplitudes first and then deriving the final probability by squaring the modulus of the sum. This character of quantum probability is sometimes referred to as 'non-additive probability', which plays an important role in quantum-like modelling beyond physics, discussed in §6.

Heisenberg's scheme essentially amounted to Hilbert-space formalism, which was introduced by von Neumann shortly thereafter and gave a more rigorous mathematical foundation to quantum mechanics, by then developed more properly by Heisenberg himself, Born and Jordan, and, differently (in terms of q-numbers), by Dirac. Heisenberg's matrices were in effect Hermitian operators in an infinite-dimensional Hilbert space over complex numbers. As noted above, Heisenberg was unaware that such objects had already been in use in mathematics for decades. Heisenberg's matrices were (re)invented by him from new physical postulates via a mathematical construction that led to an actual algebra of these matrices, which Heisenberg had to define, beginning with the multiplication rule; and as I said, even if he was aware of their existence, they would still have to be invented as a mathematical model of quantum mechanics. Besides, as I also noted, unbounded infinite matrices (necessary in dealing, as in Heisenberg, with continuous variables, such as position and momentum) were not previously studied in mathematics and were given a proper mathematical treatment by Born and Jordan. Matrix multiplication is in general non-commutative. It proved to be, arguably, the most essential feature of all mathematical models used in quantum theory and quantum-like models beyond it. There are further technical details: for example, as unbounded self-adjoint operators (defined on infinite-dimensional Hilbert spaces), these matrices do not form an algebra with respect to the composition as a non-commutative product, although some of them satisfy the canonical commutation relation. These details are, however, secondary here.

While Heisenberg, in developing his scheme, was merely not pursuing a description or representation of quantum objects and processes, in Bohr's ultimate interpretation of quantum phenomena and quantum mechanics, any analysis and representation or even conception of quantum objects and processes was, more radically, '*in principle* excluded'. As he said, responding to Einstein, who refused to entertain such a conception: 'in quantum mechanics [in this interpretation] we are not dealing with an arbitrary renunciation of a more detailed analysis of atomic phenomena, but with a recognition that such an analysis is *in principle* excluded [beyond a certain point]' [12, vol. 2, p. 62]. Bohr's interpretation stemmed from Heisenberg's approach, but it was developed, becoming progressively more radical, under the impact of Bohr's decade-long debate with Einstein and reached its ultimate version in the late 1930s. This interpretation was grounded in his argument concerning the irreducible role of measuring instruments in the constitution of quantum phenomena, which led him to his concept of phenomenon, defined by this role, and he based his ultimate interpretation on this concept. According to Bohr:

> I advocated the application of the word phenomenon exclusively to refer to the observations obtained under specified circumstances, including an account of the whole experimental arrangement. In such terminology, the observational problem is free of any special intricacy since, in actual experiments, all observations are expressed by unambiguous statements referring, for instance, to the registration of the point at which an electron arrives at a photographic plate. Moreover, speaking in such a way is just suited to emphasize that the appropriate physical interpretation of the symbolic quantum-mechanical formalism amounts only to predictions, of determinate or statistical character, pertaining to individual phenomena appearing under conditions defined by classical physical concepts [describing the observable parts of measuring instruments].
>
> [12, vol. 2, p. 64]

Bohr's appeal to classical concepts is often misunderstood. First, although indispensable, classical concepts are never sufficient for a proper account of quantum phenomena. Second, the *interaction* between quantum objects and measuring instruments is quantum (otherwise no measurement could take place), and as such, it is not amenable to a description in terms of classical concepts. Measuring instruments have both the classical strata, which we observe, and the quantum strata, through which they interact with quantum objects, and which, accordingly, we cannot observe. This interaction is 'irreversibly amplified' to the classical level of observation [12, vol. 2, p. 73]. The preparation of measuring instruments is classical and thus controllable, while the behaviour of quantum objects is not.

Bohr sees the quantum-mechanical formalism as 'symbolic' for the following reasons, indicated earlier. The formalism uses the same mathematical symbols and equations as those used in classical mechanics. However, because of the different nature of variables to which these symbols refer, this formalism does not, in his interpretation, describe the behaviour of quantum objects. It only predicts, probabilistically or statistically, what could be observed in measuring instruments under the impact of quantum objects. In present terms, in this formalism, quantum mechanics offers a non-realist mathematical model that relates to the ultimate constitution (reality) of nature. Physical quantities, through which this relation is established, are obtained in quantum measurements as *effects* of the interactions between these instruments and quantum objects. These quantities are no longer assumed to correspond to the properties of quantum objects, *even any single such property*, rather than only certain joint properties, which would be prohibited by the uncertainty relations. (I shall comment on the underlined qualification momentarily.) Bohr's concept of phenomenon '*in principle* excludes' any description of quantum objects and their behaviour [12, vol. 2, p. 62]. This exclusion, as explained earlier, also implies the irreducible difference (ultimately transcending Kant's epistemology grounded in a similar type of difference), between quantum phenomena and quantum objects, defined by means of another, very special concept. It is not possible to offer, even in principle, any representation or possible even to form a conception of their nature and behaviour.

A qualification might be in order concerning the nature of the ultimate elemental constituents of nature, elementary particles and their 'properties', some of which might appear to be ascertainable as independent of or invariant relative to measurement, and thus may appear to be permanent. The concept of elementary particle is still, more than a century since its introduction, far from being firmly established, and it would require an analysis that is beyond my scope, in part because it would involve quantum field theory. Brief remarks offered here are aimed at suggesting how this concept might be approached from the present perspective. First of all, that elementary particles cannot be distinguished from each other within each of their types (electrons, photons, quarks and so forth), while these types themselves are rigorously distinguished from each other, is consistent with the notion that the nature of elementary particles and their behaviour are beyond representation or even conception. Both features are consistently defined by the corresponding sets of effects manifested in measuring instruments, effects from which we infer the existences of elementary particles, in the first place. An elementary particle of a given type, say, an electron, is defined by a given set (potentially very large, but specific to each particle type) of possible phenomena or events observable, as effects, in measuring instruments associated with it. This set is the same for all electrons, although the correlation between any such phenomenon and any given electron can never be strictly (rather than statistically) assured, which already reflects the fact that electrons are indistinguishable from one another. To cite H. Weyl, 'the possibility that one of the identical twins Mike and Ike is in the quantum state E1 and the other in the quantum state E2 does not include two differentiable cases which are permuted on permuting Mike and Ike; it is impossible for either of these individuals to retain his identity so that one of them will always be able to say 'I'm Mike' and the other 'I'm Ike'. Even in principle one cannot demand an alibi of an electron!' [41, p. 241].[10] One cannot be certain that one encounters the same electron, say, in an experiment designed to detect it after it was emitted by a source, even in quantum-mechanical regimes, although the probability that it would be a different electron is low. The probability is much higher in high-energy regimes, governed by quantum electrodynamics, where we are just as likely to detect a positron, an electron–positron pair, or a photon. Two electrons could be distinguished by a changeable property associated with them (but, in the present view, manifested only in measuring instruments), such as their positions in space or time, momentums, energy or the directions of spins. Such variables may be subject to the uncertainty relations and complementarity. It is possible, for example, to simultaneously locate two different electrons in separate regions in space. It is not possible, however, to distinguish electrons on the basis of their (rest) mass, charge or the value of their spin. These quantities are not subject to the uncertainty relations or complementarity, although the *direction* of spin is. However, which is my main point at the moment, in the present interpretation, such properties, too, could only be associated with electrons or other particles by the corresponding set of phenomena observed in measuring instruments but not rigorously attributed to these objects, any more than any other properties.

It is crucial that the concept of phenomenon is defined by 'observations [already] *obtained* under specified circumstances' and hence refers only to already *registered* phenomena, rather than to what could be predicted. For one thing, such predictions could only be probabilistic, and hence it is never certain that the corresponding events will occur. By the same token, the concept of phenomenon entails a rigorous specification of each arrangement, determined by the type of measurement we want to make. This specification reflects the irreducibly individual, indeed unique and unrepeatable, character of each phenomenon or event. The constitutive role of these conditions can never be eliminated in considering quantum measurement or predictions. Thus, if seen independently of the *context* of its appearance, each mark on the screen in the double-slit experiment would be perceived as the same entity. Such a mark would appear the same regardless of the difference in the physical conditions and, hence, the outcome or rather

[10]The statement is cited in [42]. See also [43] for a comprehensive realist treatment of the subject from a realist perspective.

the outcomes, 'interference' or 'no interference', of the experiment are defined collectively by the two alternative set-ups of the experiment. The first is that with both slits open and no counters, which would allow us to know through which slit each quantum object passed, and the second is that when such a knowledge is possible in one way or another, even in principle rather than only actually. In Bohr's interpretation, however, each mark is part of a different individual phenomenon depending on these conditions, which are mutually exclusive. While a given single event does not allow one to establish in which setting it had occurred, the statistical distribution of the traces on the screen will always be different in these set-ups.

That we have a free choice as concerns what kind of experiment we want to perform is in accordance with the very idea of experiment, a choice that, as Bohr notes, also defines classical physics or science in general [35, p. 699]. Contrary to the case of classical physics, however, implementing our decision concerning what we want to do will allow us to make only certain types of predictions and will exclude the possibility of certain other, in Bohr's language, *complementary*, types of predictions. We actively shape what will happen, define the course of reality. In this sense, complementarity may be seen, as it was by Bohr, as a generalization of causality, in the absence of both classical causality and, in the first place, realism. The concept of complementarity is defined by Bohr as:

(a) a mutual exclusivity of certain phenomena, entities or conceptions; and yet
(b) the possibility of applying each one of them separately at any given point; and
(c) the necessity of using all of them at different moments for a comprehensive account of the totality of phenomena that we must consider.

Parts (b) and (c) of this definition are just as important as part (a), and to miss them, as is often done, is to miss much of the import of Bohr's concept. Arguably, the most significant among complementary situations are those of position and momentum measurements, and of space–time coordination and the application of momentum or energy conservation laws, which are correlative to Heisenberg's uncertainty relations and establish Bohr's interpretation of them. In Bohr's interpretation, one only deals with complementary phenomena manifested in observable parts of measuring instruments, under the impact of quantum objects, and the classical properties of these observable parts, to which and only to which, rather than quantum objects, the uncertainty relations now apply. Non-realism could in fact be inferred from complementarity, because the latter prevents us from ascertaining the complete composition of the 'whole from parts', to the degree that the latter concept applies, because the complementary parts never add up to a whole in the way they do in classical physics or relativity.

I would like to close this section by noting a more general long-term impact of Heisenberg's approach on twentieth-century and then twenty-first-century physics. This thinking not only introduced a new type of mathematical model in physics, which is an epochal achievement already, but, by the same token, also established a new way of *doing* theoretical physics. In fact, it also redefined the practice of experimental physics in quantum regimes, at least as this practice appears from the non-realist perspective adopted here.

The practice of experimental physics no longer consists, as in classical physics, in tracking the independent behaviour of the systems considered. Instead, it consists in *unavoidably* creating configurations, by now almost unbelievable in their complexity (think of the Large Hadron Collider, where the Higgs boson was recently discovered), of experimental technology that reflects the fact that what happens is *unavoidably* defined by what experiments we perform, how we interact with quantum objects, rather than only by their independent behaviour. These configurations, of course, relate to quantum objects. But they embody the effects of the interactions between quantum objects and measuring instruments, through which effects quantum objects are defined and, when possible, distinguished from one another, while always remaining beyond the reach of quantum theory and even thought. I emphasize 'unavoidably' above because, while the behaviour of classical objects is sometimes affected by experimental technology, in general, as Bohr stressed, we can observe classical physical objects without appreciably affecting them, or else we can compensate for this interference so that we can describe classical objects independently. This does not appear possible in quantum experiments.

The practice of theoretical physics no longer consists in offering an idealized mathematical description of quantum objects and their behaviour. Instead, it consists in developing mathematical machinery that is able to predict, in general (again, in accordance with what obtains in experiments) probabilistically, the effects in question, manifested as the outcomes of quantum events and of correlations between some of these events. The situation takes a more radical form in quantum field theory and experimental physics in the corresponding (high-energy) quantum regimes than in quantum mechanics and the corresponding (low-energy) experimental regimes. While this subject is beyond my scope, I might note that, while the discovery of the Higgs boson, predicted by quantum field theory, may be the most spectacular example of this situation in theoretical and experimental physics, many previous discoveries (such as those of the electroweak bosons or the top quark) equally exemplify it.[11] In the case of the Higgs boson, this technology was helped and was even made possible by computer technology, which has been for a while an essential part of mathematical modelling in physics and beyond.

5. Quantum models and quantum probability

The epistemology outlined in the preceding section entails a type of understanding of probability or statistics in quantum physics that is different from that found in classical physics, an understanding, again, implicit in Heisenberg's introduction of quantum mechanics but expressly articulated for the first time by Bohr. This understanding, however, allows for different interpretations, two of which I shall now consider. The first is a statistical interpretation, which I shall call 'the statistical Copenhagen interpretation'. The second is the Bayesian interpretation or a set of such interpretations, still only those defined by the absence of realism and, as a consequence, of the applicability of the idea of causality in individual quantum processes. It is possible to have a realist Bayesian interpretation of quantum phenomena and quantum mechanics. I am, however, only concerned with non-realist interpretations of both. As explained above, these interpretations fundamentally depart from causal understanding, including that found in classical theories that consider those classical physical systems and processes in considering which the recourse to probability or statistics becomes necessary, as in classical statistical physics, or chaos and complexity theories. In these cases, the behaviour of the physical systems considered is assumed to be causal. However, the mechanical complexity of these systems makes the recourse to probability or statistics unavoidable in predicting their behaviour. By contrast, in non-realist interpretations, there is no causality assignable even to primitive or elemental individual quantum processes, processes involving the ultimate elemental constituents of nature (elementary particles). This makes the outcome of each experiment different even if they are repeated in one and the same experimental arrangement, and unlike in classical physics, this difference cannot be reduced beyond certain limits (defined by Planck's constant, h), for example, by improving the precision of our measuring instruments. It is possible to speak of 'one and the same experimental arrangement', because, unlike the outcomes of experiments, we can control the measuring instruments involved, given that the observable parts of these instruments can be described classically. On the other hand, the state of each quantum object under investigation in each repeated experiment (say, at the time when an electron or photon is emitted from the source used) will not, in general, be identical. Under these conditions, the probabilistic character of such predictions is unavoidable even in considering primitive individual quantum processes and events, that is, the corresponding effects manifested in the measuring instruments involved. According to Bohr:

> The unrestricted applicability of the causal mode of description to physical phenomena has hardly been seriously questioned until Planck's discovery of the quantum of action,

[11] I have considered it in [44].

which disclosed a novel feature of atomicity in the laws of nature supplementing in such unsuspected manner the old doctrine of the limited divisibility of matter. Before this discovery statistical methods were of course extensively used in atomic theory but merely as a practical means of dealing with the complicated mechanical problems met with in the attempt at tracing the ordinary properties of matter back to the behaviour of assemblies of immense numbers of atoms. It is true that the very formulation of the laws of thermodynamics involves an essential renunciation of the complete mechanical description of such assemblies and thereby exhibits a certain formal resemblance with typical problems of quantum theory. So far there was, however, no question of any limitation in the possibility of carrying out in principle such a complete description; on the contrary, the ordinary ideas of mechanics and thermodynamics were found to have a large field of application also proper to atomic phenomena, and above all to offer an entirely sufficient basis for the experiments leading to the isolation of the electron and the measurement of its charge and mass. Due to the essentially statistical character of the thermodynamical problems which led to the discovery of the quantum of action [*h*], it was also not to begin with realized, that the insufficiency of the laws of classical mechanics and electrodynamics in dealing with atomic problems, disclosed by this discovery, implies a shortcoming of the causality ideal itself.

[45, pp. 94–95]

In present terms, these classical cases are handled by non-visualizable realist models (again, underlain by visualizable models of classical mechanics applied to the elemental constituents of such systems), in contrast to non-realist models of quantum mechanics, such as that of Bohr, indeed introduced in its ultimate non-realist version in the same article, and others in the spirit of Copenhagen. As Bohr notes elsewhere: '[I]t is most important to realize that the recourse to probability laws under such circumstances is essentially different in aim from the familiar application of statistical considerations as practical means of accounting for the properties of mechanical systems of great structural complexity. In fact, in quantum physics, we are presented not with intricacies of this kind, but with the inability of the classical frame of concepts to comprise the peculiar feature of indivisibility, or 'individuality', characterizing the elementary processes' [12, vol. 2, p. 34].

In each case, the wave function provides, in Schrödinger's way of putting it, 'expectation catalogues' concerning quantum experiments [26, p. 158]. Any such catalogue is reset with each new measurement, which renders previous history of measurement on the same object irrelevant as concerns our predictions from this point on [36, pp. 73–76]. The meaning of these catalogues is subject to a further interpretation, even if one follows the spirit of Copenhagen. Such interpretations may, for example, be either probabilistic, Bayesian or statistical, in the sense that I shall now explain. According to Pauli:

As this indeterminacy [that reflected in the uncertainty relations] is an unavoidable element of every initial state of a system [a quantum object] that is at all possible according to the new [quantum-mechanical] law, the development of the system even can never be determined as was the case in classical mechanics. The theory predicts only the *statistics* of the results of an experiment, when it is repeated under a given condition. Like the ultimate fact without any cause, the *individual* outcome of a measurement is, however, in general not comprehended by laws. This must in general be the case, if quantum or wave mechanics is interpreted as a rational generalization of classical physics [mechanics], which take into account the finiteness of the quantum of action [Planck's constant, *h*]. The probabilities occurring in the new laws have then to be considered to be primary, which means not deducible from deterministic [causal] laws. As an example of these primary probabilities I mention here the fact that the time at which an individual atom will undergo a certain reaction stays undetermined even under conditions where the rate of occurrence of this reaction for a large collection of atoms is practically certain.

[46, p. 32]

Pauli speaks of determinism rather than causality, but this does not affect my argument because, as explained earlier, in the case of classical mechanics both notions coincide. Most important is

Pauli's main claim: 'The theory predicts only the *statistics* of the results of an experiment, when it is repeated under a give condition. Like the ultimate fact without any cause, the *individual* outcome of a measurement is, however, in general not comprehended by laws'. Given that Pauli does not specify otherwise, this appears to include probabilistic or, in this view, *statistical* laws of quantum mechanics. Indeed, he corroborates this reading elsewhere in the context of complementarity: 'In the general case of the quantum-mechanical state of a material particle, neither the position nor the momentum is predictable with certainty; in consequence the state can be described only by *statistical* statements about the distribution of values of the results of possible measurements of position or momentum of this state. Formally these statements are embraced symbolically in a wave function, consisting of a real and an imaginary part' [46, p. 99] (Pauli's emphasis). This point has a major implication, which leads to the statistical (Copenhagen) interpretation I am about to offer.

In this interpretation, primitive individual quantum processes and events are not only beyond representation, either by means of the quantum-mechanical formalism or in general, or possibly even conception, but are also beyond probabilistic predictions. (Exact predictions are excluded automatically.) In other words, the outcome of an individual quantum experiment, a future individual quantum event, cannot be assigned probability: it is random. Only the statistics of multiply repeated (identically prepared) experiments can be predicted, which gives the corresponding meaning to the expectation catalogues provided by the formalism, say, by the wave function. In some experiments, it is possible to assign a probability to individual quantum events for all practical purposes, but not in full rigour.

It might be argued that one could speak more rigorously (rather than only provisionally) of the probability of a (future) individual quantum event, similarly to the way one does in the case of a coin toss. It might also appear or be argued that, just as in the case of a coin toss, one could, *by using the quantum-mechanical formalism*, *say*, *the wave function*, assign probabilities for individual events, defined, along Bayesian lines, in terms of 'a degree of belief', rather than in terms of frequencies or statistics. Both types of claims would appear to be implied by the standard Bayesian view (e.g. [29,33]). It is not my aim to refute these claims, or argue against Bayesian approaches to quantum theory.[12] I would like to suggest, however, that these claims pose difficulties given the data observed in quantum experiments thus far and that in any event the statistical view of quantum phenomena and quantum mechanics is consistent with the character of these data.

In addition, the statistical Copenhagen interpretation may be seen as more radical epistemologically, insofar as not even the probabilistic 'knowledge' concerning, or even the application of the concept of probability to, individual quantum events is possible. First of all, at least in most quantum situations, any verification of such individual estimates would still involve multiple events, and these individual estimates will rely on that data, reflecting these repeatable statistics. A Bayesian might contest this point in the case of quantum phenomena similarly to the way it might be and has been contested by Bayesians in the case of our estimates of the probability of a coin toss [29, pp. 317–320]. More crucial is that an individual quantum event may be beyond (rigorously) assigning it a probability at all.

Consider the double-slit experiment. It is true that in the case of the interference set-up (both slits open and there are no counters allowing us to detect through which slit each particle in question passes), we observe the interference pattern, defined by the zones of permitted or 'forbidden' impact. This pattern is strictly statistical in nature, manifesting itself only in a very large number of trials, around 70 000. The statistical nature of this pattern also reflects the fact that there is no zone of forbidden impact for any individual trial. Any given trial can leave its mark anywhere on the screen, as is clearly shown by the famous data of A. Tonomura's single electron build-up experiments [49]. Some trials (admittedly a statistically small number) can produce no impact at all. In view of these considerations, it would be difficult to speak *rigorously*

[12]The so-called quantum Bayesianism or QBism exemplifies the complexities of Bayesian thinking in quantum theory and beyond [47,48]. While it adopts a non-realist view of quantum objects and processes, and *in this respect* is in accord with Bohr's interpretation and the statistical Copenhagen interpretation, it is different from both in other respects [48]. To properly address QBism and to fairly assess its claims would require an extensive analysis that cannot be undertaken here.

of assigning a probability to an individual trial. It is true that in certain experiments, such as the Stern–Gerlach experiment, it may be possible to speak, nearly with certainty, of an object having a 50% probability of taking one path or another in the corresponding arrangement. First, however, if so, this is true only in some experiments. Second, even in these cases, there will be some trials, however small in number, in which no outcome will be registered, and unlike in classical cases, it is not a matter of outside interferences, but is something that is inherent to quantum experiments.

These circumstances would, thus, complicate, even if not exclude, the application, in quantum physics, of the Bayesian view, insofar as it refers to estimates, bets on, the outcome of single events on the basis of the information one has.[13] A given quantum phenomenon or event would, in Bohr's definition, be seen in relation to the conditions defining this multiplicity, such as one or the other set-ups of the double-slit experiment, that gives rise to the interference pattern or that in which this pattern will not appear. In any event, the corresponding interpretation, such as the one suggested here appears to be consistent with the experimental data in question.

I am, it goes without not saying, not contending that the Bayesian approach does not work in general in physics and beyond: there are many situations where it does, for example, when we need to estimate the probabilities of certain human events, as in betting on the outcome of a basketball game. Indeed, even in quantum experiments, one could make any predictions one likes or must in view of one's prior experience, which need not be based on quantum mechanics or physics in general. However, the question is that of the effectiveness of our predictions in physics, of who will do better in physics, in predicting, betting on, the outcomes of experiments. Consider, again, the double-slit experiment. One will 'win' with quantum mechanics in hand against those who do not know it or whose theory is not as good, but one would win, or would not consistently lose, only in many trials, which fact implicitly contains statistics. This is far from insignificant and is a powerful reason to use quantum mechanics for predicting the outcomes of quantum experiments. My point is only that quantum mechanics, in general, offers one no help in betting on the outcome of a single experiment, because one cannot rigorously predict what happens, although quantum mechanics can do so in practice in some experiments.

6. Quantum-like models beyond physics

Most currently used models in biology, cognitive psychology and economics are either classical-like statistical models or (probabilistic or statistical) models based in chaos and complexity theories. In many cases, these models work reasonably well. However, there also appear to be phenomena in considering which they do not appear to be effective or work at all: their predictions fail, similarly to the way such models failed in the case of quantum phenomena, which failure gave rise to quantum theory, beginning with Planck's law.

Beginning with A. Tversky and D. Kahneman's work in the 1970–1980s (e.g. [50]), certain experiments in cognitive psychology encountered situations predicting which appeared to require (non-additive) probabilistic rules akin to those encountered in quantum mechanics. This, naturally, suggested using quantum-like models in mathematical cognitive psychology and, then, economics, in part in view of the fact that certain economic behaviour, too, involves psychological factors of the type analysed by Tversky and Kahneman. Previously, these factors had not been considered in mathematical economic modelling or elsewhere in economics. Kahneman was eventually awarded a Nobel Prize in economics (Tversky died a few years earlier and thus was not eligible). At this stage of history, if one needs a model able to predict quantum-like probabilities or, more generally, address experimental situations analogous to that of quantum mechanics outside physics, one need not invent an appropriate new formalism. One already has mathematical models that could be used—those of quantum mechanics or of quantum field theory, although models based in the latter are uncommon beyond physics. It might be noted that most of the quantum-like models currently used in cognitive psychology are finite-dimensional,

[13] Of course, in the Bayesian scheme of things, there may be events to which one cannot assign probabilities, but not in general.

corresponding to those used for discrete variables in quantum mechanics, rather than those dealing with continuous variables, which require infinite-dimensional Hilbert space.

Thus far, the use of quantum-like models in these cases is motivated by the non-additive character of probability, coupled to quantum-like non-commutativity, specifically as reflected in the statistical dependence of the outcomes of certain sequences of events, such as responses to certain questions, on their order (e.g. [51,52]). This parallels the dependence of the outcome of a sequence of complementary measurements, such as those of the position and the momentum of a quantum object, on the order in which they are performed in two sets of experiments prepared identically before either measurement is performed. (Such quantum measurements, too, could be interpreted in terms of questions posed.) There are also cases in which one encounters correlations akin to those of the EPR correlations, discussed in the surveys just cited as well. One does not, however, have in these models either rigorous quantum discreteness or individuality of the processes leading to the events considered that lead to quantum theory in physics itself, or a particular motivation to use non-realist and, hence, non-causal interpretations of quantum-like models used in these situations.

Of course, in the experiments just mentioned, one still deals with discrete events, which is indeed in the nature of all probabilistic or even exact predictions. Classical-like models in physics would assume that such discrete events are continuously and causally connected, possibly, as in classical statistical mechanics or chaos and complexity theory, by a very complex and in practice intractable process, which circumstances require the recourse to probabilistic or statistical estimates. As discussed earlier, however, these models presuppose that the behaviour of individual constituents of such systems is described by causal representational models of classical mechanics, which would predict their behaviour ideally exactly were it practically possible. Models of this type, especially local ones, appear to be difficult, if not impossible, to develop or even to assume in the case of quantum phenomena, either by an interpretation of the standard quantum-mechanical formalism or by way of alternative theories, which lead to non-realist interpretations of quantum mechanics in the spirit of Copenhagen. Neither situation finds real parallels in quantum-like modelling beyond physics, because the corresponding *theories* there do not deal with individual *processes* (of whatever kind) either of the type considered in classical mechanics or of the type considered in quantum mechanics. There, moreover, this individuality manifests itself in the role of Planck's constant h, which has no equivalent in cognitive psychology or economics, or anywhere outside physics, at least thus far. In fact, the applications of quantum-like models in these fields are rarely concerned with how the data in question comes about, although it is generally assumed that there are processes, psychological (or neurological) and social, that are responsible for these data. This allows one to deal, in a quantum-mechanical-like fashion and by using a quantum-like formalism, with estimating probabilities of discrete events. This is similar to the situation that led Heisenberg to his discovery of quantum mechanics. Once again, however, there is a crucial difference, because the quantum experiments in question deal with rigorously defined ultimate individual constituents of matter, elementary particles, to which the model of classical mechanics or possibly any local classical-like model could not apply.

Moreover, whether such quantum-like models are required in these fields beyond physics, in the way they appear to be in physics, or whether classical-like models suffice remains an open question. It may, it is true, be difficult to assume that classical-like models would give quantum-like probabilities (or correlatively, be mathematically non-commutative). On the other hand, the underlying dynamics of the cognitive or psychological processes leading to the situations in question may be causal, and thus, unlike the behaviour of quantum objects (at least, again, in the present interpretation), open to an actual psychological or sociological analysis. This is because one might expect that there are psychological and social reasons for this quantum-like decision-making, and in a way the task of psychology (or psychologically oriented economics) is to understand and explain these reasons, although the research using quantum-like models generally renounces this task. It is possible, however, that an analysis of such reasons is precluded or even that one cannot assume that one could assign reasons to this decision-making.

Accordingly, to what degree the use of quantum-like models in these fields could be brought in a closer accord with the features and concepts of quantum phenomena and quantum theory considered in this article remains an open question. Thus, would discreteness of a more rigorously quantum type also find its place there (again, as against a more generally discrete nature of events about which one makes predictions)? Correlatively, would there be something like a rigorously complementary situation (or, correlatively, the uncertainty relations) corresponding to the non-commutative nature of the quantum-mechanical formalism and probabilistically non-additive character of such quantum-like models? That is, would, for example, one question of a given non-commuting sequence of two questions in the psychological experiments mentioned above automatically exclude the other? And if so, would non-realism in the spirit of Copenhagen also be possible or necessary? Bohr thought so in the case of biology and, more tentatively, in psychology. In the case of biology, he argued as follows (emphasis added):

> The existence of life must be considered as an elementary fact that cannot be explained, but must be taken as a starting point in biology, in a similar way as the quantum of action, which appears as *an irrational element* from the point of view of the classical mechanical physics, taken together with the existence of elementary particles, forms the foundation of atomic physics. The asserted impossibility of a physical or chemical explanation of the function peculiar to life would in this sense be analogous to the insufficiency of the mechanical analysis for the understanding of the stability of atoms. [53, p. 458]

In other words, the ultimate nature of biological processes may be unrepresentable or even inconceivable, and hence unavailable to realist mathematical models. Bohr's invocation of 'an irrational element' is of some interest, and I shall return to it below [4, pp. 17–18]. It is important, and was one of Bohr's points, that this approach may be adopted even if the nature of biological processes is not *physically* quantum in the sense of life being a physically quantum effect. In this case, it would be automatically unrepresentable or inconceivable in Bohr's interpretation. At stake here, however, or in quantum-like modelling in psychology or neuroscience are *parallel*, rather than connected, situations that may require the same mathematical models.[14] This view also implies a potentially emergent nature of biological processes, as not reducible to physical ones, classical or quantum, but rather as unrepresentable or inconceivable in their own right. At least, this nature would not be 'mechanically' described as a theory and, correspondingly, not represented by its mathematical models, just as quantum objects and processes were not described (rather than placed beyond any description or conception) by Heisenberg in his discovery of quantum mechanics. There is, again, a crucial difference, because in quantum mechanics, such causal connections are difficult and, in interpretations in the spirit of Copenhagen, impossible to assume, in the latter case by virtue of the fact that a realist model of how quantum phenomena come about is, to return to Bohr's locution, '*in principle* excluded' [12, vol. 2, p. 63]. As indicated above, the situation on this score is far less clear in biology, psychology or economics, where such connections would be molecular in the first case, neurological or psychological in the second, and psychological, sociological, political or economic in the third. Indeed, while still entertaining in principle a similar argument, Bohr, too, was even more tentative regarding psychology than he was regarding biology [2, pp. 158–166].

A recent article by E. Haven and A. Khrennikov provides an intriguing example of a possible role of quantum-like discreteness in market economics [55].[15] A phenomenon of quantum tunnelling occurs when a particle tunnels through a barrier that it would not be able to surmount if it behaved classically and is thus a quantum phenomenon *par excellence*. The process itself cannot be observed. (In the present view, no quantum process could be.) We only deal with observable effects of this process, specifically with the fact that there is a non-zero probability that

[14] Arguments for such connections have been advanced, prominently by R. Penrose, from his first major work on the subject [54] on. While Penrose's theory primarily concerns the neurobiology of the brain and consciousness, Penrose also links his argument to the possibility that life itself is a quantum effect.

[15] See [56] for a more general discussion of quantum-like modelling in economics and other social sciences.

a particle can be found beyond the barrier, which is to say that the corresponding measurement will register an impact of this particle on the measuring instrument beyond the barrier. Thus, we deal with two discrete phenomena, relatable only by probabilistic predictions concerning the second on the basis of the first, in the absence, at least in the present interpretation, of any representation or even conception of the process that physically connects these two phenomena.

Haven and Khrennikov argue that the cases of arbitrage in finance could be mathematically modelled by a quantum-like model analogously to the way quantum tunnelling is modelled in quantum mechanics. 'Arbitrage' is the practice of taking advantage of a price difference between two or more markets, striking a combination of matching deals that capitalize on the imbalance, the profit being the difference between the market prices. An arbitrage is a transaction that involves no negative cash flow at any probabilistic or temporal state and a positive cash flow in at least one state; in simple terms, it is the possibility of a risk-free profit at zero cost. Ideally, an arbitrage is risk-free. In practice, there are always risks in arbitrage, sometimes minor (such as fluctuation of prices decreasing profit margins) and sometimes major (such as devaluation of a currency or derivative). In most ideal theoretical models, an arbitrage involves taking advantage of differences in price of a single asset or identical cash flows.

If, however, arbitrage can be modelled analogously to quantum tunnelling, one might, given the essential quantum character of quantum tunnelling, expect features more rigorously analogous to those of quantum phenomena more readily than in other situations beyond physics where quantum-like models are used. Haven and Khrennikov themselves are primarily concerned with the use of the mathematical formalism of the quantum type in predicting the probabilities involved, rather than with discreteness or the epistemology of the situation. They do offer, however, some considerations concerning discreteness. They say:

> We believe that the equivalent of quantum discreteness in this paper corresponds to the idea that each act of arbitrage is a discrete event corresponding to the detection of a quantum system after it passed … the barrier. In reality arbitrage opportunities do not occur on a continuous time scale. They appear at discrete time spots and often experience very short lives. We would like to argue that it is the tunnelling effect which is closely associated to the occurrence of arbitrage. This argument is linked to Proposition 5 below (which gives a necessary mathematical condition of the existence of arbitrage). We also mentioned the wave function in the discussion above, and quantum discreteness is narrowly linked with quantum probabilities. Those are very difficult and important issues which need addressing. We hope to make this part of future research. In light of this discussion, we believe it is appropriate to call the tunnelling effect we try to emulate here, as being 'quantum-mechanical-like'.
>
> [55, p. 4095].

This remark allows, at least in principle, for a, correlatively, quantum-like discrete and non-realist (hence, again, non-causal) interpretation of arbitrage. There remains, however, some space for debate whether arbitrage requires such an interpretation. Haven and Khrennikov do not appear to take a non-realist view concerning the situation, or leave the question open.[16]

One might, again, argue that whether we can or cannot mathematically *represent* the economic processes themselves involved in arbitrage, rather than only *predict* certain probabilities involved, is secondary. A mere suspension of realism suffices: as in the case of cognitive experiments mentioned above, one is merely not concerned with this representation or, a conceptual-mathematical realist mapping, any more than Heisenberg was with representing the behaviour of quantum objects in deriving his matrix mechanics. One is only concerned with predicting the probabilities of certain future events of arbitrage given the preceding situation and the data involved, analogously dealing even with the individual events of arbitrage, although in this case the underlying dynamics, even if indescribable, are multiple, rather than individual, as it is in quantum mechanics, which is, again, a crucial difference. The question here is, again, twofold.

[16]In [27], mentioned in footnote 7, Khrennikov argued for a classical-like model at the ultimate level of the constitution of nature, an argument that informs this analysis of arbitrage.

First, it is that of the necessity or relative effectiveness of such models as against more classical-like models previously used, to some degree in contrast to the situation that obtains in quantum mechanics, where the difficulties of having realist and causal models, at least, again, local ones, remains formidable and are acknowledged even by the proponents of realism. Second, and more significantly, the question as to what degree such models pertain to the fundamental nature of economic processes and phenomena, however these are configured in terms of forces shaping them (economic, psychological, social, political or other). In sum, the question is what is the ultimate character of the processes considered in economics or other theories beyond physics, and whether these processes can be represented or even (ultimately) conceived of.

This is, obviously, an enormous question, hardly answered in physics either, and I would not presume to be able to answer it. I would, however, venture a surmise that our brains may work, at least sometimes, in accordance with all three features—discreteness, non-realism and quantum (non-additive) probability—and thus, without relying on hidden causality but only on the quantum-like play of probabilities and correlations.[17] One can speak here of a Bayesian brain. But, as against rational Bayesian agents, associated with the term Bayesian in cognitive psychology or economics (against which association quantum-like models in cognitive psychology and economics are advanced as well), this kind of Bayesian agents need not always function rationally. In this sense, Bohr's invocation of 'an irrational element' in non-realist models, an irrational element rationally accounted for by the theories using these models, could apply quite directly. It is used by him more metaphorically in the case of quantum mechanics or biology. It may, accordingly, be possible to have a quantum-like model and a theory that would allow one to predict the outcomes of decision-making and other psychological events as governed, on Bayesian lines, by the information involved in making them, although we cannot, in principle, have access to all of this information, even the most crucial piece.[18] Nor can those who make these decisions have this access, given the role of the unconscious in making them, and that this unconscious is not causal in its functioning, in the way, for example, S. Freud saw it, although his thinking was more complex on this point as well, and even, on occasion, quantum-like [57]. He even thought that the future developments of biology, 'truly a land of unlimited possibilities', could 'blow away the whole of our artificial structure of [psychoanalytic] hypotheses' [58, p. 54].

7. Conclusion: the spirit of experimentation

Our best Bayesian bet, in any of the fields here considered, might well be on *experimenting* with mathematical models, whatever their nature, in confronting actual experiments, and thus treating mathematics itself as experimental technology of thought. This may well be the greatest lesson physics, from Galileo and Newton on, and especially quantum physics, taught us. Taking advantage of and bringing together both main meanings of the word 'experiment' (as a test and as an attempt at an innovative creation), one might argue that, while not without help from nature, the practice of quantum physics is the first practice of physics or science that is both, and jointly and interactively, *fundamentally* experimental and *fundamentally* mathematical. That need not mean that this practice has no history; quite the contrary, beginning, again, with Galileo's and Newton's work. Both were also experimentalists, both in the conventional sense and in this sense under discussion at the moment. Lagrange's and Hamilton's analytical mechanics, Maxwell electrodynamics, the thermodynamics of Maxwell, Boltzmann, Gibbs and Planck, and Einstein's relativity are all major events of this history of experimenting with mathematical models in physics. Nevertheless, this experimentation acquires a new form with quantum mechanics and then extends to higher level quantum theories, and the new understanding of the nature of quantum phenomena and thus experimental quantum physics (in the usual sense).

The practice of quantum physics is fundamentally experimental because we no longer track, as we do in classical physics or relativity, the independent behaviour of the systems considered,

[17]This, again, need not imply the neurological processes of the brain are physically quantum. See footnote 14.

[18]This suggestion does not depend on the applicability of a Bayesian, as against a frequentist, statistical approach, to quantum mechanics itself.

track what happens in any event, by however ingenious experiments. Instead, we *define* what will happen in the experiments we perform, by how we *experiment* with nature by means of our experimental technology, even though and because we can only predict what will happen probabilistically or statistically.[19] Thus, in the double-slit experiment, the two alternative set-ups of the experiment, whether we, respectively, can (by way of using one experimental device or another) or cannot know, even in principle, through which slit each particle, say, an electron, passes, we obtain two different outcomes of the statistical distributions of the traces on the screen (with which each particle collides). Or, thus also giving a rigorous physical and philosophical meaning to the uncertainty relations, we can set up our apparatus so as to measure and correspondingly predict, again, probabilistically or statistically, either the position or the momentum of a given quantum object, but never both together. Either case requires a separate experiment, incompatible with the other, rather than merely representing an arbitrary selection of either type of measurement within the same physical situation, by tracking either one of its aspects or the other in the way we do in classical mechanics. There, this is possible because we can, at least in principle, measure and assign simultaneously both quantities within the same experimental arrangement. In quantum physics we cannot.

By the same token, quantum physics is also fundamentally mathematical, because the mathematical formalism of the theory is not in the service of tracking, by way of a mathematical representation, what would have happened anyhow, which would shape the formalism accordingly, but is in the service of predictions required by our experiments. The mathematical formalism of quantum theory is able to predict correctly the experimental data in question without offering any description or representation at all of the physical processes responsible for these data. As a result, as I have argued here, beginning with Heisenberg's discovery of quantum mechanics, quantum theory established radically new relationships between mathematics and physics, vis-à-vis those, essentially realist in character, found in classical physics and relativity. Quantum mechanics is strictly a theory of predictions concerning the outcomes of possible future experiments on the basis of previously performed experiments.

It also follows that, as quantum physics experiments with nature by using mathematics and technology, quantum theory experiments with mathematics itself, more so than in classical physics. This is because we invent, in the way Heisenberg did, effective mathematical schemes of whatever kind they might be and however far they may be from our general phenomenal intuition, rather than proceed by refining mathematically our phenomenal representations of nature, which process limits us in classical physics or even (albeit to a lesser a degree) in relativity. The experimentally mathematical character of theoretical physics, introduced by Heisenberg in the process of his discovery of quantum mechanics in 1925, has defined quantum theory ever since and continues to do so. But it need not be limited to physics. The emergence of quantum-like models beyond physics is an affirmation of this spirit of experimenting with mathematics in understanding the world, natural and human.

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References

1. Heisenberg W. 1930 *The physical principles of the quantum theory*. (Transl. by K Eckhart, FC Hoyt). New York, NY: Dover.
2. Plotnitsky A. 2012 *Niels Bohr and complementarity: an introduction*. Berlin, Germany: Springer.

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3. Plotnitsky A, Khrennikov A. 2015 Reality without realism: on the ontological and epistemological architecture of quantum mechanics. *Found. Phys.* **95**, 1269–1300. (doi:10.1007/s10701-015-9942-1)
4. Plotnitsky A. 2014 Are quantum-mechanical-like models possible, or necessary, outside quantum physics? *Phys. Scr.* **T163**, 014011. (doi:10.1088/0031-8949/2014/T163/014011)
5. Cartwright N. 1983 *How the laws of physics lie*. Oxford, UK: Oxford University Press.
6. Hacking I. 1983 *Representing and intervening: introductory topics in the philosophy of natural science*. Cambridge, UK: Cambridge University Press.
7. Cartwright N. 1999 *The dappled world: a study of the boundaries of science*. Cambridge, UK: Cambridge University Press.
8. Galison P. 1997 *Image and logic: a material culture of microphysics*. Chicago, IL: University of Chicago Press.
9. Latour B. 1999 *Pandora's hope: essays on the reality of science studies*. Cambridge, MA: Harvard University Press.
10. Hacking I. 2000 *The social construction of what?* Cambridge, MA: Harvard University Press.
11. Deleuze G, Guattari F. 1994 *What is philosophy?* (Transl. by H Tomlinson, G Burchell). New York, NY: University Press.
12. Bohr N. 1987 *The philosophical writings of Niels Bohr*, vol. 3. Woodbridge, CT: Ox Bow Press.
13. Von Neumann J. 1932 *Mathematical foundations of quantum mechanics*. (Transl. by RT Beyer). Princeton, NJ: Princeton University Press.
14. Heisenberg W. 1925 Quantum-theoretical re-interpretation of kinematical and mechanical relations. In *Sources of quantum mechanics* (ed. BL Van der Waerden), pp. 261–277. New York, NY: Dover.
15. Mehra J, Rechenberg H. 2001 *The historical development of quantum theory*, vol. 6. Berlin, Germany: Springer.
16. Einstein A. 1949 *Autobiographical notes*. (Transl. by AP Schillp). La Salle, IL: Open Court.
17. Butterfield J, Isham CJ. 2001 Spacetime and the philosophical challenge of quantum gravity. In *Physics meets philosophy at the Planck scale: contemporary theories of quantum gravity* (eds C Callender, N Huggett), pp. 33–89. Cambridge, UK: Cambridge University Press.
18. Miller AI. 1978 Visualization lost and regained: the genesis of quantum theory in the period 1913–1927. In *On aesthetics in science* (ed. J Wechsler), pp. 72–102. Cambridge, MA: MIT Press.
19. Gieser S. 2005 *The innermost kernel: depth psychology and quantum physics. Wolfgang Pauli's dialogue with C. G. Jung*. Berlin, Germany: Springer.
20. Newton SI. 1999 (1687) *The principia: mathematical principles of natural philosophy*. (Transl. by AB Cohen, A Whitman). Berkeley, CA: University of California Press.
21. Kuhlman M. 2014 Quantum field theory. In *The Stanford encyclopedia of philosophy* (ed. E Zalta). See http://plato.stanford.edu/entries/quantum-field-theory/.
22. Kant I. 1997 *Critique of pure reason*. (Transl. by P Guyer, AW Wood). Cambridge, UK: Cambridge University Press.
23. Einstein A, Podolsky B, Rosen N. 1935 Can quantum-mechanical description of physical reality be considered complete? In *Quantum theory and measurement* (eds JA Wheeler, WH Zurek), pp. 138–141. Princeton, NJ: Princeton University Press.
24. Wittgenstein L. 1924 *Tractatus logico-philosophicus*. (Transl. by CK Ogden). London, UK: Routledge.
25. Plotnitsky A. 2011 'Dark materials to create more worlds': on causality in classical physics, quantum physics, and nanophysics. *J. Comput. Theor. Nanosci.* **8**, 983–997. (doi:10.1166/jctn.2011.1778)
26. Schrödinger E. 1935 The present situation in quantum mechanics. In *Quantum theory and measurement* (eds JA Wheeler, WH Zurek), pp. 152–167. Princeton, NJ: Princeton University Press.
27. Khrennikov A. 2012 Quantum probabilities and violation of CHSH-inequality from classical random signals and threshold type detection scheme. *Prog. Theor. Phys.* **128**, 31–58. (doi:10.1143/PTP.128.31)
28. Einstein A. 1906 Theorie der Lichterzeugung und Lichtabsorption. *Ann. Phys.* **20**, 199–206. (doi:10.1002/andp.19063250613)
29. Jaynes ET. 2003 *Probability theory: the logic of science*. Cambridge, UK: Cambridge University Press.

30. Khrennikov A. 2009 *Interpretations of probability*. Berlin, Germany: de Gruyter.
31. Háyek A. 2014 Interpretation of probability. In *Stanford encyclopedia of philosophy* (ed. EN Zalta). Winter 2014 edn. See http://plato.stanford.edu/archives/win2012/entries/probability-interpret/.
32. Jaynes ET. 2003 *Probability theory: the logic of science*. Cambridge, UK: Cambridge University Press.
33. De Finetti B. 2008 *Philosophical lectures on probability*. (Transl. by H Hosni). Berlin, Germany: Springer.
34. Sophocles. 1984 *The three Theban plays: Antigone, Oedipus the King, and Oedipus at Colonus*. (Transl. by R Fagles). New York, NY: Penguin.
35. Bohr N. 1935 Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **48**, 696–702. (doi:10.1103/PhysRev.48.696)
36. Plotnitsky A. 2009 *Epistemology and probability: Bohr, Heisenberg, Schrödinger and the nature of quantum-theoretical thinking*. Berlin, Germany: Springer.
37. Weyl H. 1918 *The continuum: a critical examination of the foundation of analysis*. (Transl. by S Pollard, T Bole). New York, NY: Dover.
38. Galilei G. 2010 *Dialogues concerning two new sciences*. (Transl. by H Crew, A De Salvio). New York, NY: Cosimo Classics.
39. Einstein A. 1936 Physics and reality. *J. Franklin Inst.* **221**, 349–382. (doi:10.1016/S0016-0032(36)91047-5)
40. Maxwell JC. 1965 *The scientific papers of James Clerk Maxwell* (ed. WD Niven). New York, NY: Dover.
41. Weyl H. 1931 *The theory of groups and quantum mechanics*. (Transl. by HP Robertson). London, UK: Methuen.
42. French S. 2011 Identity and individuality in quantum theory. In *The Stanford encyclopedia of philosophy* (ed. E Zalta). See http://plato.stanford.edu/archives/sum2011/entries/qt-idind/.
43. Jaeger G. 2013 *Quantum objects: non-local correlation, causality and objective indefiniteness in the quantum world*. Berlin, Germany: Springer.
44. Plotnitsky A. 2015 A matter of principle: the principles of quantum theory, Dirac's equation, and quantum information. *Found. Phys.* **95**, 1222–1268. (doi:10.1007/s10701-015-9928-z)
45. Bohr N. 1938 The causality problem in atomic physics. In *The philosophical writings of Niels Bohr: causality and complementarity, supplementary papers*, vol. 4 (eds J Faye, HJ Folse), pp. 94–121. Woodbridge, CT: Ox Bow Press.
46. Pauli W. 1994 *Writings on physics and philosophy*. Berlin, Germany: Springer.
47. Fuchs C, Mermin ND, Schack R. 2013 An introduction to QBism with an application to the locality of quantum mechanics. (http://arxiv.org/abs/1131.5253)
48. Mermin ND. 2014 Why QBism is not the Copenhagen interpretation and what John Bell might have thought of it. (http://arxiv.org/abs/1409.2454v1)
49. Tonomura A, Endo J, Matsuda T, Kawasaki T, Ezawa H. 1989 Demonstration of single-electron buildup of an interference pattern. *Am. J. Phys.* **57**, 117–120. (doi:10.1119/1.16104)
50. Tversky A, Kahneman D. 1974 Judgment under uncertainty: heuristics and biases. *Science* **185**, 1124–1131. (doi:10.1126/science.185.4157.1124)
51. Dzhafarov EN, Kujala JV. 2012 Selectivity in probabilistic causality: where psychology runs into quantum physics. *J. Math. Psychol.* **56**, 54–63. (doi:10.1016/j.jmp.2011.12.003)
52. Pothos E, Busemeyer JR. 2013 Can quantum probability provide a new direction for cognitive modeling? *Behav. Brain Sci.* **36**, 255–274. (doi:10.1017/S0140525X12001525)
53. Bohr N. 1933 Light and life. *Nature* **131**, 457–459. (doi:10.1038/131457a0)
54. Penrose R. 1995 *The emperor's new mind*. Oxford, UK: Oxford University Press.
55. Haven E, Khrennikov A. 2013 Quantum-like tunnelling and levels of arbitrage. *Int. J. Theor. Phys.* **52**, 4083–4099. (doi:10.1007/s10773-013-1722-0)
56. Haven E, Khrennikov A. 2013 *Quantum social science*. Cambridge, UK: Cambridge University Press.
57. Plotnitsky A. 2004 The unthinkable: nonclassical theory, the unconscious mind, and the quantum brain. In *Brain and being: at the boundary between science, philosophy, language, and arts* (eds G Globus, K Pribram, G Vitiello), pp. 29–46. Amsterdam, The Netherlands: Benjamin.
58. Freud S. 1961 *Beyond the pleasure principle*. (Transl. by J Strachey). New York, NY: W. W. Norton.