3.6 Derivatives of Log Functions

$\frac{d}{dx}(\log_a x) = ?$ $\longrightarrow$ want y'

let $y = \log_a x$

then $x = a^y$

differentiate implicitly

$$\frac{d}{dx}(x) = \frac{d}{dx}(a^y)$$

$$1 = \ln a \cdot a^y \cdot y'$$

so $y' = \dfrac{1}{\ln a \cdot a^y} = \dfrac{1}{\ln a \cdot x}$

$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{\ln a \cdot x}}$$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

$$\boxed{\frac{d}{dx}(\ln u) = \frac{1}{u}u'}$$

<u>example</u> : Find $f'(1)$ if $f(x) = \ln(3+e^{12x})$

$$f'(x) = \frac{1}{3+e^{12x}} \cdot \frac{d}{dx}(3+e^{12x})$$

$$= \frac{1}{3+e^{12x}} \cdot (e^{12x} \cdot 12)$$

$$f'(x) = \frac{12e^{12x}}{3+e^{12x}}$$

$$f'(1) = \boxed{\frac{12e^{12}}{3+e^{12}}}$$

example: Find the equation of the tangent line to $y = \ln(x^2 - 4x + 1)$ at $(4, 0)$

find slope: $y' = \frac{1}{x^2 - 4x + 1} \cdot (2x - 4)$

at $x = 4$, $y' = \frac{2(4) - 4}{(4)^2 - 4(4) + 1} = 4$

line eq.: $y - y_1 = m(x - x_1)$

$y - 0 = 4(x - 4)$

$\boxed{y = 4x - 16}$

What is the derivative $y = x^x$

(power rule only applies to x^n where n is a constant)

use <u>logarithmic differentiation</u>

$$y = x^x$$

take ln on both side

$$\ln y = \ln x^x = x \cdot \ln x$$

find y' be implicit differentiation

$$\frac{1}{y} y' = x \cdot \frac{1}{x} + \ln x \cdot 1$$

$$= 1 + \ln x$$

$$y' = y(1+\ln x) \quad \text{but } y = x^x$$

so $\boxed{y' = x^x(1+\ln x)}$

example:

$$y = x^{\cos x}$$

$$\ln y = \ln x^{\cos x} = \cos x \cdot \ln x$$

$$\frac{1}{y} y' = \cos x \cdot \frac{1}{x} + \ln x \cdot -\sin x$$

$$\boxed{y' = x^{\cos x}\left(\frac{\cos x}{x} - \ln x \cdot \sin x\right)}$$

<u>example</u> : $y = (x^5+x)^2 (x^3+4)^4$

product + chain

$$y' = (x^5+x)^2 \cdot 4(x^3+4)^3 (3x^2) + (x^3+4)^4 \cdot 2(x^5+x)(5x^4+1)$$

---

can use log differentiation, too

$$y = (x^5+x)^2 (x^3+4)^4$$

$$\ln y = \ln\left[(x^5+x)^2 (x^3+4)^4\right]$$

$$= \ln (x^5+x)^2 + \ln (x^3+4)^4$$

$$\ln y = 2 \ln (x^5+x) + 4 \ln (x^3+4)$$

$$\frac{1}{y} y' = 2 \cdot \frac{1}{x^5+x} \cdot (5x^4+1) + 4 \cdot \frac{1}{x^3+4} \cdot 3x^2$$

$$\frac{1}{y} y' = \frac{10x^4+2}{x^5+x} + \frac{12x^2}{x^3+4}$$

$$\boxed{y' = (x^5+x)^2(x^3+4)^4\left(\frac{10x^4+2}{x^5+x} + \frac{12x^2}{x^3+4}\right)}$$

<u>example</u> $y = \sqrt{\dfrac{x-1}{x^8+1}} = \left(\dfrac{x-1}{x^8+1}\right)^{1/2}$

w/o log diff

$$y' = \frac{1}{2}\left(\frac{x-1}{x^8+1}\right)^{-1/2} \cdot \frac{d}{dx}\left(\frac{x-1}{x^8+1}\right)$$

$$= \frac{1}{2}\left(\frac{x-1}{x^8+1}\right)^{-1/2} \cdot \frac{(x^8+1)(1) - (x-1)(8x^7)}{(x^8+1)^2}$$

<u>example</u> $$y = \left(\frac{x-1}{x^8+1}\right)^{1/2}$$

with log diff this time

$$\ln y = \ln\left(\frac{x-1}{x^8+1}\right)^{1/2}$$

$$\ln y = \frac{1}{2}\ln\left(\frac{x-1}{x^8+1}\right) = \frac{1}{2}\left(\ln(x-1) - \ln(x^8+1)\right)$$

$$\frac{1}{y}y' = \frac{1}{2}\left(\frac{1}{x-1}\cdot 1 - \frac{1}{x^8+1}\cdot(8x^7)\right)$$

$$\boxed{y' = \frac{1}{2}\left(\frac{x-1}{x^8+1}\right)^{1/2}\left(\frac{1}{x-1} - \frac{8x^7}{x^8+1}\right)}$$