4.4 Indeterminate Forms and l'Hospital's Rule

recall $\lim_{x \to 0} \frac{\sin x}{x}$ appears to go to $\frac{0}{0}$ ← indeterminate form

we used geometry to show $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$$\lim_{x \to \infty} \frac{2x^2+5x-15}{1x^2+10}$$ appears to go to $\frac{\infty}{\infty}$

$= 2$

l'Hospital's Rule

(Guillaume de l'Hospital 1661-1704)

really should be called Bernoulli's Rule

(Johann Bernoulli 1667-1748)

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad \text{if} \quad \lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

$$\frac{0}{0} \text{ or } \frac{\infty}{\infty} \quad \text{or} \quad \lim_{x \to a} f(x) = \infty \quad \text{and} \quad \lim_{x \to a} g(x) = \infty$$

$$\text{then} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

(l'Hôpital)

$$\lim_{x\to 0} \frac{\sin x}{x} \to \frac{0}{0}$$ so l'Hospital's Rule applies

$$\overset{L}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1$$

$$\lim_{x\to\infty} \frac{2x^2+5x-15}{x^2+10} \to \frac{\infty}{\infty}$$ so l'Hospital's ok to use

$$\overset{L}{=} \lim_{x\to\infty} \frac{4x+5}{2x} \to \frac{\infty}{\infty}$$ l'Hospital again

$$\overset{L}{=} \lim_{x\to\infty} \frac{4}{2} = 2$$

one-sided limits are ok, too

__example__

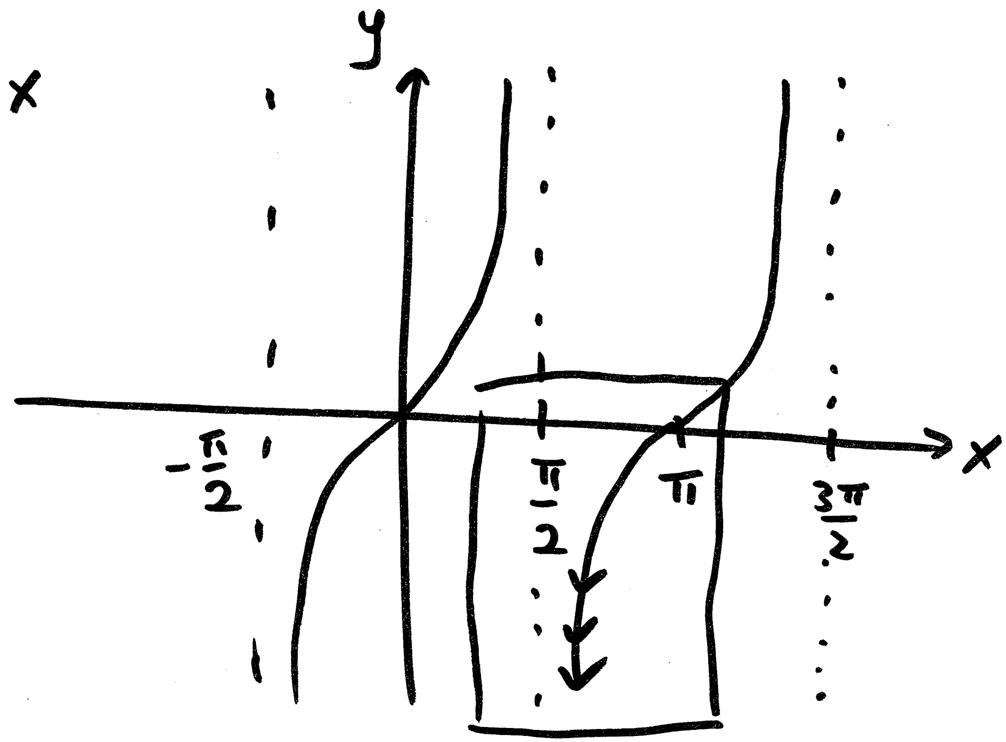
$$\lim_{x \to \left(\frac{\pi}{2}\right)^+} \frac{\cos x}{1 - \sin x} \to \frac{0}{0}$$

$$\overset{L}{=} \lim_{x \to \left(\frac{\pi}{2}\right)^+} \frac{-\sin x}{-\cos x} \to \frac{1}{0}$$

l'Hospital's __NOT__ applicable

$$= \lim_{x \to \left(\frac{\pi}{2}\right)^+} \tan x$$

$$= -\infty$$



example

$$\lim_{x \to 0} \frac{\sin 2x}{\tan 5x} \to \frac{0}{0}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

$$\overset{L}{=} \lim_{x \to 0} \frac{2 \cos 2x}{5 \sec^2 5x} \to \frac{2}{5}$$

$$= \boxed{\frac{2}{5}}$$

<u>example</u> $\lim_{x \to \infty} x \sin\left(\frac{1}{x}\right) \to \infty \cdot 0$ indeterminate product

NOT actually the number zero (is a limit)

Somehow transform to $\frac{\infty}{\infty}$ or $\frac{0}{0}$

change $\sin\left(\frac{1}{x}\right)$ to $\frac{1}{\csc\left(\frac{1}{x}\right)}$

~~$\lim_{x \to \infty} x \cdot \csc($~~

$$\lim_{x \to \infty} \frac{x}{\csc\left(\frac{1}{x}\right)} \to \frac{\infty}{\infty} \quad \text{ok}$$

$$\longrightarrow \lim_{x \to \infty} \frac{1}{\frac{1}{x}} \sin\left(\frac{1}{x}\right) = \lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \to \frac{0}{0}$$

$$\lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \quad x^{-1}$$

$$\lim_{x\to\infty} \frac{x}{x} = 1$$

$$\stackrel{L}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\cdot \cancel{-x^{-2}}}{\cancel{-x^{-2}}}$$

if things cancel, cancel BEFORE using l'Hospital's

$$= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \boxed{1}$$

example

$$\lim_{x \to 0} (\csc x - \cot x) \to \infty - \infty$$ indeterminate difference

$\csc x = \dfrac{1}{\sin x}$, $\cot x = \dfrac{\cos x}{\sin x}$

Somehow turn into $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$

$$\lim_{x \to 0} \sin x = 0$$

$$\lim_{x \to 0} \frac{1}{\sin x} = \infty$$

$$= \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right)$$

$$= \lim_{x \to 0} \left(\frac{1 - \cos x}{\sin x} \right) \to \frac{0}{0}$$

$$\overset{L}{=} \lim_{x \to 0} \frac{\sin x}{\cos x} = 0$$

example

$$\lim_{x \to 0} (1-2x)^{1/x} \to 1^{\infty}$$

$\to$ or $1^{-\infty} = \frac{1}{1^{\infty}}$

NOT the number one
(limit goes to one)

Somehow, $\frac{0}{0}$ or $\frac{\infty}{\infty}$

let $y = (1-2x)^{1/x}$

so we want : $\lim_{x \to 0} y$

get varieble out of exponent $\to$ ln

$$\ln y = \ln (1-2x)^{1/x}$$

$$= \frac{1}{x} \cdot \ln(1-2x) = \frac{\ln(1-2x)}{x} \to \frac{0}{0}$$

can work w/ $\lim_{x \to 0} \ln y$ w/ l'Hospital's

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1-2x)}{x}$$

$$\stackrel{L}{=} \lim_{x\to 0} \frac{\frac{1}{1-2x}\cdot -2}{1} = \lim_{x\to 0} \frac{-2}{1-2x} = -2$$

want to $\lim_{x\to 0} y$

$$\lim_{x\to 0} \ln y = -2$$

$$\lim_{x\to 0} \underbrace{e^{\ln y}}_{y} = \boxed{e^{-2}}$$