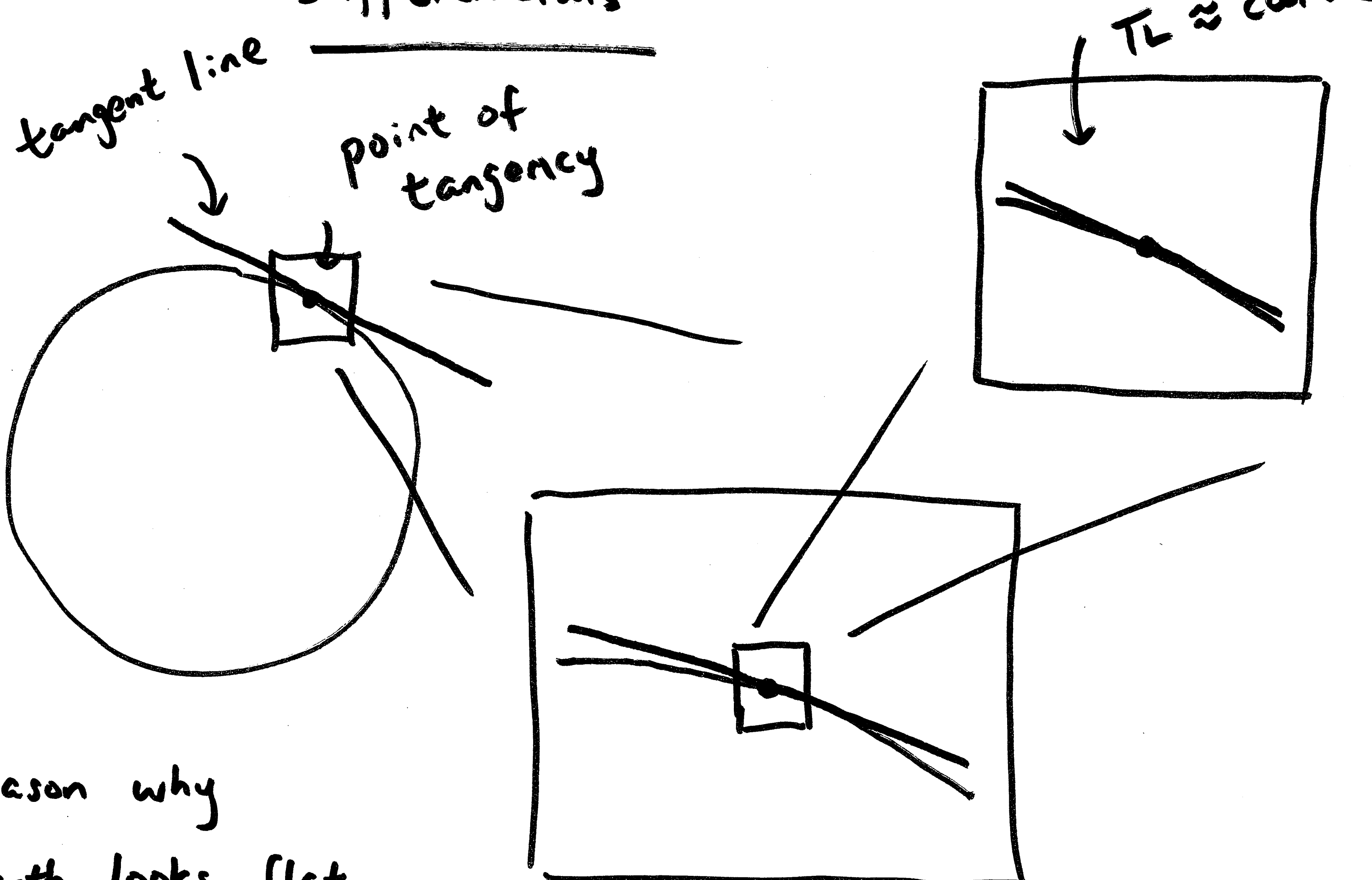
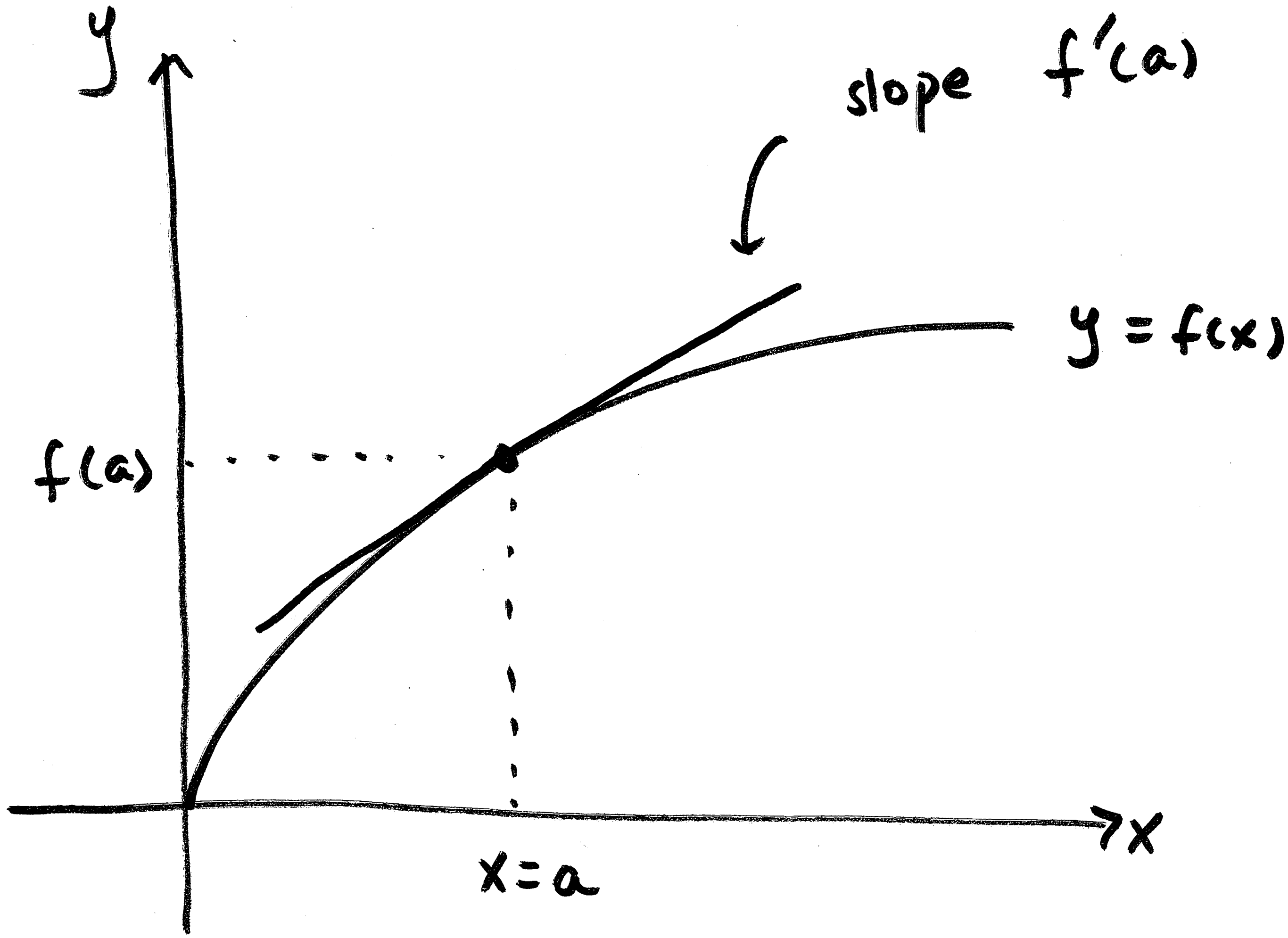
3.10 Linear Approximation and Differentials



Reason why Earth looks flat even though it is round.



equation of tangent line

$y - f(a) = f'(a)(x-a)$

slope $= f'(a)$

point : $(a, f(a))$

$$y = f(a) + f'(a)(x-a) \approx f(x) \text{ if } x \text{ is near } a$$

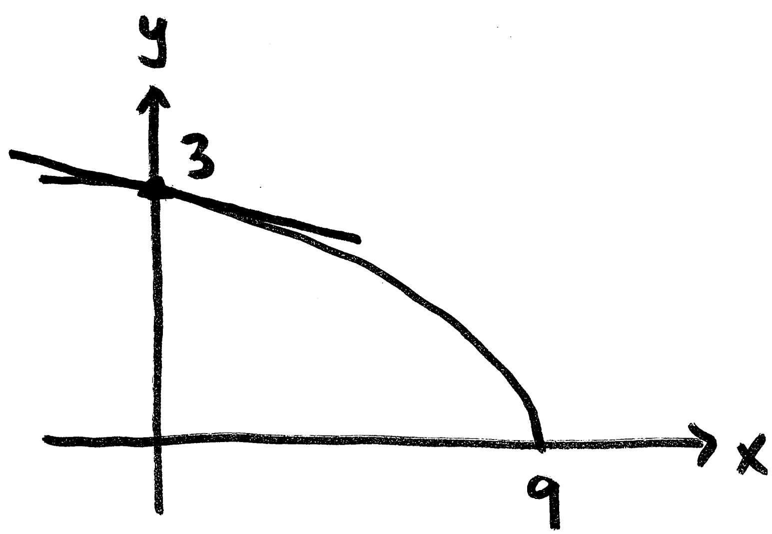
Linear / Tangent Line Approximation

$$L(x) = f(a) + f'(a)(x-a)$$

near $x = a$, can use $L(x)$ in place of (often more complicated) $f(x)$

example Find the linear approx. of $f(x) = \sqrt{9-x}$ near $a = 0$

$\sqrt{x}$



$$f(x) = \sqrt{9-x} = (9-x)^{1/2} \qquad a = 0$$

$$f(a) = f(0) = \sqrt{9} = 3$$

$$f'(x) = \frac{1}{2}(9-x)^{-1/2}(-1) = \frac{-1}{2\sqrt{9-x}}$$

$$f'(a) = f'(0) = \frac{-1}{2\sqrt{9}} = -\frac{1}{6}$$

$$L(x) = f(a) + f'(a)(x-a)$$

$$\boxed{L(x) = 3 - \frac{1}{6}x} \approx \sqrt{9-x} \quad \underline{\underline{\text{near}}}\ x = 0$$

at $x = 0.1$ $\qquad L(0.1) = 3 - \frac{1}{6}(0.1) = 2.9833$

true value (calculator) $\sqrt{8.9} = 2.983287$

at $x = 5$ $\qquad L(5) = 2.1667$

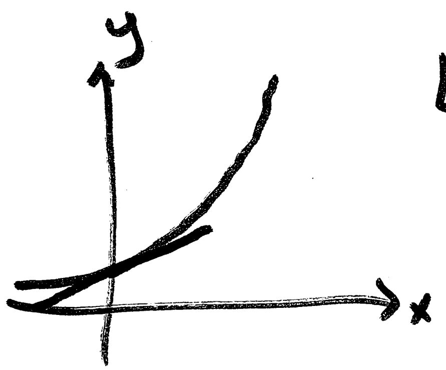
true value $\sqrt{9-5} = \sqrt{4} = 2$

example estimate $e^{0.015}$

take advantage of $e^0 = 1$

Since 0.015 is near 0, build an approx. at $x=0$ and estimate $e^{0.015}$

$f(x) = e^x \qquad a = 0$



$L(x) = f(a) + f'(a)(x-a)$

$f(a) = e^0 = 1$

$f'(x) = e^x$

$f'(a) = e^0 = 1$

$L(x) = 1 + x$

$e^{0.015} \approx L(0.015) = 1.015$

true value (calculator) : $e^{0.015} = 1.015113$

example

estimate $\sqrt{99.9}$

we know $\sqrt{100} = 10$

$f(x) = \sqrt{x}$ $\quad\quad a = 100$

$L(x) = f(a) + f'(a)(x-a)$

$f(a) = \sqrt{100} = 10$

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ $\quad\quad f'(a) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$

$L(x) = 10 + \frac{1}{20}(x-100)$

$$\boxed{L(x) = 5 + 0.05x}$$

$\sqrt{99.9} \approx 5 + 0.05(99.9) \approx 9.995$

true $\sqrt{99.9} = 9.9949987$

Differential

$$y = f(x)$$

$$\frac{dy}{dx} = f'(x)$$

treat $\frac{dy}{dx}$ like a fraction

multiply both sides by dx

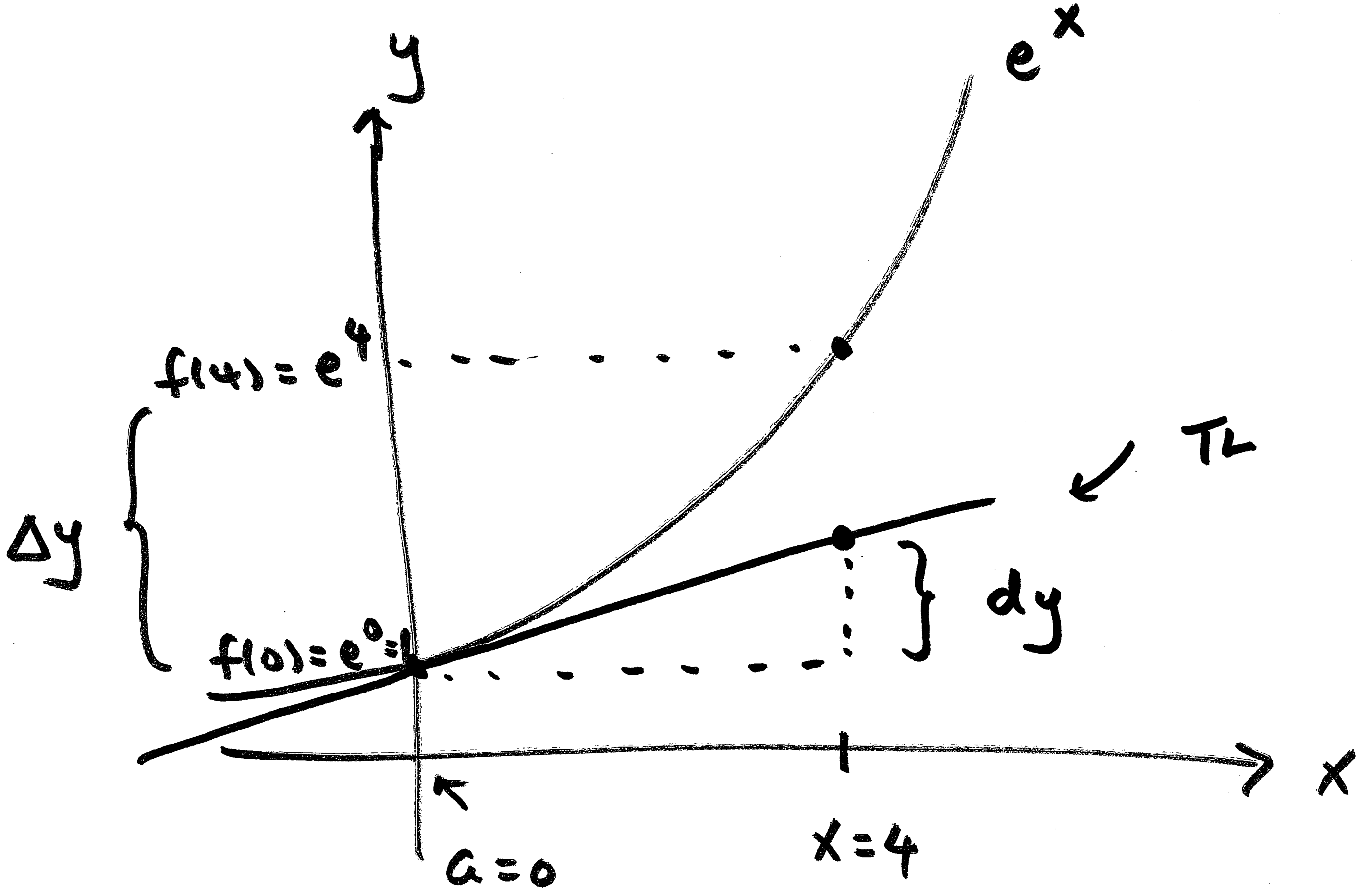
$$\boxed{dy = f'(x)\,dx}$$

"differential"

example: $f(x) = \sqrt{x} = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$$\boxed{dy = \frac{1}{2}x^{-1/2}\,dx}$$



$\Delta y = f(4) - f(0) = e^4 - 1 \nearrow$ true

$dy = L(4) - L(0) \longrightarrow$ tangent line approx.

$\Delta y \approx dy$ if x is <u>near</u> a

(Δx is <u>small</u>)

<u>example</u> Use the differential to estimate $e^{0.015}$

We know $e^0 = 1$ start at $x = 0$

$\Delta y = e^{0.015} - e^0 = e^{0.015} - 1$

use dy to estimate Δy

$y = e^x$

$\frac{dy}{dx} = e^x$

$dy = e^{x} dx$

$= e^0 (0.015) = 0.015$

$dx = \Delta x = 0.015 - 0$

$= 0.015$

x = starting value of x (same as "a")

$= 0$

$dy = e^{0.015} - 1 = 0.015$

$$\boxed{e^{0.015} = 1.015}$$