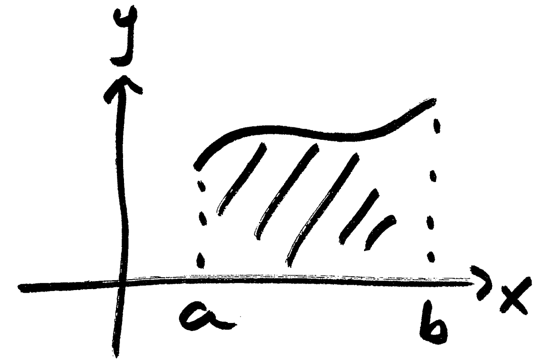
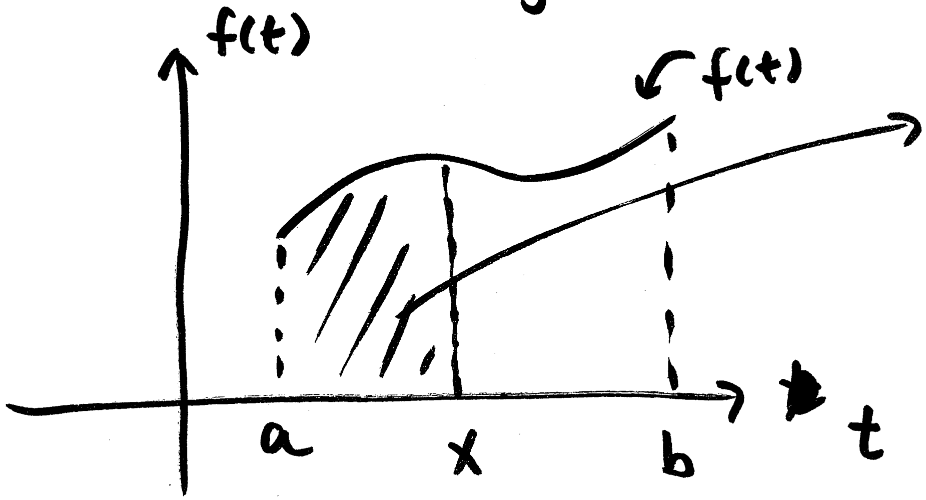
5.3 The Fundamental Theorem of Calculus

recall $\int_a^b f(x)\,dx$ is the area under $f(x)$ from $x=a$ to $x=b$



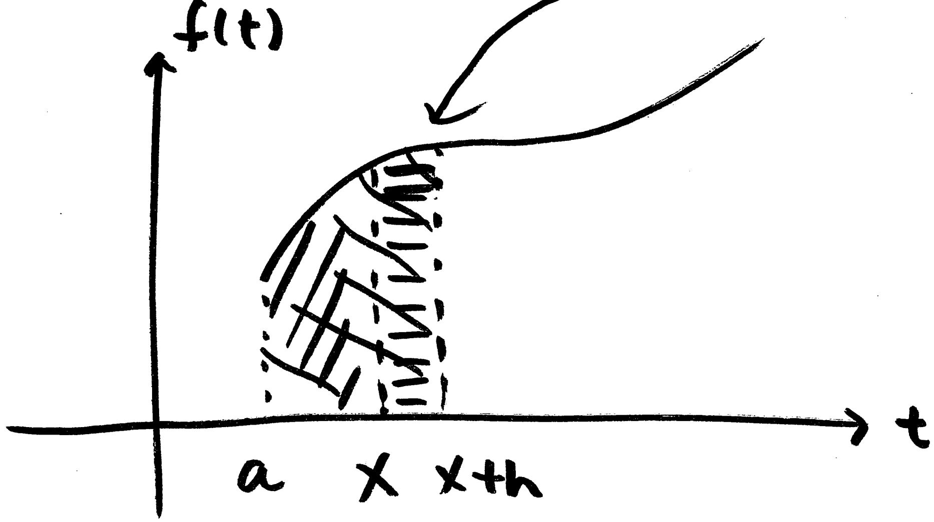
generalize it to get an area function



area accumulated from $t=a$ to $t=x$

$$\boxed{A=\int_a^x f(t)\,dt}$$

define $g(x) = \int_a^x f(t)\,dt$ what is $g'(x)$



$g(x) = \int_a^x f(t)\,dt$ (blue)

$g(x+h) = \int_a^{x+h} f(t)\,dt$ (~~reed~~ red)

$g(x+h) - g(x) \approx h \cdot f(x)$ → roughly a rectangle with width h and height f(x) (if h is small)

$$\frac{g(x+h) - g(x)}{h} \approx f(x)$$

$$\underbrace{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}_{g'(x)} = f(x)$$

So $g(x) = \int_a^x f(t)\,dt$

has derivative $g'(x) = f(x)$

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

Fundamental Theorem of Calculus (Part 1)

↳ differentiation (slope) and integration (area) are connected (inverse process of each other)

<u>example</u>

$$\frac{d}{dx}\int_{1}^{x}\frac{1}{t^3+1}\,dt$$

$$=\frac{1}{x^3+1}$$

<u>example</u>

$$\frac{d}{dx}\int_{\pi}^{x}\sqrt{1+\sec t}\,dt$$

$$=\sqrt{1+\sec x}$$

<u>example</u>

$$\frac{d}{dx}\int_{x}^{\pi}\sqrt{1+\sec t}\,dt$$

need x in the upper limit

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$=\frac{d}{dx}\left(-\int_{\pi}^{x}\sqrt{1+\sec t}\,dt\right)$$

$$=-\sqrt{1+\sec x}$$

example

$$\frac{d}{dx}\int_1^{e^x} \ln t \, dt$$

note upper limit is not x but some function of x

we need to use chain rule

let $u = e^x$

$$\frac{d}{dx}\int_1^{u} \ln t \, dt$$

but we can find $\frac{d}{du}\int_1^{u} \ln t \, dt = \ln u$

recall $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$$\frac{d}{dx}\int_1^{e^x} \ln t \, dt = \left(\frac{d}{du}\int_1^{u} \ln t \, dt\right) \cdot \frac{d}{dx}(e^x)$$

$$\frac{d}{dx}\int_1^{e^x} \ln t \, dt = (\ln u)\, e^x \quad \text{but} \quad u = e^x$$

$$= (\ln e^x)\cdot e^x$$

$$= \boxed{x e^x}$$

example

$$\frac{d}{dx}\int_0^{\tan x} \sqrt{t+\sqrt{t}} \, dt$$

let $u = \tan x$

then find $\frac{d}{du}\int_0^{u} \sqrt{t+\sqrt{t}}\, dt = \sqrt{u+\sqrt{u}}$

then $\frac{du}{dx} = \frac{d}{dx}\tan x = \sec^2 x$

$$= \sqrt{u+\sqrt{u}} \cdot \sec^2 x = \boxed{\sqrt{\tan x + \sqrt{\tan x}} \cdot \sec^2 x}$$

Now find a way to calculate $\int_a^b f(x)\,dx$

exactly w/o using rectangles.

define $g(x) = \int_a^x f(t)\,dt$

by Fundamental Theorem of Calculus 1

$g'(x) = f(x)$ $\quad$ $g(x)$ is antiderivative of $f(x)$

let's call it $F(x) = g(x) + C$

$g(a) = \int_a^a f(t)\,dt = 0$

$F(a) = g(a) + C = C$

$F(b) = g(b) + C$ $\quad$ $g(b) = \int_a^b f(x)\,dx$

we want $g(b)$

$$F(b) - F(a) = [g(b) + c] - [c]$$

$$= g(b) = \int_a^b f(x)\,dx$$

so $$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$ Fun. Theorem of calc. Part 2

where $F(x)$ is an antideriv. of $f(x)$

<u>example</u> $$\int_0^5 x\,dx$$ $f(x)$

find $F(x) = \frac{1}{2}x^2 + c$

$$F(5) = \frac{1}{2}(5)^2 + C = \frac{25}{2} + C$$

$$F(0) = \frac{1}{2}(0)^2 + C = C$$

$$\int_0^5 x\,dx = \underbrace{\frac{25}{2} + C}_{F(5)} - \underbrace{C}_{F(0)} = \boxed{\frac{25}{2}}$$

example

$$\int_{\pi/6}^{\pi} \sin x\,dx$$

antideriv. of $-\cos x$

$$= -\cos x \Big|_{\pi/6}^{\pi} = \underbrace{-\cos(\pi)}_{F(b)} - \underbrace{-\cos(\pi/6)}_{F(a)}$$

$$= \boxed{1 + \frac{\sqrt{3}}{2}}$$