4.2 The Mean Value Theorem

local max/min $\rightarrow$ occur at $f'(c) = 0$

or $f'(c)$ DNE

(c : critical number)

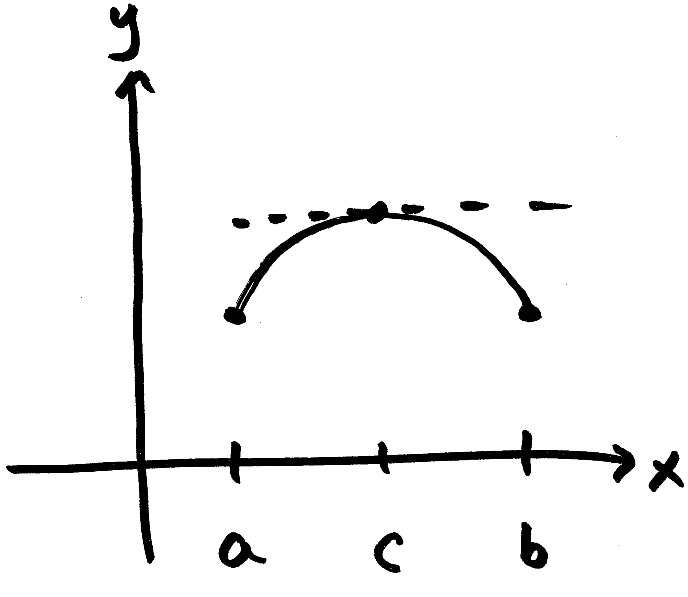
How do we know these c's exist?

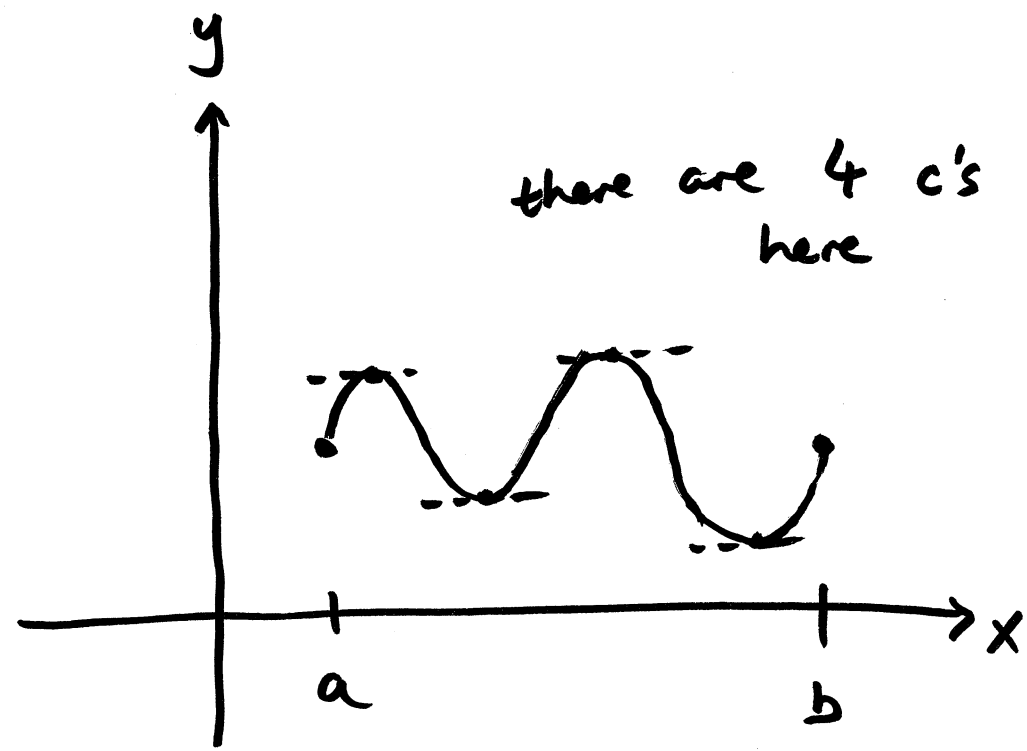
Rolle's Theorem (Michel Rolle 1652-1719)

Suppose f satisfies the following

1. continuous on $[a, b]$
2. differentiable on (a, b)
3. $f(a) = f(b)$

Then there $\underbrace{\text{is a}}_{\text{at least}}$ c in (a, b) where $f'(c) = 0$


y
x
a
c
b


y
there are 4 c's here
x
a
b

example $f(x) = 5 - 48x + 4x^2$ $[5, 7]$

verify $f(x)$ satisfies Rolle's Theorem and find c

1. continuous on $[5, 7]$

   yes, a polynomial

2. differentiable on $(5, 7)$

   yes, derivative of a polynomial is a polynomial, so deriv. defined everywhere

3. $f(5) = f(7)$

   $f(5) = 5 - 48(5) + 4(5)^2$

   $f(7) = 5 - 48(7) + 4(7)^2$

   yes

Rolle's Theorem says there is a c

so $f'(c) = 0$

$$f'(x) = -48 + 8X = 0 \longrightarrow c = 6$$

a generalization of Rolle's Theorem is the <u>Mean Value Theorem</u>

<u>Mean Value Theorem</u> (Joseph-Louis Lagrange 1736-1813)

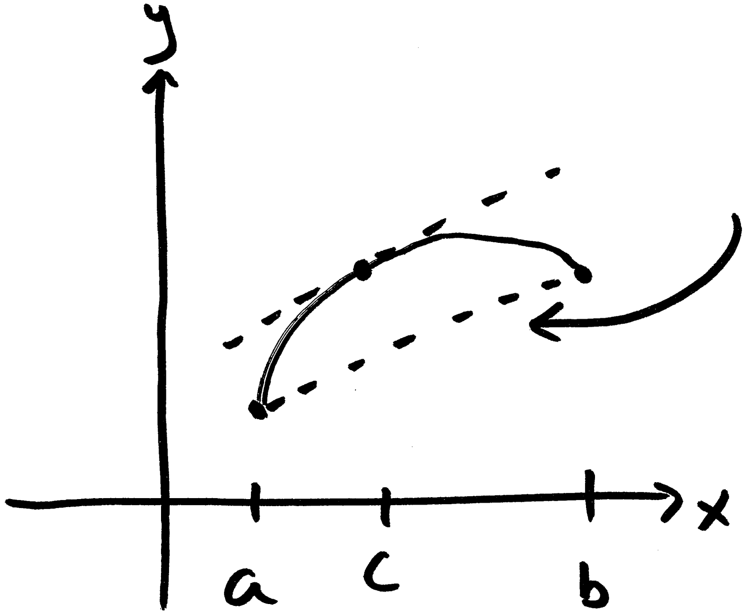
$f(x)$ satisfies

1. continuous on $[a, b]$
2. differentiable on (a, b)

then there exists at least a c where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

slope: $\dfrac{y_2 - y_1}{x_2 - x_1}$



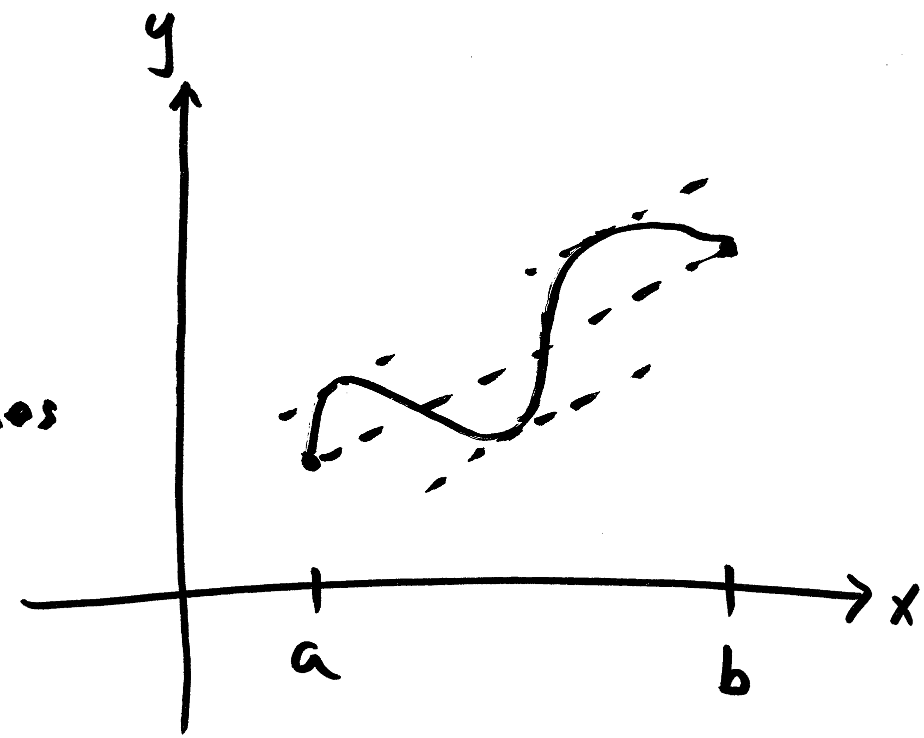
has slope $\frac{f(b)-f(a)}{b-a}$

what happens if

$f(a)=f(b)$

$\rightarrow$ this theorem becomes

Rolle's



<u>example</u> $f(x) = \sqrt[3]{x}$ on $[0, 27]$

1. continuous on $[0, 27]$?

yes, $\sqrt[3]{x}$ is defined on $(-\infty, \infty)$

2. differentiable on $(0, 27)$

$$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}} = \frac{1}{3(x^{1/3})^2}$$

yes, $f'(0)$ DNE but we are not going to 0

$$\frac{f(27) - f(0)}{27 - 0} = \frac{3}{27} = \frac{1}{9}$$

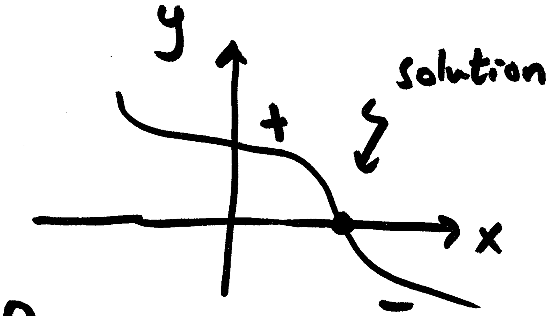
$f'(x) = \frac{1}{9}$ somewhere on $(0, 27)$

$$\frac{1}{3x^{2/3}} = \frac{1}{9} \qquad x^{2/3} = 3 \qquad x^2 = 27$$

$$\boxed{x = \sqrt{27} = c}$$

<u>example</u> Show that $2x + \cos x = 0$ has exactly one real solution.

There is a solution, then prove it is the <u>only</u> solution



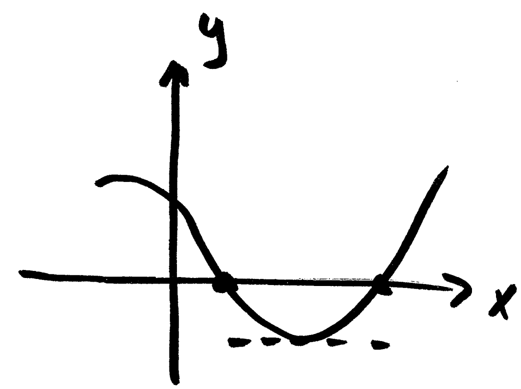
let $f(x) = 2x + \cos x$

$f(\pi) = 2\pi + \overset{-1}{\bullet} = 2\pi - 1 > 0$

$f(-\pi) = -2\pi\ \overset{-1}{\bullet} = -2\pi < 0$

so there is a # between $-\pi$ and π where $f(x)$ crosses x-axis

<u>If</u> there are two solutions, then by Rolle's, somewhere $f'(x) = 0$



$f'(x) = 2 - \sin x$

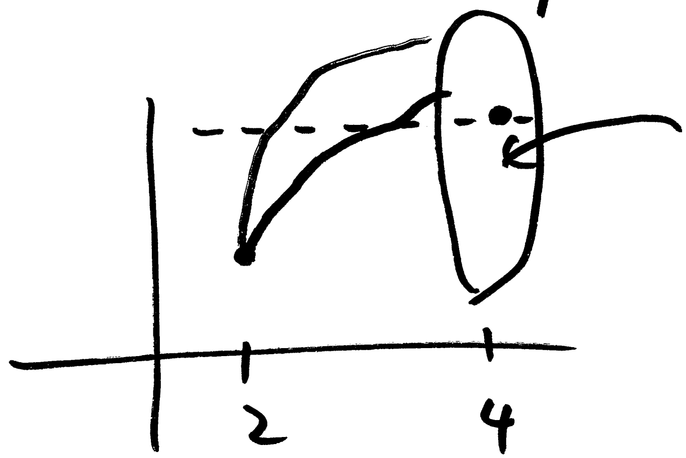
if $f' = 0$, then there is at least one more solution

can it ever be zero? NO! because

$-1 \le \sin x \le 1$

so $f'(x) > 0$ for all x

$\rightarrow$ by Rolle's no other solutions example If $f(2)=15$ and $f'(x) \geq 2$ for $2 \leq x \leq 4$, how small can $f(4)$ possibly be?



use Mean Value Theorem

$$f'(c) = \frac{f(b)-f(a)}{b-a} \geq 2$$

$$\frac{f(4)-15}{4-2} \geq 2$$

$$\boxed{f(4) \geq 19}$$