Combined Games

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Abstract

What happens when you play Chess and Tic-Tac-Toe at the same time? What about Mancala, Othello and Checkers? Playing multiple games together as a Combined Game introduces an entirely new, infinite set of games to analyze and play. We use traditional Game Theory tools to examine combined pairs of small, well-solved games, yielding information on the optimal strategies for playing each combined game. The results of our analysis reveal patterns for two Normal Games or any amount of Meiseré Games being played as one Combined Game and suggest that such patterns exist when combining other classes of games.

1 An Introduction to Combined Games

What exactly is a *Combined Game*? The definition is derived and generalized from one's intuition: a game where you combine at least two other games. For example, playing two games of Tic-Tac-Toe at once is a combined game; playing Nim and Checkers is another combined game. Such combinations create completely new games to explore. In this section, we formalize and define the concept of a *Combined Game* and introduce properties combinations of games exhibit.

1.1 Introduction

Let us begin by introducing and defining the notion of a *Combined Game*. In our examination, we greatly restrict the space of *Combined Games* to a smaller, more manageable space.

1.1.1 What are *Combined Games*?

A *Combined Game* G is defined as the following:

Definition 1. *A* Combined Game G *is a finite set of* n *sub-games, where* $n \geq 2$*. We assume every* $g \in G$ *has the following properties:*

- *Two Player,*
- *Sequential Play,*
- *Perfect Information,*
- Well-Solved/Well-Studied.

We define a *Combined Game* in this way to simplify our analysis; in general, a combined game may have any number of players, consist of simultaneous or sequential play, and may or may not be well-solved. In particular, we study *well-solved* combinations to observe whether or not winning strategies and patterns from the sub-games carry over to the combined games space. We also only want to study games with perfect information; i.e., no games of chance, so that our analysis is again much simpler.

1.1.2 Rules of *Combined Games*

In addition to the above properties, a Combined Game G has the following rules:

1. On a given turn, the current player may make one legal move on any $g \in G$.
2. The current player in a Combined Game G must skip their turn if and only if for every $g \in G$, that player has no legal moves on g.
3. The current player may forfeit one $g \in G$ if and only if that player forfeits all $h \in G$, where h is a sub-game of G.
4. A *Combined Game* must be sequential; i.e., it must respect the sequential play of the sub-games.
5. The winner of a *Combined Game* G is the player who has won the most sub-games $g \in G$.

It is important to mention that if skipping turns was allotted, then a combined game may never end, as both players refuse to move on a bad board position, as we shall note in our analysis of general combinations of games. Additionally, we take this rule from non-combined games to keep the notion of a sequential game consistent.

Similarly, we forbid the forfeiting of only one game to prevent a player from forfeiting a bad-position game so they force the other player to move on a bad position game, resulting in every combined game of even size to end in a draw. We shall prove this result in Section 2. The prohibition of forfeiting a single game makes the optimal strategy more interesting and the games more fun for players.

1.1.3 Example of Playing a Combined Game

Here is an example of a Combined Game of Hexapawn and Tic-Tac-Toe. For those unfamiliar with Hexapawn, the game and its rules are explained in Section 1.3.1.

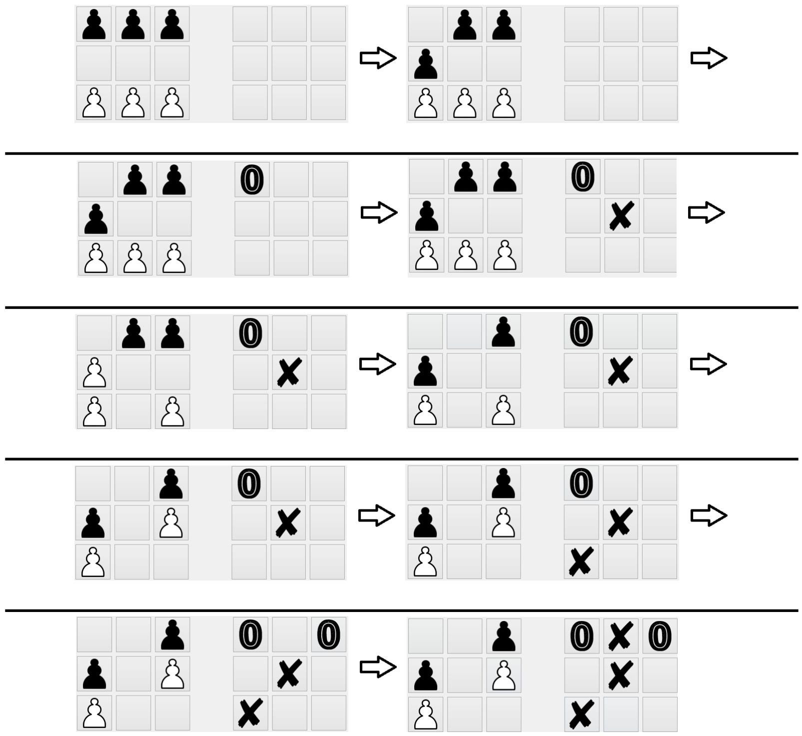

Figure 1: *In this example, each arrow indicates the resulting board from a player moving. The first board is the opening board, then the next board is the result of Player 1 moving on Hexapawn, and the next board is the result of Player 2 moving on Tic-Tac-Toe, and so on and so forth.*

1.2 Game Theory Notations and Definitions

This section outlines definitions and notations from Game Theory that are necessary for our analysis of Combined Games.

The definitions are intuitive: any player knows these definitions, but maybe not the formal names for them. When playing a game, each player has his or her own strategy which they believe leads them to victory. This is known formally as a *Winning Strategy.*

Winning Strategy A *winning strategy* of a game is a strategy s such that if a player follows s, that player wins the game.

Now, a player's strategy need not be a winning strategy; many strategies exist that do not lead a player to victory. However, most players wish to win the games they play, so the players attempt to only make moves that help that player win the game and their opponent lose. This is known formally as *Optimal Play.*

Optimal Play A player P plays a game *optimally* if P only makes moves to attempt to win the game given the available information. [2, pg. 3]

In later Sections when we start proving statements about the optimal strategy for a *Combined Game*, all of our proofs operate on the over-arching assumption that all players play optimally. Thus, understanding what it means to have Optimal Play is crucial to understanding how the winning strategies work.

N-Position A board is said to be an *N-Position* if the Next Player to make a move has the winning strategy.

Another way of describing an N-Position is to say that the current player has the winning strategy.

P-Position A board is said to be a *P-Position* if the Previous Player to move has the winning strategy.

Another way of describing a P-Position is to say that the player who just made a move has the winning strategy. Note that in the analysis of *Combined Games*, we refer to a player based on whether or not they are the current player, rather than Player 1 and Player 2. Our analysis uses Player 1 and Player 2 as labels for players, but always say which player is the current player and which is the previous player.

Game Tree A *Game Tree* T is a directed graph where each node is a game board and each edge is a move to another game board. Additionally, all leaves are P-Positions, all nodes preceding a P-Position are N-Positions, and all nodes preceding an N-Position are either a P-Position or an N-Position.

In order to determine whether or not a node is a P-Position or an N-Position, we begin by noting that all leaves are P-Positions, and all parent nodes of the leaves are N-Positions. To determine which parent nodes are N-Positions or P-Positions, we examine the known properties of N- and P-Positions; in particular, a P-Position can only become an N-Position, and an N-Position may become either a N-Position or P-Position. Translating that to the relationship between nodes and parent nodes, we say that a parent node is a P-Position if and only if all of its children are N-Positions. Otherwise, it is an N-Position. This information is extremely useful for labeling purposes of game trees—whether it be by hand or by a computer program—and allows us to find certain interesting paths in a game tree, including Maximal and Maximum N-Position Paths (see Section 1.4).

1.3 Observed Game Combinations

To begin analyzing the nature of Combined Games, we start by picking combinations of small and simple games. We briefly describe each game analyzed and state the which player has the winning strategy. [1]

1.3.1 Game 1: Hexapawn

We focus on the 3x3 case. Players each have three chess pawns, and move them according to the rules of chess. A player wins by moving their pawn to the board side opposite their own or by making the last legal move. For the 3x3 case, assuming optimal play, Hexapawn is a P-Position. See Figure 2 in Section 1.3.6 for an example of playing Hexapawn.

1.3.2 Game 2: Tic-Tac-Toe

Players alternate drawing an 'X' or an 'O' on a 3x3 board. The player who draws three of their symbol in a row is the winner. Assuming optimal play, this game always ends in a tie.

1.3.3 Game 3: Poison

Played with a pile of objects greater than 0, players alternate removing one or two objects from the top of the pile. The player who removes the last object loses. Given a pile of size X, the position of the game is:

- an N-Position if $X \equiv 0, 2$ $(mod\ 3)$.

- a P-Position if $X \equiv 1 \ (mod \ 3)$.

See Figure 3 in Section 1.3.6 for an example of a game of Poison.

1.3.4 Game 4: Cupcake

Played exactly the same as Poison, except the player that removes the last object wins. Given a pile of size M, the position of the game is:

- an N-Position if $M \equiv 1, 2 \ (mod \ 3)$.
- a P-Position if $M \equiv 0 \ (mod \ 3)$.

See Figure 4 in Section 1.3.6 for an example of a game of Cupcake.

1.3.5 Game 5: Brussels Sprouts

Brussels Sprouts is an example of a *Pre-Determined Game*, which we define in Section 3. Players start with an number of crosses and play by connecting tips of crosses together without crossing edges. Once an edge is formed, a dash is drawn through the middle of that edge. A player wins if they are the last player make an edge. Let n be the number of initial crosses. Then:

- If n is even, Brussels Sprouts is a P-Position.
- If n is odd, then Brussels Sprouts is an N-Position.

See Figure 5 in Section 1.3.6 for an example of a game of Brussels Sprouts.

1.3.6 Examples of the Outlined Games

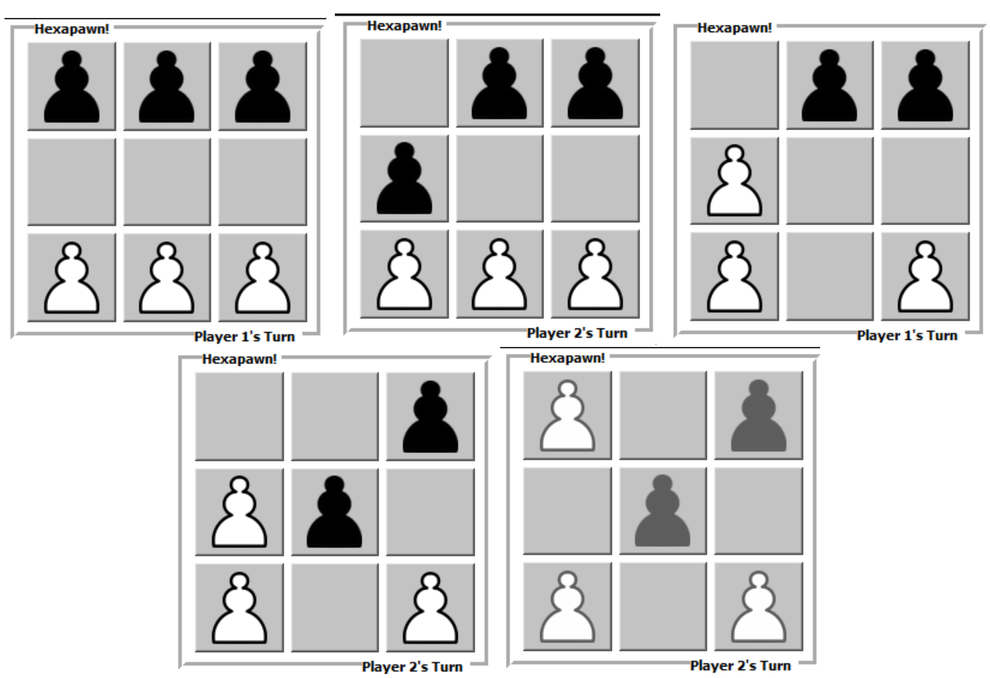

Figure 2: *This figure outlines a game of Hexapawn.*

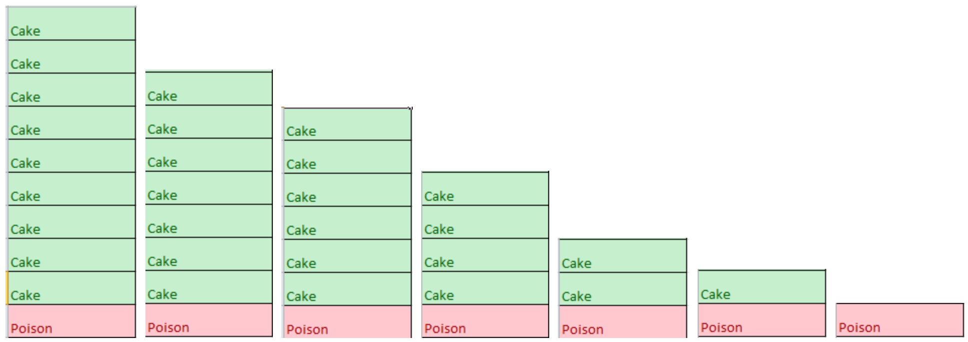

Figure 3: *This figure outlines a game of Poison of size 10.*

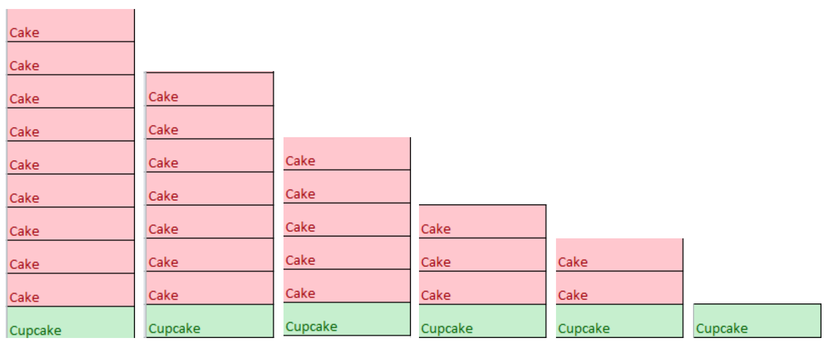

Figure 4: *This figure outlines a 10-Piece game of Cupcake.*

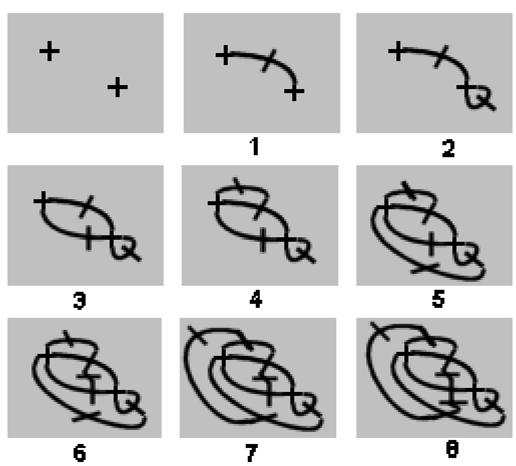

Figure 5: *This figure outlines a game of Brussels Sprouts with two initial crosses. [3]*

1.4 Observed Properties of Combined Games

The properties in this section are derived from observing various Combined Games. Our observations show that Combined Games in which one or more of these properties exist have optimal strategies that depend on these properties. These observations appear in the proofs of optimal strategies for Combined Games (Section 2).

The Tic-Tac-Toe Effect

Property 1. *A game G is said to have the* Tic-Tac-Toe Effect *(TTTE) if:*

1. *G optimally ends in a draw*
2. *The first player to move on G is the last player to move on G.*

This property was initially observed as a pattern resulting from the Adaptive Learning implementation of the combined game Hexapawn&Tic-Tac-Toe. We then generalized the effect, realizing its importance to optimal play of a combined game.

The Skip-and-Lose Principle

Property 2. *Given a game board g of a game G, if:*

- *Player A makes a move on board g, which results in board g',*
- *Player B skips making a move on board g',*
- *Player A make a move on board g', resulting in board g",*

and all resulting boards from g" result in a loss for Player B, then g is said to have the Skip-and-Lose Principle *(SLP).*

In early states of Adaptive Learning and our own observations, we found that certain moves always resulted in at least one loss for a player. The observed patterns define the Skip-and-Lose Principle, and these patterns often lead to a player's complete loss of a combined game, not just one sub-game.

Maximum N-Position Path

Property 3. *Let T be a game tree for an N-Position Normal Game. Then, T contains a* Maximum N-Position Path $P = p_0 p_1 ... p_n$ *from the opening position* p_0 *of T to a terminal position* p_n *such that for all* $0 \leq \imath \leq n-1$, $p_\imath$ *is an N-Position and* p_n *is a P-Position.*

Note that not all N-Position Normal Games have a Maximum N-Position Path.

Maximal N-Position Path

Property 4. *Let T be a game tree for an N-Position Normal Game.. Then T contains a* Maximal N-Position Path $P = p_0 p_1 ... p_k$ *from the opening position* p_0 *of T to a position* p_k *such that* p_k *is a P-Position (not necessarily terminal) and for all* $0 \leq \imath \leq k-1$, $p_\imath$ *is an N-Position.*

Note that every Maximum N-Position Path is a Maximal N-Position Path, and that every N-Position Normal Game has a Maximal N-Position Path of length $\ell \geq 1$.

The last two properties are more akin to definition than actual properties like the Skip-and-Lose Principle and the Tic-Tac-Toe Effect. However, in normal game analysis (outside of combined game analysis), these properties are implicit in analysis; often they are never mentioned in the discussion of optimal strategies. In the context of Combined Games, these two properties become essential to understanding the optimal strategy of combinations of two Normal Games (Section 2.5).

2 An Examination of Combinations of Games with Results of Analysis

In the previous section, we define what a *Combined Game* is, as well as outline rules and properties of such games, and define traditional and necessary terms for analyzing games that we apply to *Combined Games*. In this section, we discuss the Combinations of Games we examine, and particular how we approach examining the Combinations of Games. Then for each Combination, we present the results of our examination.

2.1 Combination One: Hexapawn & Tic-Tac-Toe

The first combination we examine is Hexapawn & Tic-Tac-Toe. We begin with these two games because they are simple and well-solved, with relatively small game trees (compared to games such as chess). Figure 1 in the previous section outlines a game of Hexapawn & Tic-Tac-Toe.

The examination began with actually playing the combination of Hexapawn & Tic-Tac-Toe and trying to make educated guesses as to what the winning strategy could be, taking into consideration the winning strategies of both Hexapawn and Tic-Tac-Toe as individual games. By playing this combination many times, it was clear that because Tic-Tac-Toe ends in a draw, understanding it's role int he winning strategy would be equivalent to finding the winning strategy. This observation led to the defining of the Tic-Tac-Toe Effect (Section 1.4).

Additionally, as we began to outline the game tree for Hexapawn & Tic-Tac-Toe, the branching factor and depth of the tree became too great. This lead to the use of an Adaptive Learning Program to try to learn whether Hexapawn & Tic-Tac-Toe was an N-Position or P-Position. By examining game after game of Hexapawn & Tic-Tac-Toe played by the program, a pattern began to emerge: the last player to move on Tic-Tac-Toe was the winning player. With this information, we formulated a winning strategy that we proved to be the winning strategy of the combined game of Hexapawn & Tic-Tac-Toe.

We prove the strategy below:

Theorem 1. *In the Combined Game of Hexapawn and Tic-Tac-Toe, Player 1 has the winning strategy, which is to play on Tic-Tac-Toe first, then play on whatever board Player 2 moves on.*

Proof. There are two things that must be proven: 1. That there exists a winning strategy. 2. The winning strategy belongs to Player 1, and that strategy is to play on Tic-Tac-Toe first, then to move on whatever board Player 2 moves on.

1. Because Hexapawn cannot end in a tie, and Tic-Tac-Toe optimally ends in a draw, at most 1 player will receive 1 point. Thus, Hexapawn & Tic-Tac-Toe (H&TTT) cannot end in a tie. Because H&TTT cannot end in a tie and is a progressively bounded combinatorial game, there exists a winning strategy.

2. Assuming optimal play, the opening move on H&TTT must be on the Tic-Tac-Toe board because the first move on Hexapawn is a P-Position. Thus Player 1 moves on Tic-Tac-Toe first. Now, by the Tic-Tac-Toe Effect, if Player 2 continues playing Tic-Tac-Toe, Player 2 is the player to make the first move on Hexapawn, which guarantees Player 2's loss. If Player 2 moves on Hexapawn, and if Player 1 follows and also moves on Hexapawn, Player 2 loses H&TTT since Tic-Tac-Toe ends in a tie and Player 1 wins Hexapawn. But, if Player 2 moves on Hexapawn, then Player 1 moves on Tic-Tac-Toe, then Player 2 moves on Hexapawn a second time, by the Skip-And-Lose Principle, Player 2 wins Hexapawn and Player 1 wins Tic-Tac-Toe, which leaves H&TTT as a tie game. However, since Player 1 plays optimally, if Player 2 moves on Hexapawn, Player 1 follows. Additionally, if Player 2 moves on Tic-Tac-Toe, Player 1 follows because of optimal play. Thus, Player 1 has the winning strategy.

□

2.2 Combination Two: Hexapawn & Poison

The examination of Hexapawn & Poison came from the desire to examine the more general combined game of a Normal and a Meiseré game. Again, the first examination was playing the game over and over again, trying to see if there was a hidden pattern holding information about a winning strategy. This combination again led us to use an Adaptive Learning Program to try and find a pattern.

Initial results of the Adaptive Learning Program revealed that playing on Poison didn't have an effect on who won Hexapawn. We questioned this pattern and began to think more deeply about the problem, and it was later we discovered that the Adaptive Learning Program was incorrect and did not properly adapt. However, our thoughts and hand-drawing of the game tree for what we believed to be key states of the game revealed the following winning strategy:

Theorem 2. *The Combined Game of Hexapawn and Poison (of size $n > 0$) always ends in a draw.*

Proof. First, if one player has a winning strategy, playing the sub-games simultaneously has the same effect as playing the two sub-games one by one. Assume that Player A has a winning strategy. If Player A follows his winning strategy, he wins both sub-games because neither Poison or Hexapawn is able to draw and if Player A only wins only one game then he must lose the other. Since player A has winning strategy, he would not allow himself to skip one move in one sub-game. Furthermore, Player A is always able to prevent himself from skipping one move in one game by choosing the game Player B moved on. As a whole, because Player A has the winning strategy, he chooses to play the game the Player 2 plays. In that case, playing the two sub-games simultaneously has the same effect as playing the two sub-games one by one.

Second, if one player has a winning strategy, he would force the other player to take both the first move in Hexapawn and the last move in poison. In a Hexapawn, the second player to move has winning strategy, and in Poison, the player who takes the last move loses. If Player A has winning strategy, he would win both the games. Moreover, since playing the two sub-game simultaneously has the same effect as playing the two games one by one; if one player makes the first move in Hexapawn, he would become Player 1 in Hexapawn forever and he would lose the sub-game. In that case, Player A would try to force the other player to take the first move in Hexapawn. To sum up, if Player A has winning strategy, he would force the other player to take both the first move in Hexapawn and the last move in Poison.

Finally, no one is able to force the other player to take both the first move in Hexapawn and the last move in poison. If both players play optimally, no one would like to take the first move in Hexapawn. The winning strategy holder must try and force the other player to take that move. However, the only way to force the other player to play first in Hexapawn is to take the last move in Poison. In another words, if the winning strategy holder wants to win the Hexapawn game, he has to take the poison piece in Poison. In that case he would lose Poison, but win Hexapawn. Thus, the combined game of Hexapawn and Poison ends in a draw. □

2.3 Combination Three: Tic-Tac-Toe & Poison

The examination of Tic-Tac-Toe & Poison arose from the question of how the a game with the Tic-Tac-Toe Effect combined with a Meiseré game would change the winning strategy, much like how the Effect affected Hexapawn & Tic-Tac-Toe.

The winning strategy for this combination was revealed from simply playing the game over and over, while keeping in mind the winning strategy for Poison and the Tic-Tac-Toe Effect. This meant we did not use Adaptive Learning to try and find a pattern; the winning strategy became quite clear after a few trials.

Theorem 3. *In the Combined Game of Tic-Tac-Toe and Poison of any size $n \geq 1$,*

- *If $n \equiv 0, 1$ (mod 3), the game is an N-Position.*

- *If $n \equiv 2$ $(mod\ 3)$, the game is a P-Position.*

Proof. We proceed by induction, with base cases $n = 1$ and $n = 2$.

Case 1: $n = 1$: If $n = 1$, then Player 1 plays on Tic-Tac-Toe because moving on Poison yields a loss. The same follows for Player 2. By the Tic-Tac-Toe Effect, when Tic-Tac-Toe is finished, Player 2 must take from Poison, and thus Player 1 wins.

Case 2: $n = 2$: If $n = 2$, then Player 1 has two moves: take one from the Poison or move on Tic-Tac-Toe. If he moves on Poison, then this case is reduced to $n = 1$, and thus Player 2 wins. If Player 1 moves on Tic-Tac-Toe, Player 2 follows by the Skip-and-Lose Principle. Thus, when Tic-Tac-Toe is finished, Player 2 begins on Poison. He takes one from Poison, leaving Player 1 to lose Poison. Thus, Player 2 wins.

Inductive Hypothesis: Assume true for Poison of size n: Consider the Combined Game of Tic-Tac-Toe & Poison of size $n + 1 = k$. If $k \equiv 2$ $(mod\ 3)$ then $(k - 1) \equiv 1$ $(mod\ 3)$ and $(k - 2) \equiv 0$ $(mod\ 3)$, which are N-Positions. Because Poison is a game such that every N-Position can move to at least one P-Position, k must be a P-Position. If $k \equiv 1$ $(mod\ 3)$, then $(k - 1) \equiv 0$ $(mod\ 3)$ and $(k - 2) \equiv 2$ $(mod\ 3)$. Because $k - 1$ is an N-Position and $k - 2$ is a P-Position and because every P-Position can only move to an N-Position, k must be an N-Position. The argument is similar when $k \equiv 0$ $(mod\ 3)$. Thus, we have shown which player has the winning strategy for $n + 1$.

□

2.4 Combination Four: Cupcake & Cupcake

Cupcake & Cupcake was the first combination examined where no sub-game had the Tic-Tac-Toe Effect. Further, it is the first Combined Game where a draw is possible, assuming optimal play. We again began analysis by playing various sizes of Cupcake & Cupcake, and the results are noted in the following table:

Game 1, Position	Game 2, Position	Overall Position
1,N	1,N	D
1,N	2,N	D
2,N	2,N	D
1,N	3,P	N
2,N	3,P	N
3,P	3,P	P
1,N	4,N	D

Table of Cupcake & Cupcake board positions.

A pattern quickly emerged, which formulated the theorem for which player has the winning strategy:

Theorem 4. *In the Combined Game of Cupcake (of size $m > 0$) & Cupcake (of size $n > 0$), then the winning player follows the presented pattern:*

1. *If both piles are P-Positions, then the game is a P-Position.*
2. *If both piles are N-Positions, then the game is a draw.*
3. *If one pile is an N-Position, and one pile is a P-Position, then the game is an N-Position.*

Originally, this theorem was proven as part of our initial results; however, upon closer examination of the proof, there were no specific qualities of Cupcake that were used in the proof. It was determined that Cupcake & Cupcake is a specific case of a Combined Game of Two Normal Games. Thus, the above theorem is a corollary of the theorem for Two Normal Games, presented in Section 2.5.

2.5 Combination Five: Two Normal Games

The analysis of Cupcake & Cupcake led to the formulation of this theorem, which is the first general theorem for *Combined Games* that was formulated. Before the formulation of the proof, we tested the pattern suggested by the theorem with many combinations of normal games, and found that the pattern held true. However, it is important to note that we avoided games that optimally end in a draw because of the Tic-Tac-Toe Effect.

With the collected data from testing the pattern, the foundation for the proof of this theorem became clear, and is outlined in the following proof:

Theorem 5. *In a Combined Game G of two* Normal *sub-games, g_1 and g_2, such that g_1, g_2 cannot end in a draw, we have:*

1. *If g_1 and g_2 are P-Positions, then G is a P-Position.*
2. *Without loss of generality, if g_1 is an N-Position and g_2 is a P-Position, then G is an N-Position.*
3. *If g_1 and g_2 are N-Positions, then:*
   (a) *if both g_1 and g_2 contain a Maximum N-Position Path, then G is a draw.*
   (b) *if both g_1 and g_2 do not contain Maximum N-Position Paths, then the position of G depends on the lengths $\imath$, $\jmath$ of the Maximal N-Position Paths of g_1 and g_2.*

Proof. Let $g_1, g_2 \in G$.

1. Let g_1 and g_2 be P-Positions. WLOG, assume Player 1 moves on g_1, resulting in g_1', which is an N-Position. Thus, Player 2 moves g_1' to g_1", where g_1" is a P-Position. Thus Player 1 must move on g_1" or g_2, which are both P-Positions. Following this process, when g_1 is finished, Player 2 is the winner, and whatever state g_2 is currently in is a P-Position. Thus, Player 2 wins g_2. Because Player 2 wins both g_1 and g_2, G is a P-Position.
2. WLOG, let g_1 be an N-Position and g_2 be a P-Position. Then, assuming optimal play, Player 1 moves g_1 to g_1', where g_1' is a P-Position. Then, Player two must begin on g_1' and g_2, which are both P-Positions. Thus, as we have shown previously, Player 2 loses on both g_1' and g_2. Thus, G is an N-Position.
3. Let g_1 and g_2 be N-Positions. Then, moving on g_1 or g_2 results in g_1' and g_2'. WLOG, assume Player 1 plays optimally and moves g_1 to g_1'.
   (a) Assume both g_1 and g_2 contain a Maximum N-Position Path.
   Case 1: g_1' is a P-Position Let g_1' be a P-Position. As such, the combined game G' = $\{g_1$',$g_2\}$ is an N-Position, and thus Player 2 wins. However, because Player 1 moves optimally, he does not move g_1 to a P-Position.
   Case 2: g_1' is an N-Position Let g_1' be an N-Position. Now, Player 2 must move on g_1' or g_2. Assuming Player 2 plays optimally, he does not move g_1' or g_2 to a P-Position by the argument in Case 1. Thus, both Player 1 and Player 2 continuously reduce g_1 and g_2 to N-Positions. WLOG, assume Player 1 wins g_1. Then, the current state of g_2 is an N-Position, and thus Player 2 wins g_2. Because both players win a game, G ends in a draw.

(b) Assume both g_1 and g_2 do not contain a Maximum N-Position Path. Each of g_1 and g_2 have a Maximal N-Position Path of lengths $\imath$, $\jmath$ respectively.

Case 1: both $\imath$, $\jmath$ are even If $\imath$, $\jmath$ even, then the number of steps in g_1 and g_2 to the last N-Position of the Maximal N-Position Paths are odd. WLOG assume g_1 is reduced to such an N-Position first, denoted as g_1'. Then, since $\imath - 1$ is odd, Player 1 makes the move to g_1'. Then, Player 2 moves on g_2. Since $\jmath - 1$ is odd, Player 2 makes the move to g_2', the last N-Position in the Maximal N-Position Path for g_2. Then Player 1 is forced to move either g_1' or g_2' to a P-Position. Thus, by Part 2 from above, G is a P-Position.

Case 2: both $\imath$, $\jmath$ are odd If $\imath$, $\jmath$ are odd, then then the number of steps in g_1 and g_2 to the last N-Position of the Maximal N-Position Paths are even. WLOG assume g_1 is reduced to such an N-Position first, denoted as g_1'. Then, since $\imath - 1$ is even, Player 2 makes the move to g_1'. Then, Player 1 moves on g_2. Since $\jmath - 1$ is even, Player 2 makes the move to g_2'. Then Player 1 is forced to move either g_1' or g_2' to a P-Position. Thus, by Part 2 from above, G is a P-Position.

Case 3: $\imath$ is even, $\jmath$ is odd WLOG, assume $\imath$ is even and $\jmath$ is odd. Then, since $(\imath-1)+(\jmath-1)$ is odd, the number of steps needed to reach both g_1' and g_2' (as used in Cases 1 and 2) is odd. Thus, Player 2 is forced to move either g_1' or g_2' to a P-Position. Thus, by Part 2 from above, G is an N-Position.

□

2.6 Combination Six: Poison & Poison

For an easier understanding of Poison & Poison, we assumed the Combined Game is simply two piles of Poison, where a player can only take from one pile at a time. Again, to understand this combination, many trials of different sizes were tested. Much to our surprise, the following theorem outlines the pattern we found to exist for every trial of Poison & Poison we played—no matter the size.

Theorem 6. *The Combined Game of Poison (of size $m > 0$) & Poison (of size $n > 0$) always ends in a draw.*

Proof. Because Poison is a Meiseré game, optimal play assumes both players avoid picking the last item in a pile. Thus, if one pile in Poison & Poison has only one piece remaining, if there is more than one piece in the other pile, the players take pieces from that pile, until just 1 remains. Thus, each player must take the one poison from one of the piles. Therefore, each player receives 1 point, which means that Poison & Poison ends in a draw. □

This result was completely unexpected; no matter the size of the two piles, optimal play always ended in a draw. As apparent from the proof, Poison being a Meiseré game is the reason for this result.

The proof behind the result of Poison & Poison led us to ask the question: does this hold true for a combined game of Two Meiseré games? As we began to test this question, another question arose: what if we have more than two games of Poison in our combined game? What if we hade more than two Meiseré games? Section 2.7 answers these questions.

2.7 Combination Seven: M Meiseré Games

The idea behind the winning strategy for a combined game of M Meiseré games stems from the analysis of the questions posed at the end of Section 2.6. Our analysis again began with testing various trials of different amounts of Meiseré games being played in combination; we concluded that observing these combinations is equivalent to observing M Meiseré games with only one move remaining. The results of the analysis culminate in the following theorem:

Theorem 7. *In a Combined Game G of M* Meiseré *sub-games, such that each sub-game cannot end in a tie, the following is true:*

1. *If M is even, then G always ends in a draw,*
2. *If M is odd, then the winner of G depends on the status of the last sub-game b of G such that b is the only sub-game that is not one move away from a terminal position.*

Proof. 1. If M is even, we can examine the case of $M = 2$ first, with sub-games g_1 and g_2. In a Meiseré game, the optimal strategy is to avoid making the last move. Assume Player 1 and Player 2 play according to the optimal strategy. As such, let g_1 and g_2 be the starting positions of the two sub-games, and let g_1* and g_2* denote a game state of g_1 and g_2, respectively, that is one move away from a terminal position. After some rounds of play, WLOG assume g_1 has been moved to g_1*, and g_2 has been moved to some g_2' such that $g_2' \neq g_2*$. Because Player 1 and Player 2 follow the optimal strategy, neither player moves on g_1*; each player moves on g_2', until g_2' is moved to g_2*. Because g_1 and g_2 cannot end in a tie, each player loses one of the sub-games, g_1 or g_2. Thus, G is a draw.
It follows that if there is an even amount of M sub-games such that $M > 2$, this pattern holds; thus, G ends in a draw.

2. If M is odd, then we can reduce the game to the case of $M - 1$ Meiseré games and 1 Meiseré game that we denote as b, where $M - 1$ is even. By the proof of the even case, we know that each of the $M - 1$ games are reduced to a state that is one move away from a terminal position. Assume that b has not been reduced to such a state, but assume b has been moved to some state b'. Because the $M - 1$ sub-games give each player an equal amount of points, the winner is determined by the position of b'. If b' is an N-Position, then G is an N-Position; if b' is a P-Position, then G is a P-Position. Thus, the winner of G is determined by b.

□

The results of this theorem allowed us to answer the question presented in Section 2.6 of more than two games of Poison being played in combination:

Theorem 8. *In the Combined Game of X games of Poison (X-Poison) of size $n_1, n_2, \cdots, n_X$ where $X \geq 2$ and $n_\imath \geq 1$ for $1 \leq \imath \leq X$:*

1. *If N is even, then X-Poison is a draw.*
2. *If N is odd, then the winner of X-Poison is dependent on the number of pieces p in the last pile with more than one piece*
   (a) *If $p \equiv 0, 2$ (mod 3), then X-Poison is an N-Position.*
   (b) *If $p \equiv 1$ (mod 3), then X-Poison is a P-Position.*

Proof. Because X-Poison is a Combined Game of M-Meiseré games, it follows from the proof of the M-Meiseré Combined Game that the above strategy is correct. □

While we explored specific examples of *Combined Games* in this section and proved which player has the winning strategy (if one exists), the most important results are the two general theorems about Combined Normal Games (Section 2.5) and Combined Meiseré Games (Section 2.7). While they do not describe every possible combination in the *Combined Games* space we've defined, they include a large subset of games from the space.

In the next section, we discuss in depth a special and interesting case of the Theorem of Two Normal Games from Section 2.5.

3 Combinations of Pre-Determined Games

In the exploration of *Combined Games*, the most interesting results come from examining combinations of Pre-Determined Games. In this section, we define what it means to be a Pre-Determined Game, various properties of Pre-Determined Games, introduce definitions to simplify analysis of Combined Pre-Determined Games, and present the results of our analysis.

3.1 What is a *Pre-Determined Game*?

A *Pre-Determined Game* is defined as the following:

Definition 2. *A* Pre-Determined Game *D is a game that has a fixed number of moves, and no action done by a player can change this number.*

In our analysis, we concern ourselves only with Normal Pre-Determined Games.

3.2 Properties of *Pre-Determined Games*

To begin the analysis of Combined Pre-Determined Games, we formalize known properties of Pre-Determined Games, as they are necessary to understanding how such a game works. The properties are as follows:

- The winner of the game is known from the starting board,
- The game board positions follow a strict alternating nature; i.e., an N-Position may only lead to a unique P-Position, and a P-Position may only lead to a unique N-Position,
  - In other words, the game tree for a Pre-Determined Game is *degenerate.*
- Any Pre-Determined Game can be represented as *The Take-Away Game.*

The first property is the essence of a Pre-Determined Game: no matter what either player does—playing optimally or not—the game is decided from the initial state and cannot be changed. This property necessarily implies the second property, or that the game tree of any Pre-Determined Game is degenerate. That is, the tree is simply a path from the single start node to the single end node.

The final property stems from our need to easily represent a Pre-Determined Game in a computer for use with Adaptive Learning. We define *The Take-Away Game* in the next section.

3.3 Definitions

This section defines two ideas needed to simplify the analysis of Combined Pre-Determined Games.

Definition 3. *The* Take-Away Game *is a game where players are given a collection of n objects, and a turn consists of a player removing one object. The winner is the player who removes the last object.*

All Pre-Determined Games are representable as *The Take-Away Game.* It is easy to see that if n is odd, then *The Take-Away Game* is an N-Position, and if n is even, then *The Take-Away Game* is a P-Position. Note this is assuming the Pre-Determined Game is Normal; if Meiseré, then the opposite holds true.

Definition 4. *A game is said to be* trivial *if the game is over after one turn.*

The first definition is a result of our desire to easily represent a Pre-Determined Game in a computer; in particular, to study Brussels Sprouts & Brussels Sprouts (Section 1.3.5) with Adaptive Learning. The idea of a trivial game stems from our want to define the *boring* games; the games that aren't really games at all. Note that a trivial game is a special case of a Pre-Determined game; in particular, it is a Pre-Determined Game of size 1.

3.4 An Examination of Combinations of Pre-Determined Games with Results of Analysis

The core of our analysis and desire to research Combined Pre-Determined Games comes from the question of whether or not two or more Pre-Determined Games played in combination remain Pre-Determined. In this section, we present the surprising results of our research into this question. We draw our general results for Pre-Determined Games from our analysis of the Combined Games of Brussels Sprouts (Section 1.3.5).

Theorem 9. *Let G be a Combined Game comprised of two games of Brussels Sprouts, g_1 and g_2. Then, G has the following properties:*

1. *if g_1 and g_2 are P-Positions, then G is a P-Position.*
2. *WLOG, if g_1 is an N-Position, and g_2 is a P-Position, then G is an N-Position.*
3. *if g_1 and g_2 are N-Positions, then G is a P-Position.*

Proof. It is important to note that the total number of moves on a given game of Brussels Sprouts is $5n - 2$, where n $(n \geq 1)$ is the number of initial crosses [1]. In other words, no game of Brussels Sprouts is trivial. This is important for the proof of Theorem 11. Additionally, note that Brussels Sprouts & Brussels Sprouts is a special case of the Combined Game of Two Normal Games.

1. Assume g_1 and g_2 are P-Positions. Then, by Theorem 5, Part 2, G is a P-Position.
2. WLOG, assume g_1 is an N-Position and g_2 is a P-Position. Then Player 1 has two choices:

   Case 1: Player 1 Moves on g_1. Then, g_1 is reduced to g_1', a P-Position. Then, by Part 1, Player 1 wins. Thus G is an N-Position.

   Case 2: Player 1 Moves on g_2. Since the previous case results in a win for Player 1, moving on g_2 would not be optimal, contradicting Player 1 playing optimally. Thus, this case is impossible.
3. Assume g_1 and g_2 are N-Positions. Then, WLOG assume Player 1 moves g_1 to g_1', a P-Position. Then by Case 1 of Part 2, Player 2 wins. Thus, G is a P-Position.

□

This theorem was born from examining the results of the Adaptive Learning Program implemented for Brussels Sprouts & Brussels Sprouts; in the output, we found that if both of the sub-games were of size 3 or less, then the Combined Game was Pre-Determined. However, we tested this pattern by hand and found that the sub-games needed to be of size 1 each, and that any greater size made the Combined Game not Pre-Determined. This gave us the basis for the presented proof.

The results of this proof led to two general theorems about Combined Pre-Determined Games of size 2.

Theorem 10. *Let D be a Combined Pre-Determined Game of size 2. Then, D is also Pre-Determined if and only if d_1 and d_2 are trivial.*

Proof. We prove both directions of the theorem.

1. Let d_1 and d_2 be trivial games. As such, a player may only make one move on each board. Thus, Player 1 wins a board, and Player 2 wins a board. Thus, D is always a draw. Therefore, D is Pre-Determined.
2. Assume that D is Pre-Determined.

Case 1: WLOG assume d_1 is non-trivial. (a) Assume d_1 is an N-Position. Then, if Player 1 moves on d_2, Player 2 can only move on d_1, an N-Position. Thus, D is a draw. Now if Player 1 moves on d_1 instead, d_1' is a P-Position. Then, Player 2 can move on d_2, resulting in Player 1 losing d_1'. Thus, D is a P-Position. This contradicts D being Pre-Determined, as a player action changed the outcome of the game.

(b) Assume d_1 is a P-Position. Then, if Player 1 moves on d_2, Player 2 must move on d_1, a P-Position. Thus, Player 1 wins d_1 and D is an N-Position. Now if Player 1 moves on d_1 instead, d_1' is an N-Position. If Player 2 moves on d_2, then Player 1 wins d_1'. If Player 2 moves on d_1', we have the previous part, which results in either D being a draw or D being a P-Position. This contradicts D being Pre-Determined.

Case 2: Assume both d_1 and d_2 are non-trivial. Then, both d_1 and d_2 proceed until one of them becomes trivial. Then, we have Case 1. Thus contradicting D being Pre-Determined.

Thus, if D is Pre-Determined, then d_1 and d_2 are trivial.

□

This theorem help answer our driving question behind researching Combined Pre-Determined Games: does a Combined game of Pre-Determined Games remain Pre-Determined? Theorem 10 leads us to the corollary:

Corollary 1. *Let D be a Combined Pre-Determined Game of size 2. Then, D is not Pre-Determined if and only if at least one of d_1 and d_2 is non-trivial. In particular, D has a winning strategy.*

This Corollary and Theorem 10 show that for any interesting (i.e., non-trivial) Combined Pre-Determined Games of size 2, there is an actual strategy and that Combined Game is not Pre-Determined, a completely unexpected result. Thus, we have a theorem about the winner of a Combined Game of two non-trivial Pre-Determined Games.

Theorem 11. *Let D be Combined Game of two Pre-Determined Games d_1 and d_2, where at least one sub-game is non-trivial. Then,*

1. *if d_1 and d_2 are P-Positions, then D is a P-Position.*
2. *WLOG, assume d_1 is an N-Position and d_2 is a P-Position. Then, D is an N-Position.*
3. *if d_1 and d_2 are N-Positions, then D is P-Position.*

Proof. Recall that for any Pre-Determined Game, there is an equivalent Take-Away Game. Thus, Brussels Sprouts is equivalent to a Take-Away Game that is equivalent to some other Pre-Determined Game. Thus, the proof is exactly as in Theorem 9. □

As a consequence of Theorems 10, 11, and Corollary 1, we can now prove a general result about a Combined Pre-Determined Game of size N, the fifth general result in the field of *Combined Games.*

Theorem 12. *Let D be a Combined Pre-Determined game of size N. Then, D is Pre-Determined if and only if all sub-games are trivial.*

Proof. We prove both directions of the theorem.

1. If d_1, d_2, ... , d_N are trivial, then we proceed by induction on N:

   Base Case: $N = 2$. By Part 1 of Theorem 10, D is trivial.

Inductive Hypothesis: Assume true for d_k, where $2 \leq k \leq N$. Now assume we have a game G with $k+1$ trivial sub-games. If Player 1 moves on any one of them, we have G' with k trivial sub-games. By hypothesis, G' is Pre-Determined. and since Player 1 can move on any of the $k+1$ trivial sub-games of G to reach G', G is also Pre-Determined. Thus, if all sub-games are trivial, then G is Pre-Determined.

2. Assume D is Pre-Determined and assume that at least one $d_\imath$, $0 \leq \imath \leq N$ is not trivial. Then, we have a similar argument as in Theorem 10, Part 2. Thus, all $d_\imath$ are trivial.

□

This theorem gives us another corollary, which is our sixth general result found in the *Combined Games* field:

Corollary 2. *Let D be a Combined Pre-Determined Game of size N. Then D is not Pre-Determined if and only if at least one of $d_\imath$, $0 \leq \imath \leq N$, is non-trivial, where $d_\imath$ is a Pre-Determined sub-game. In particular, D has a winning strategy.*

This corollary generalizes Corollary 2 and follows from the proof of Theorem 12, showing that for any interesting Combined Pre-Determined Game of size $N \geq 2$, there exists a winning strategy. In other words, where a given Pre-Determined Game can hardly be called a game since players have no effect on the outcome, a Combined Game of non-trivial Pre-Determined Games becomes interesting as players now have an effect on the outcome.

The last Combined Pre-Determined Game we examine is the Combined Game of Three Brussels Sprouts. We analyzed this Combined Game in hopes that it would reveal a pattern of the more general case of the Combined Game of Three Normal Games. However, our analysis of the results of Three Brussels Sprouts failed to yield any connection to the general case of Three Normal Games, but we were able to prove which player has the winning strategy for Three Brussels Sprouts.

Theorem 13. *Let D be a Combined Game of 3 Brussels Sprouts, d_1, d_2, and d_3 such that at least one sub-game is not trivial. Then,*

1. *if d_1, d_2, and d_3 are P-Positions, then D is a P-Position.*

2. *WLOG, if d_1 is an N-Position and d_2, d_3 are P-Positions, then D is an N-Position.*

3. *if d_1, d_2, and d_3 are N-Positions, then D is an N-Position.*

4. *WLOG, if d_1, d_2 are N-Positions and d_3 is a P-Position, then D is a P-Position.*

Proof. Note that by Corollary 2, there exists a winning strategy for D. It just depends on the sub-games.

1. Let d_1, d_2, and d_3 be P-Positions. Then, any move Player 1 makes reduces a board to an N-Position. WLOG, assume Player 1 moves d_1 to d_1', an N-Position. Then Player 2 has two choices:

   Case 1: Player 2 moves on d_1'. Then, d_1' is reduced to an P-Position, back to 3 P-Positions. If Player 1 and Player 2 continued this repetition, the resulting board is d_2 and d_3, with Player 1 as the next player. Thus, by Theorem 9, Player 2 wins. Thus, D is a P-Position.

   Case 2: WLOG, Player 2 moves on d_2. Then, d_2 is reduced to d_2', an N-Position. Then, Player 1 can reduce the board to 3 P-Positions again, and follow the strategy outlined in Case 1, thus resulting in a win for Player 1. However, since Player 2 plays optimally, this case is impossible.

   Thus, D is a P-Position.

2. WLOG, let d_1 be an N-Position and let d_2, d_3 be P-Positions. Then, playing optimally, Player 1 reduces d_1 to a P-Position. Then, we have Part 1, with Player 2 as the next player. Thus, Player 1 wins and D is an N-Position.

3. Let d_1, d_2, and d_3 be N-Positions. Then WLOG assume Player 1 reduces d_1 to d_1', a P-Position.

   Case 1: WLOG, Player 2 moves on d_2. Then, d_2 becomes d_2', a P-Position. Then, we have two P-Positions and an N-Position, with Player 1 as the next player. By Part 2, Player 1 wins. So by optimal play, this case is impossible.

   Case 2: Player 2 moves on d_1' Then, the game is reduced to 3 N-Positions. After continuing this process, the board consists of 2 N-Positions, with Player 2 as the next player to move. Thus, by Theorem 9, Player 1 wins and D is an N-Position.

4. WLOG, let d_1 and d_2 be N-Positions and d_3 be a P-Position.

   Case 1: Player 1 moves on d_3. Then we have the case of 3 N-Positions. Thus by Part 3, Player 2 is the winner and thus D is a P-Position.

   Case 2: WLOG, Player 1 moves on d_1. Then, d_1 is reduced to d_1', a P-Position. Then, we have the case of one N-Position and two P-Positions. Thus, by Part 2, Player 2 is the winner and thus D is a P-Position.

□

4 Conclusions and Final Remarks

This paper represents the first steps into the field of *Combined Games* research, and with it, methods for which one may approach problems in this field. We found that playing the Combined Game in question many times with many different people yields a large amount of data that is useful for finding out which player has the winning strategy, what the winning strategy could be, and putting new restrictions on the space of *Combined Games* as a whole. Then when simply playing the Combined Game stops revealing information, Adaptive Learning is useful for finding patterns in the game being played, as well as figuring out which player has the winning strategy and what that strategy is. Note that Adaptive Learning need not be entirely correct—as was the case for many of our Adaptive Learning Programs—, as useful information can still be extracted from the data collected.

We believe the properties introduced and general theorems proved in this research will help to greatly advance and potentially simplify future research into the field of *Combined Games*. We hope too that advancements in *Combined Games* research can provide new tools to the field of Combinatorial Game Theory, or researching normal, non-combined games, as well as new tools for *Combined Games* as well.

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