Hardness of Computing the Biclique Partition Number

Alexander R. Block

March 6, 2017
Introduction

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  - Minimum number of *bicliques* needed to partition the edges of a graph $G$, denoted $bp(G)$
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  - Minimum number of bicliques needed to partition the edges of a graph $G$, denoted $bp(G)$
  - Note that $G$ can be any graph
  - Bicliques are complete bipartite graphs, denoted $K_{n,m}$
Introduction (Examples)

$K_4$
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Introduction

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Graham-Pollak Theorem: $\text{bp}(K_n) = (n - 1)$. All proofs are algebraic and no purely combinatorial proof is known [GP72, Tve82, Pec84, Vis08, Vis13]
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- Showed $bp(G) \geq \max\{n_+(A(G)), n_-(A(G))\}$ [Witsenhausen, 1980s]
- Known that $n_-(A(K_n)) = (n - 1)$
- Can partition $K_n$ into $(n - 1)$ stars
  - A star is a biclique of the form $K_{1,i}$ for some positive integer $i$
Applications of Biclique Partition

- Graham and Pollak that a problem on loop switching in networking is equivalent to partitioning a multigraph, yielding their celebrated result [GP71, GP72, Tai13]

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\text{INPUT: } n, r, k \in \mathbb{N} \text{ with } k \leq r \leq n
\]

\[
\text{MINIMIZE: size of } \mathcal{F} := \{ f_i : [n] \rightarrow [r] \}
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\text{CONSTRAINT: } \forall K \subseteq [n] \text{ with } |K| = k, \exists i \text{ such that } f_i|_K \text{ is injective}
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Asking for unique \(i\) and \(r = k = 2\) asks for \(bp(K_n)\)

Connections to the nondeterministic state complexity of finite automata, namely used as a lower bound method [GH06]

Plays a role in analysis of HLA reaction matrices used in biology [NMW A78]
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  - Minimum (amount of) leakage that kills the possibility of key agreement
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  - Bicliques are useless for KA (because 0 mutual information)
  - Roughly corresponds to the *biclique partition number*

- Close connections to communication complexity and circuit lower bounds [HS12, NW95, Raz92]
Finding **bp is Hard**

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- We show a proof for bipartite graphs and another for general graphs.
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- We show a proof for bipartite graphs and another for general graphs.
- Proof for bipartite graphs is a reduction from the \textit{vertex clique problem}

\begin{tabular}{|l|}
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\textbf{GIVEN}: Graph $G$ with $V(G) = \{v_1, \ldots, v_n\}$ \\
\textbf{DETERMINE}: Fewest number of cliques which include all of $V(G)$ \\
\hline
\end{tabular}

- Proof for general graphs is a reduction from the \textit{vertex cover problem}
Proof of NP-Completeness of bp for bipartite graphs

- Suppose we are given a $G$ in the instance of the \textit{vertex clique problem} described before and we want to answer the following question

<table>
<thead>
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Proof of NP-Completeness of \textbf{bp} for bipartite graphs

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\textbf{GIVEN:} Bipartite graph $G$
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\end{center}

- Construct $G' = (L, R, E')$ with $L = \{x_1, \ldots, x_n\}$, $R = \{y_1, \ldots, y_n\}$ and $E' = \{(x_i, y_i): \forall i\} \cup \{(x_i, y_j): (v_i, v_j) \in E(G)\}$
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- Let $H' = \{(x_i, y_j): i = 1, \ldots, n\}$ be the set of edges to be covered.
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Proof of NP-Completeness of \( \text{bp} \) for bipartite graphs

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- If \( C' \) is a biclique of \( G' \) which includes edges \((x_{j_1}, y_{j_1}), (x_{j_2}, y_{j_2}), \ldots, (x_{j_k}, y_{j_k})\), then by construction it must be the case that \( \{v_{j_1}, \ldots, v_{j_k}\} \) is a clique in \( G \).
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- Let $H' = \{(x_i, y_j) : i = 1, \ldots, n\}$ be the set of edges to be covered.

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- If $C'$ is a biclique of $G'$ which includes edges $(x_{j_1}, y_{j_1}), (x_{j_2}, y_{j_2}), \ldots, (x_{j_k}, y_{j_k})$, then by construction it must be the case that $\{v_{j_1}, \ldots, v_{j_k}\}$ is a clique in $G$.

- So the minimum number of cliques that cover all vertices in $G$ is equal to the minimum number of bicliques of $G'$ needed to cover the edges in $H'$.
Proof of NP-Completeness of \(bp\) for general graphs

- Suppose we are given a graph \(G\) and need to find a vertex cover of size \(k \leq |V(G)|\)
Proof of NP-Completeness of $\text{bp}$ for general graphs

- Suppose we are given a graph $G$ and need to find a vertex cover of size $k \leq |V(G)|$
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- Suppose we are given a graph $G$ and need to find a vertex cover of size $k \leq |V(G)|$
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- Notice that $\alpha(G') = \alpha(G) + |E|$
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- Notice that \( \alpha(G') = \alpha(G) + |E| \)
- Thus, \( \text{bp} (G') = \alpha(G') = \alpha(G) + |E| \)
- So \( \alpha(G) \leq k \) if and only if \( \text{bp} (G') \leq k + |E| \)
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  - The proof is not very insightful, so it will be skipped in this talk
Chalermsook, Heydrich, Holm, and Karrenbauer [CHHK14] proves an approximation algorithm for bp with approximation guarantee of $O\left(\frac{n_L}{\sqrt{\log(n_L)}}\right)$, where $|L| = n_L$ and the input graph is bipartite. For our purposes, assume $|L| = |R| = n$. 
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- Choose parameter $r$ (to be fixed later) and partition $L$ into $n/r$ subsets of size $r$ ($L_1, \ldots, L_{n/r}$)
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- Each biclique from each \( L_i \) are put together and from a biclique cover of the whole graph
Nearly Tight Approximability for $bp$

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  - Given $L_i$, run a brute force algorithm over all $2^r$ subsets and enumerate all $r$-tuples of each subset
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Chalermsook et al. also give an approximation with respect to the number of edges $m$, which has guarantee

$$O\left(\frac{m \log^2 \log m}{\log^3 m}\right)$$
Open Problems

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    \[ \sqrt{n - 1} \leq bp_2(K_n) \leq \lfloor \sqrt{n} \rfloor + \lfloor \sqrt{n} \rfloor - 2 \] [Alo97, HS12]
  - Easy to ask: what is $bp_t(K_n)$ for constant $t$?
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- Are there approximation algorithms with better guarantees?

Chalermsook et al. [CHHK14] give better guarantees if $\mathbf{NP} \not\subseteq \mathbf{BPTIME}(2^{\text{polylog} n})$ (Bounded Error Probabilistic Time).

How close is $\text{bp}$ to Wyner's Common Information?

How good of an approximation is one to the other?

How close are $\text{bp}$ and the biclique cover number ($\text{bc}$)?

Known that $\text{bc} \leq \text{bp}$

[Pin14] This relation may be quite loose:

\[ \text{bp}(\mathcal{K}_n) \geq 2^{\text{bc}(\mathcal{K}_n)} - 1 - \frac{1}{2^{\text{bc}(\mathcal{K}_n)}} \] (note that $\text{bc}(\mathcal{K}_n) = \lceil \log n \rceil$)

\[ \text{bp}(\mathcal{G}) \leq \frac{1}{16} \left( 3^{\text{bc}(\mathcal{G})} - 1 \right) \]
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How close is \( bp \) to Wyner’s Common Information?
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How close are \( bp \) and the \textit{biclique cover number} (\( bc \))?
  ▶ Known that \( bc \leq bp \)
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    - $\text{bp} \left( G \right) \leq \frac{1}{2} \left( 3^{\text{bc} \left( G \right)} - 1 \right)$
Conclusions

- The biclique partition number is a fertile, rich area of research in mathematics with many connections to other fields.
- Determining $bp$ and $bc$ is an NP-Hard problem.
  - Even for bipartite graphs.
- $bp$ and $bc$ are NP-Hard to approximate as well.
  - Even for bipartite graphs.
- Still many open problems in relation to $bp$ and $bc$. 

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