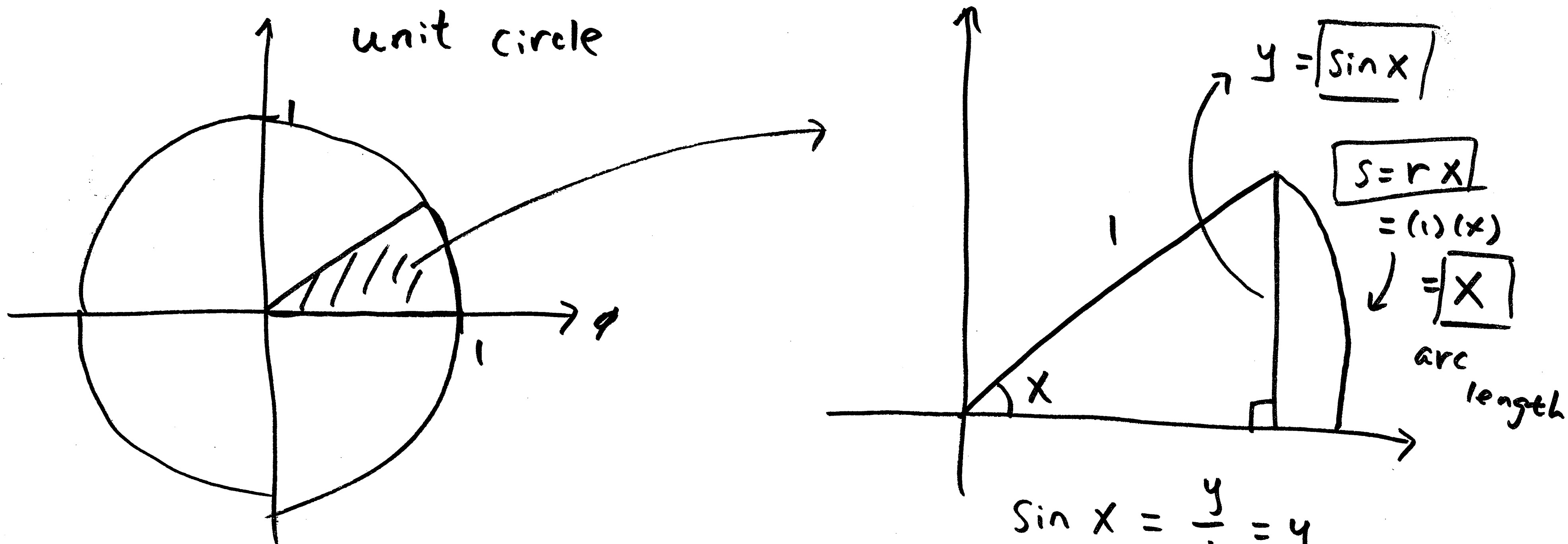
3.3 Derivative of Trig Functions

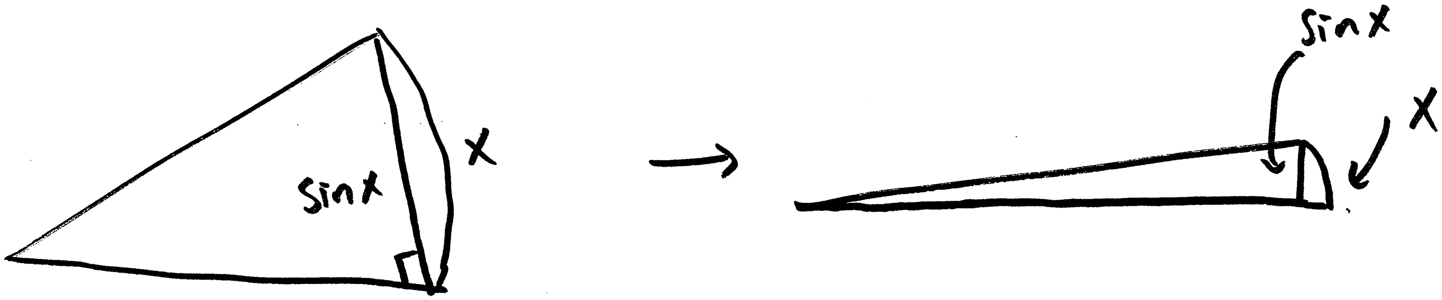
need two special limits

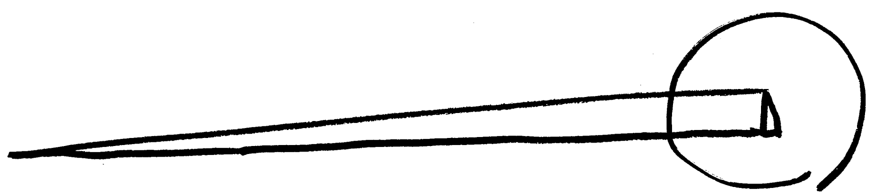
$$\lim_{x \to 0} \frac{\sin x}{x} \to \frac{0}{0} \qquad \lim_{x \to 0} \sin x = 0$$



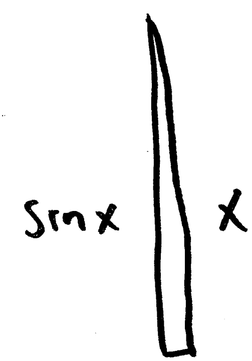
$$\sin x = \frac{y}{1} = y$$

as $x \to 0$, wedge becomes smaller



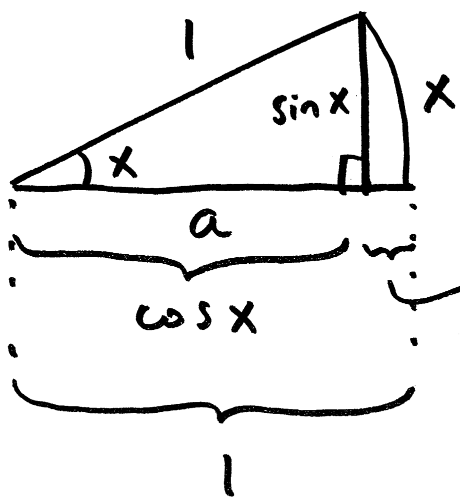


as $x \to 0$ $\quad \sin x \approx x$



$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

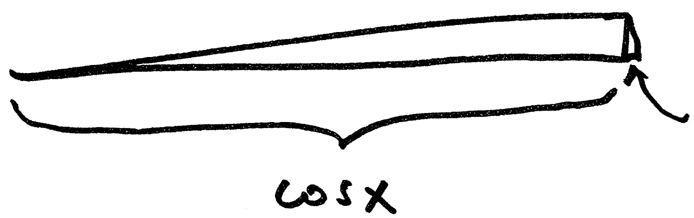
$$\lim_{x\to 0} \frac{\cos x - 1}{x} \to \frac{0}{0}$$



$$\cos x = \frac{a}{1} = a$$

$1 - \cos x$

let $x \to 0$



$1 - \cos x$

↳ goes to zero

$\cos x \approx 1$

$\cos x - 1$ approches zero faster than x so $\boxed{\lim_{x\to 0} \frac{\cos x - 1}{x} = 0}$

$f(x) = \sin x \qquad f'(x) = ?$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \lim_{h \to 0} \frac{\cos(x)\sin(h)}{h}$$

$$= \lim_{h \to 0} \sin(x) \cdot \overbrace{\frac{\cos(h) - 1}{h}}^{\to 0} + \lim_{h \to 0} \cos(x) \overbrace{\frac{\sin(h)}{h}}^{\to 1}$$

$$= \cos(x)$$

so $\boxed{\frac{d}{dx} \sin x = \cos x}$

Similarly, $\boxed{\dfrac{d}{dx}(\cos x) = -\sin x}$

Summary

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

all co-functions have negative derivatives.

recover $\frac{d}{dx} \tan x = \sec^2 x$

by using quotient rule $\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2$$

$$= \boxed{\sec^2 x}$$

example

$$y = \sec(x)\cot(x)$$

product rule

$$y' = \sec(x) \cdot -\csc^2(x) + \cot(x) \cdot \sec(x)\tan(x)$$

$$= \boxed{\sec(x) \cdot [-\csc^2(x) + 1]}$$

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another way: $y = \frac{1}{\cos x} \cdot \frac{\cos x}{\sin x} = \frac{1}{\sin x} = \csc x$

$$y' = \boxed{-\csc x \cot x}$$

these are the same even though they look different

How is $\sec(x) \cdot \underbrace{\left[-\csc^2(x)+1\right]} = -\csc x \cot x$?

$\downarrow$

$-\cot^2 x$ (since $\cot^2 x + 1 = \csc^2 x$)

$-\sec(x)\cot^2(x) = -\csc(x)\cot(x)$

$-\dfrac{1}{\cancel{\cos(x)}} \dfrac{\cos^{\cancel{2}}(x)}{\sin^2(x)}$

$-\dfrac{\cos(x)}{\sin(x)} \cdot \dfrac{1}{\sin(x)}$

$-\cot(x) \cdot \csc(x)$ ✓

example: Find y'' if $y = e^x \sin x$

$$y' = e^x \cos x + e^x \sin x = e^x(\cos x + \sin x)$$

$$y'' = \cancel{-e^x \sin x} + e^x \cos x + e^x \cos x + \cancel{e^x \sin x}$$

$$= \boxed{2e^x \cos x} = 2\cos x\, e^x = \cos x \cdot 2e^x$$

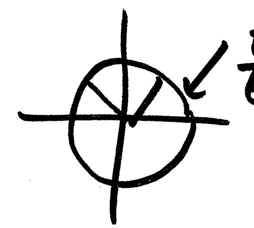
example : Where on the graph of

$y = x + 2\cos x$ is the tangent line horizontal?

$\hookrightarrow \quad y' = 0$

$y' = 1 - 2\sin x = 0$

$\sin x = \frac{1}{2}$



solutions in I, II

$x = \frac{\pi}{6} + 2\pi n$ $x = \frac{5\pi}{6} + 2\pi n$

I   II $n = 0, 1, 2, 3, \ldots$

example:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \to 0} \frac{\sin(5x)}{x} = ?$$

$$\left(\lim_{x \to 0} \frac{\sin(5x)}{(5x)} = \lim_{x \to 0} \frac{\sin y}{y} \quad \text{where } y = 5x \right)$$

$$\lim_{x \to 0} \frac{\sin(5x)}{(5x)} \cdot 5 = \boxed{5}$$