Lesson 10   2.8 The Derivative as a Function

last time, we saw $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ and

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

now we will relax the constraint of $x=a$ and find derivative at <u>all</u> x by changing a to x

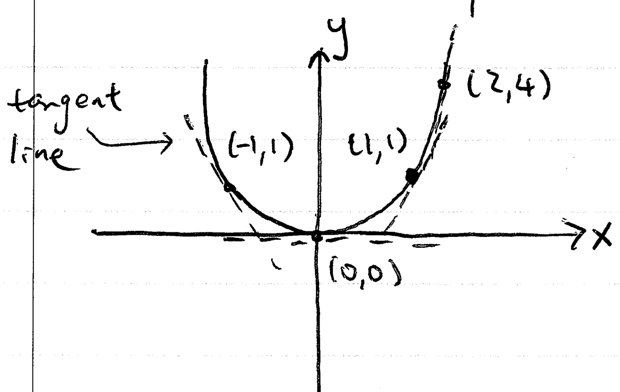
Obviously, 2nd form is easier to change (first form gives $\frac{0}{0}$)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

this gives us slope of the tangent line at any x where the function is <u>differentiable</u> (more on this later)

now let's compare graphs of $f(x)$ and $f'(x)$

start with a simple one: $f(x) = x^2$



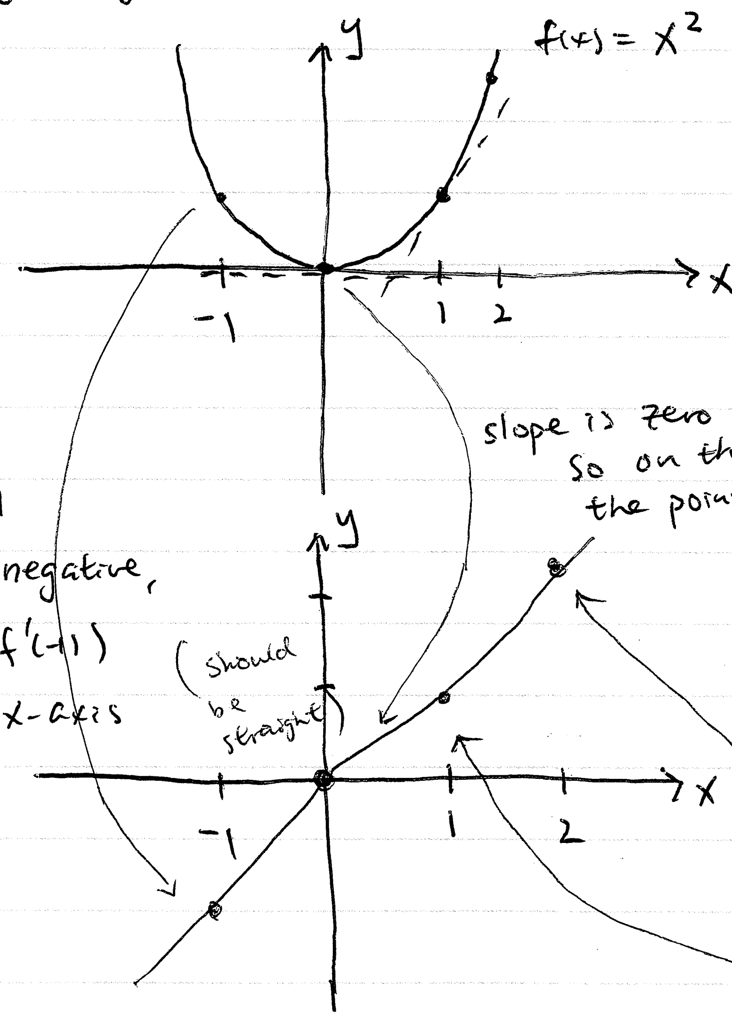
note that the tangent line is horizontal at $x=0$

so $f'(0)=0$

and slope of tangent line at $x=-1$ is negative
but is the same <u>number</u> to the slope at $x=1$
finally, the slope at $x=2$ is the largest (most steep)

ranking of slopes from highest to lowest : $f'(2)$, $f'(1)$, $f'(0)$, $f'(-1)$

let's try to graph $f'(x)$ from $f(x)$



slope is zero at $x=0$
so on the graph of $f'(x)$
the point is $(0,0)$ since
$f'(0)=0$

at $x=-1$
$f'(-1)$ is negative,
so $(-1, f'(-1))$
is below x-axis

then by inspection,
$f'(2)$ is greater
than $f'(1)$
so $(2, f'(2))$
is higher than
$(1, f'(1))$

now we connect the dots : note between $x=0$ and $x=1$
the tangent line slope <u>increases</u> from 0 to something
greater, and continues doing so (so value of $f'(x)$
keeps going up)

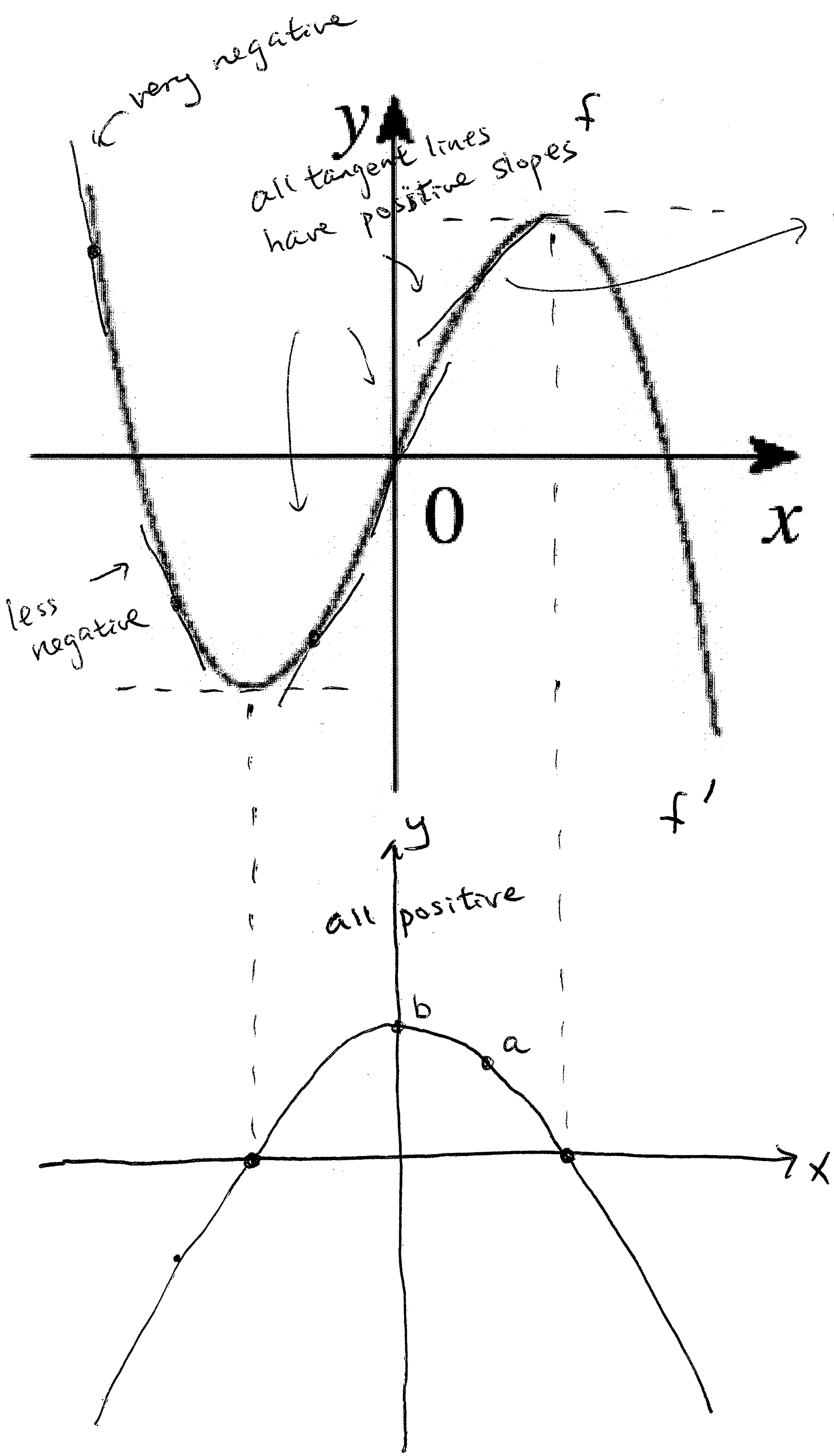
Another example

First, find where slope of tangent line is zero, these are x-intercepts on f' (because $f'=0$)

Now note the middle part has tangent lines with positive slopes

→ so $f'(x)$ is entirely above x-axis in the middle (starts and ends on x-axis)

Left-most portion: the graph above shows all tangent lines have negative slopes, so $f'(x) < 0$ on this interval



not as steep as the one at origin

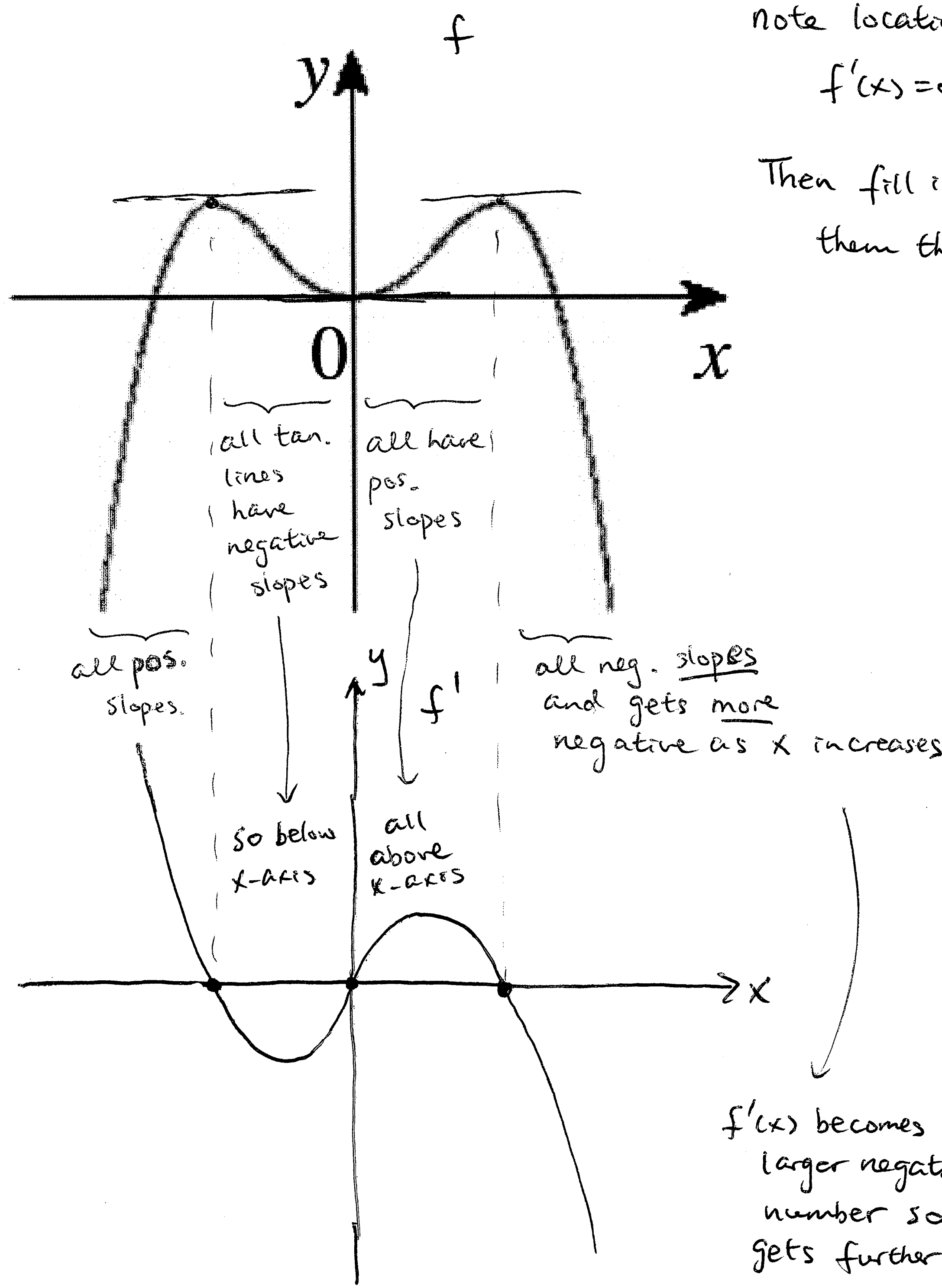
(so point a on bottom graph is lower than point b)

Keep in mind: the bottom graph is tracking value of $f'(x)$ (steepness of tangent line)

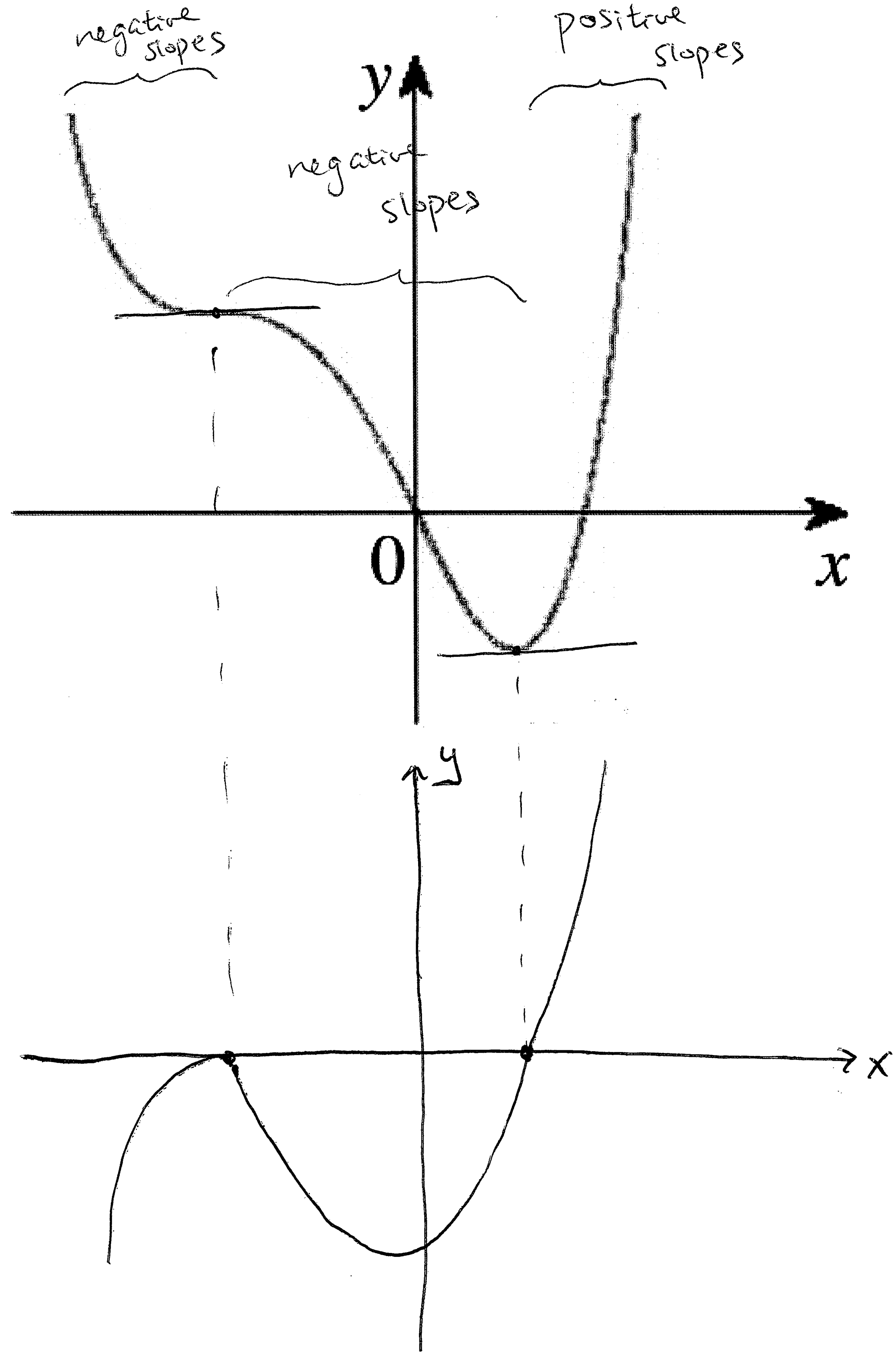
Right-most portion: Same idea as on the left-most portion

note locations of
$f'(x)=0$ first.

Then fill in between
them the f' graph

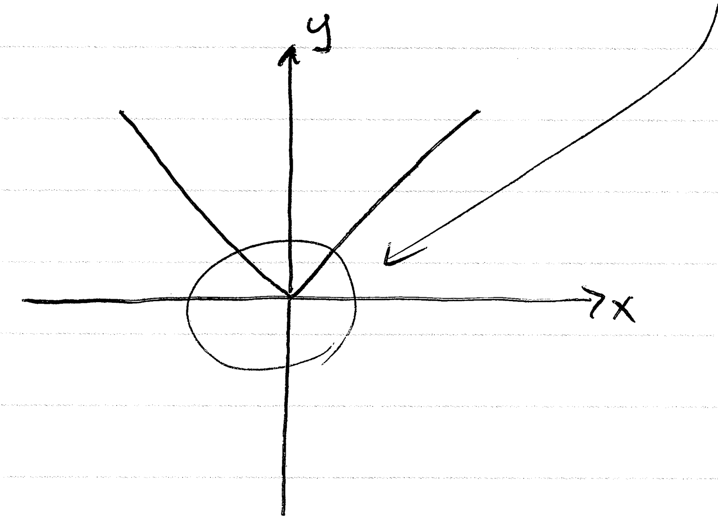


$f'(x)$ becomes
larger negative
number so
gets further below


negative slopes
negative slopes
positive slopes
y
0
x
y
x

If $f'(x)$ exists at a point, then the function is said to be <u>differentiable</u> (<u>to differentiate</u> is another way of saying "find a derivative")

Now take a look at $f(x) = |x|$ at $(0, 0)$



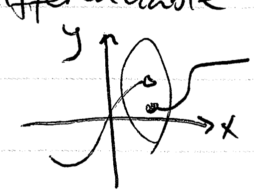
at the sharp corner (cusp), we cannot place the tangent line that has the same slope as the graph

$$\left(\text{also, } \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ DNE}\right)$$

So $f'(x)$ is not defined there → not differentiable

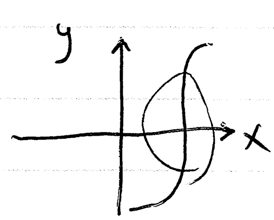
function is also not differentiable at :

1) any discontinuity



3) cusps / corners (see above)

2) vertical tangent (since $f'(x)$ is undefined)



If a function is differentiable, then it is automatically continuous

BUT, the converse is NOT TRUE - NOT all continuous functions are differentiable (see $|x|$)