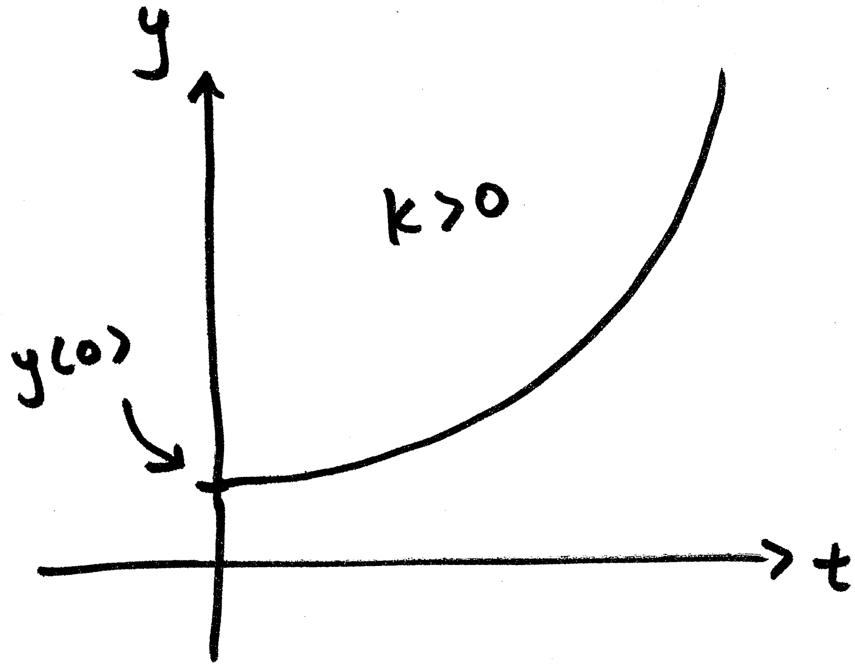
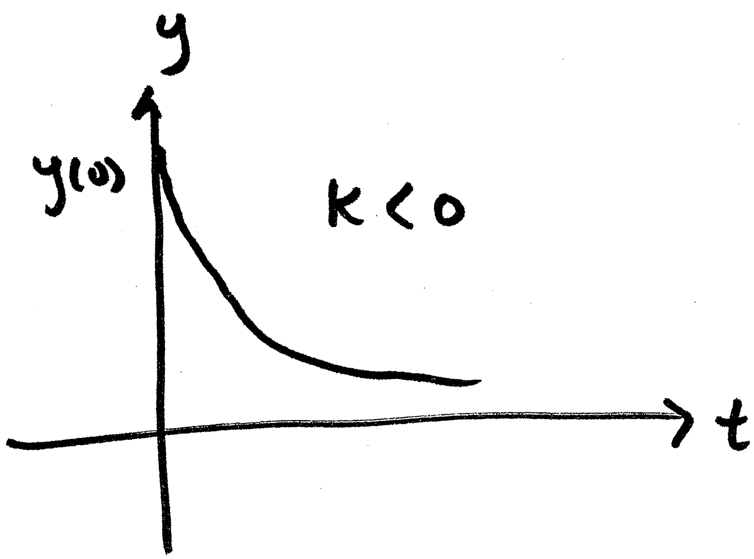
3.8 Exponential Growth / Decay



$$y(t) = \underbrace{y(0)}_{\text{initial amount}} e^{\overbrace{k}^{}t}$$

initial amount

relative growth/decay constant (rate)



$k > 0 \rightarrow$ growth

$k < 0 \rightarrow$ decay

$$y(t) = y(0)\,e^{kt}$$

$$\frac{dy}{dt} = \underbrace{y(0)\,e^{kt}}\cdot k \qquad \text{but} \quad y(0)\,e^{kt} = y$$

$$\boxed{\frac{dy}{dt} = ky}$$

rate of change is
some multiple of ~~current~~
size / population

A common inhabitant of human intestines is the bacterium *Escherichia coli*. A cell of this bacterium in a nutrient-broth medium divides into two cells every 20 minutes. The initial population of a culture is 60 cells.

(a) Find the relative growth rate.

(b) Find an expression for the number of cells after t ~~hours~~.

minutes

$$\frac{dP}{dt} = kP \longleftrightarrow \boxed{P(t) = P(0)e^{kt}}$$

$P(0) = 60$

divides into 2 every 20 minutes

$P(0) = 60$ $\boxed{P(20) = 120}$

sub in

t in minutes

a). $120 = 60e^{k \cdot 20}$

$2 = e^{20k}$

$\ln 2 = \ln e^{20k} = 20k$

$$k = \frac{\ln 2}{20} \approx \underbrace{0.0347}_{\text{relative growth rate (3.47\%)}}$$

b). $\boxed{P(t) = 60e^{0.0347t}}$

alternatively. $k = \frac{\ln 2}{20}$

$$p(t) = 60\, e^{\frac{\ln 2}{20} t}$$

$$= 60 \left(e^{\ln 2}\right)^{\frac{1}{20} t}$$

$$= \boxed{60\, (2)^{t/20}}$$

time to double

doubling

$\frac{dP}{dt} = kP$

A bacteria culture grows with constant relative growth rate. The bacteria count was 400 after 2 hours and 25,600 after 6 hours.

$k = ?$

(a) What is the relative growth rate? Express your answer as a percentage.

(b) What was the intitial size of the culture?

(c) Find an expression for the number of bacteria after t hours.

(d) Find the number of cells after 4.5 hours.

(e) Find the rate of growth after 4.5 hours.

(f) When will the population reach 50,000?

a). $p(t) = p(0)e^{kt}$ t in hours

$p(2) = 400 = p(0)e^{2k} \rightarrow$ ①

$p(6) = 25600 = p(0)e^{6k} \rightarrow$ ②

divide ② by ①

$$\frac{25600}{400} = \frac{\cancel{p(0)}e^{6k}}{\cancel{p(0)}e^{2k}} = e^{4k}$$

$64 = e^{4k}$ $\ln 64 = \ln e^{4k} = 4k$

$$k = \frac{\ln 64}{4} = \boxed{1.0397}$$

103.97 %

b). $K = 1.0397$

$p(t) = p(0)\, e^{1.0397t}$ $\quad$ $p(2) = 400$

$400 = P(0)\, e^{1.0397(2)}$

$$P(0) = \frac{400}{e^{1.0397(2)}} = \boxed{50}$$

c). $P(t) = 50\, e^{1.0397t}$

d). $P(4.5) = 50 e^{1.0397(4.5)} = 5381$ (rounded)

e). take a deriv. to find $p'(t)$

or $\frac{dp}{dt} = kp \longrightarrow$ at $t = 4.5$ $\frac{dp}{dt} = 1.0397 \cdot 5381$

$= \boxed{5595}$

Strontium-90 has a half-life of 28 days.

(a) A sample has a mass of 50 mg initially. Find a formula for the mass remaining after t days.

(b) Find the mass remaining after 40 days.

(c) How long does it take the sample to decay to a mass of 2 mg?

(d) Sketch the graph of the mass function.

half-life: time to lose half of initial amount half-life 28 days t: in days

a). $m(t) = m(0)\, e^{kt}$

$m(t) = 50\, e^{kt}$

$25 = 50\, e^{28k}$ $\frac{1}{2} = e^{28k}$

$\ln \frac{1}{2} = 28k$ $k = -0.02476$

$$\boxed{m(t) = 50\, e^{-0.02476t}}$$

b). $m(40) = 50\, e^{-0.02476(40)} = 19$

c~~d~~). $2 = 50\, e^{-0.02476t}$

$\frac{1}{25} = e^{-0.02476t}$ $\ln \frac{1}{25} = -0.02476t$

$t = \boxed{130}$ days

Newton's Law of Cooling

$$\frac{dT}{dt} = k\,(T - T_s)$$

temp. of surrounding

T: temp

t: time

big difference → fast rate

$$\boxed{T(t) = [T(0) - T_s]\,e^{kt} + T_s}$$

In a murder investigation, the temperature of the corpse was 32.5°C at 1:30 PM and 30.3°C an hour later. Normal body temperature is 37.0°C and the temperature of the surroundings was 20.0°C. When did the murder take place?

$T(t) = [T(0) - T_s]\, e^{kt} + T_s$

$t = 0 \rightarrow$ 1:30 pm $\quad T(0) = 32.5$

$T(1) = 30.3$

find t when $T = 37$

$T_s = 20$

$T(t) = (32.5 - 20)\, e^{kt} + 20$

$30.3 = 12.5\, e^{k} + 20 \longrightarrow k = -0.19358$

$37 = 12.5\, e^{-0.19358t} + 20 \longrightarrow \underbrace{t = -1.6}_{\text{1.6 hrs before 1:30 pm}}$

$\rightarrow$ 11:54 AM