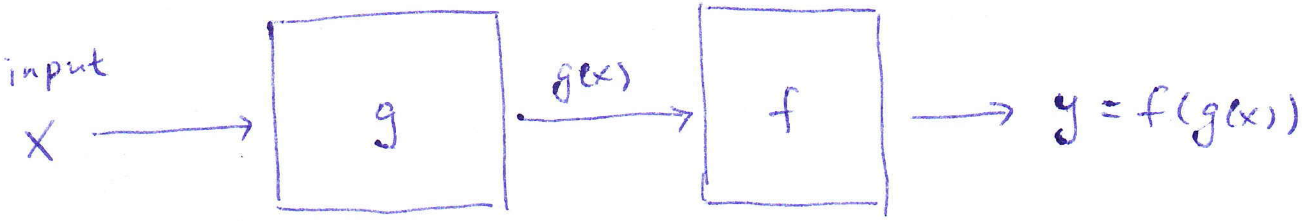
3.4 The Chain Rule (Part One)

Composite function (function of function)

$y = h(x) = f(g(x))$

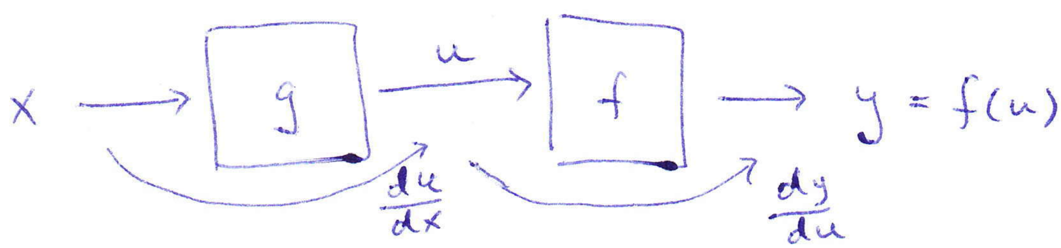
here, $g(x)$ is the input to $f(x)$.



We want to find derivative of y with respect to x

$\frac{dy}{dx}$ → how is y affected by a change in x?

see schematic again, let $u = g(x)$ this time



first, note y is affected by u, so we need to account for $\frac{dy}{du}$

then take into account of how u is affected by x : $\frac{du}{dx}$

now we "chain" these together by multiplying

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

note, if we could "cancel" du on the right, we will get $\frac{dy}{dx}$ back.

in prime notation with $u = g(x)$

$$\boxed{y' = f'(g(x))\, g'(x)}$$

example:

$$y = \sqrt[3]{1+4x}$$

this could be seen as $f(g(x))$ or $f(u)$

with $u = 1+4x$ and $f(u) = \sqrt[3]{u}$

check: if $u = 1+4x$ and $f(u) = \sqrt[3]{u}$

then $f(u) = f(1+4x) = \underbrace{\sqrt[3]{1+4x} = y}_{\text{what was given}}$

find $\frac{dy}{dx}$: rewrite: $y = (1+4x)^{1/3} = u^{1/3}$ $u = 1+4x$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Since $y = u^{4/3}$, $\frac{dy}{du} = \frac{1}{3} u^{-2/3}$

$u = 1+4x$ $\qquad \frac{du}{dx} = 4$

so $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{3} u^{-2/3} \cdot 4$

but $u = 1+4x$, so

$$\boxed{\frac{dy}{dx} = \frac{4}{3}(1+4x)^{-2/3} = \frac{4}{3(1+4x)^{2/3}}}$$

example

$y = (x^2-5x+10)^3$

here, $u = x^2-5x+10$ ("inside" function)

$y = f(u) = u^3$ ("outside function)

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2 \cdot (2x-5)$$

$$= \boxed{3(x^2-5x+10)^2(2x-5)}$$

Note: the two examples above demonstrate

that $\frac{d}{dx}(\square)^n = n(\square)^{n-1}\frac{d}{dx}(\square)$

$$\frac{d}{dx}\left[(1+4x)^{1/3}\right] = \underbrace{\frac{1}{3}(1+4x)^{-2/3}}_{\substack{\text{like the}\\ \text{usual power rule}\\ \text{with } 1+4x \text{ playing}\\ \text{the role of "x"}}}(4)$$

← deriv. of $1+4x$

<u>example</u> $y = \sin(9x)$

as usual, recognize $u = 9x$ (inside)

$y = \sin u$ (outside)

then $y' = \frac{dy}{du}\frac{du}{dx} = \cos u \cdot 9 = \boxed{9\cos(9x)}$

so the rule you knew: $\frac{d}{dx}\sin x = \cos x$

is generalized as $\frac{d}{dx}\sin u = \cos u \cdot \frac{du}{dx}$

with u being any differentiable function

→ true for rules of other trig functions too!

now we know the chain rule, we can have more options

example: $y = \dfrac{1}{(x^2+1)^2}$

we could use the quotient rule

d-low

$$y' = \frac{(x^2+1)^2(0) - (1)\boxed{(2)(x^2+1)(2x)}}{(x^2+1)^4}$$

$$= \frac{4x(x^2+1)}{(x^2+1)^4} = \boxed{\frac{4x}{(x^2+1)^3}}$$

or, we could rewrite first:

$$y = (x^2+1)^{-2}$$

$$y' = (-2)(x^2+1)^{-3}(2x)$$

$$= \boxed{-4x(x^2+1)^{-3}}$$

Same

If numerator is a constant, rewrite first is always easier recall e is special since $\frac{d}{dx}(e^x) = e^x$

but what about e^{2x} ?

$y = e^{2x}$ is equivalent to $u = 2x$
$y = e^u$

so $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \cdot 2 = 2e^u = 2e^{2x}$

thus: $\boxed{\frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}}$

Now let's see what we can do about

$y = a^x$ where $a \neq e$ (e.g. $y = 2^x$)

note $y = a^x = e^{\ln a^x}$ since $e^{\ln x} = x$

rewrite: $y = e^{\ln a^x} = e^{x \ln a}$ recall $\ln a^x = x \ln a$

$= e^{(\ln a)x}$

here we now have $u = (\ln a)x$
$y = e^u$

so $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

$= e^{u} \cdot \ln a$

Keep in mind $\ln a$ is a <u>number</u>, so $\dfrac{d}{dx}\left[(\ln a)x\right] = \ln a$

(like $\dfrac{d}{dx}(2x) = 2$)

but $u = (\ln a)x$

$\dfrac{dy}{dx} = e^{(\ln a)x} \cdot \ln a$ $\quad \left(e^{\ln a^x} = e^{x(\ln a)} = e^{(\ln a)x}\right)$

$= e^{\ln a^x} \cdot \ln a$ $\quad \left(e^{\ln a^x} = a^x\right)$

$= a^x \cdot \ln a$

~~in general~~ summary: $\boxed{\dfrac{d}{dx}(a^x) = (\ln a)\,a^x}$

↑ this formula works even if $a = e$ since $\ln e = 1$

generalize: $\boxed{\dfrac{d}{dx}(a^u) = (\ln a)\,a^u \cdot \dfrac{du}{dx}}$

<u>example</u>: $y = 9^{2x^2-1}$

$\boxed{y' = (\ln 9)\,9^{2x^2-1} \cdot (4x)}$