5.5 Substitution Rule (part 2)

tricky substitutions

<u>example</u> $\int \frac{x}{1+x}\,dx$

cannot divide term by term like

if $\frac{1+x}{x}$ then

$\frac{1+x}{x} = \frac{1}{x} + \frac{x}{x} = \frac{1}{x} + 1$

two parts: x vs $1+x$

what if $u = x$ $\quad \frac{du}{dx} = 1 \quad du = dx$

$\int \frac{u}{1+u}\,du$ back to where we started → wrong u

$$\int \frac{x}{1+x}\,dx$$

try $\boxed{u = 1+x}$

$$\frac{du}{dx} = 1$$

$$du = dx$$

must be changed to in terms of u

$$\int \frac{x}{u}\,du$$

from $u = 1+x$

$$x = u-1$$

$$= \int \frac{u-1}{u}\,du = \int\left(\frac{u}{u} - \frac{1}{u}\right)du = \int\left(1 - \frac{1}{u}\right)du$$

$$= u - \ln|u| + C = \boxed{1+x-\ln|1+x|+C}$$

<u>example</u> $\int x^3 (x^2+1)^4 \, dx$

compare parts : x^3 vs x^2+1

try $u = x^3$

$\frac{du}{dx} = 3x^2$

$du = 3x^2 \, dx$

$$\int u \underbrace{(x^2+1)^4}_{} dx$$

difficult to sub out

or impossible

<u>wrong u</u>

$$\int x^3 (x^2+1)^4 \, dx$$

let $u = x^2+1$

$$\frac{du}{dx} = 2x$$

$$du = 2x \, dx$$

$$\int x^3 \cdot u^4 \, dx$$

$$= \int x^2 \cdot u^4 \cdot x \, dx$$

$$= \int \frac{1}{2} (u-1) \, u^4 \, du$$

$$= \int \frac{1}{2} (u^5 - u^4) \, du = \frac{1}{2}\left(\frac{u^6}{6} - \frac{u^5}{5}\right) + C$$

$$= \boxed{\frac{1}{12}(x^2+1)^6 - \frac{1}{10}(x^2+1)^5 + C}$$

how to get rid of $x^3 dx$?

$$x^3 dx = \underbrace{x^2}_{u-1} \cdot \underbrace{x \, dx}_{\frac{du}{2}}$$

Definite Integrals w/ subs

<u>example</u>

$$\int_0^2 \sqrt{1+2x}\,dx = \int_0^2 (1+2x)^{1/2}\,dx$$

$u = 1+2x$

$\frac{du}{dx} = 2$

$du = 2\,dx$

$$= \int_{0 \to x=0}^{2 \to x=2} u^{1/2}\,\frac{du}{2}$$

mismatch in variable (u) and limits (x)

$$= \int_{x=0}^{x=2} \frac{1}{2} u^{1/2}\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Bigg|_{x=0}^{x=2}$$

get u out BEFORE plugging in numbers

$$= \frac{1}{3}(1+2x)^{3/2}\Big|_{x=0}^{x=2}$$

$$= \frac{1}{3}(5)^{3/2} - \frac{1}{3}(1)^{3/2}$$

$$= \boxed{\frac{1}{3}(5)^{3/2} - \frac{1}{3}}$$

Alternate way: change limits to refer to u

$$\int_0^2 \sqrt{1+2x}\, dx$$

$u = 1+2x$

$du = 2\, dx$

old upper limit $x = 2 \longrightarrow u = 1+2(2) = 5$

old lower limit $x = 0 \longrightarrow u = 1+2(0) = 1$

new integral:

$$\int_{u=1}^{u=5} \frac{1}{2} u^{1/2} \, du$$

no x anywhere

FINISH problem in $\underline{\underline{u}}$

$$= \frac{1}{2} \, \frac{2}{3} u^{3/2} \Big|_{u=1}^{u=5}$$

$$= \frac{1}{3}(5)^{3/2} - \frac{1}{3}(1)^{3/2}$$

$$= \boxed{\frac{1}{3}(5)^{3/2} - \frac{1}{3}}$$

__example__

$$\int_e^{e^2} \frac{dx}{x \ln x}$$

$$\underbrace{e \le x \le e^2}_{\text{positive, so don't need absolute value in ln}}$$

$$= \int_e^{e^2} \frac{1}{x \ln x}\, dx$$

$$\cancel{x=} \qquad = \int_e^{e^2} \frac{1}{x} \cdot \frac{1}{\ln x}\, dx$$

$$u = \ln x$$

$$du = \frac{1}{x}\, dx$$

change limits: $x = e^2 \Rightarrow u = \ln e^2 = 2$

$x = e \Rightarrow u = \ln e = 1$

$$\int_e^{e^2} \frac{1}{\ln x} \cdot \frac{1}{x}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$x = e^2 \rightarrow u = 2$

$x = e \rightarrow u = 1$

$$= \int_1^2 \frac{1}{u}\,du = \ln u \Big|_1^2 = \ln 2 - \ln 1$$

$$= \boxed{\ln 2}$$

Geometric interpretation of substitution

$$\int_0^1 \sqrt{1+x}\,dx$$

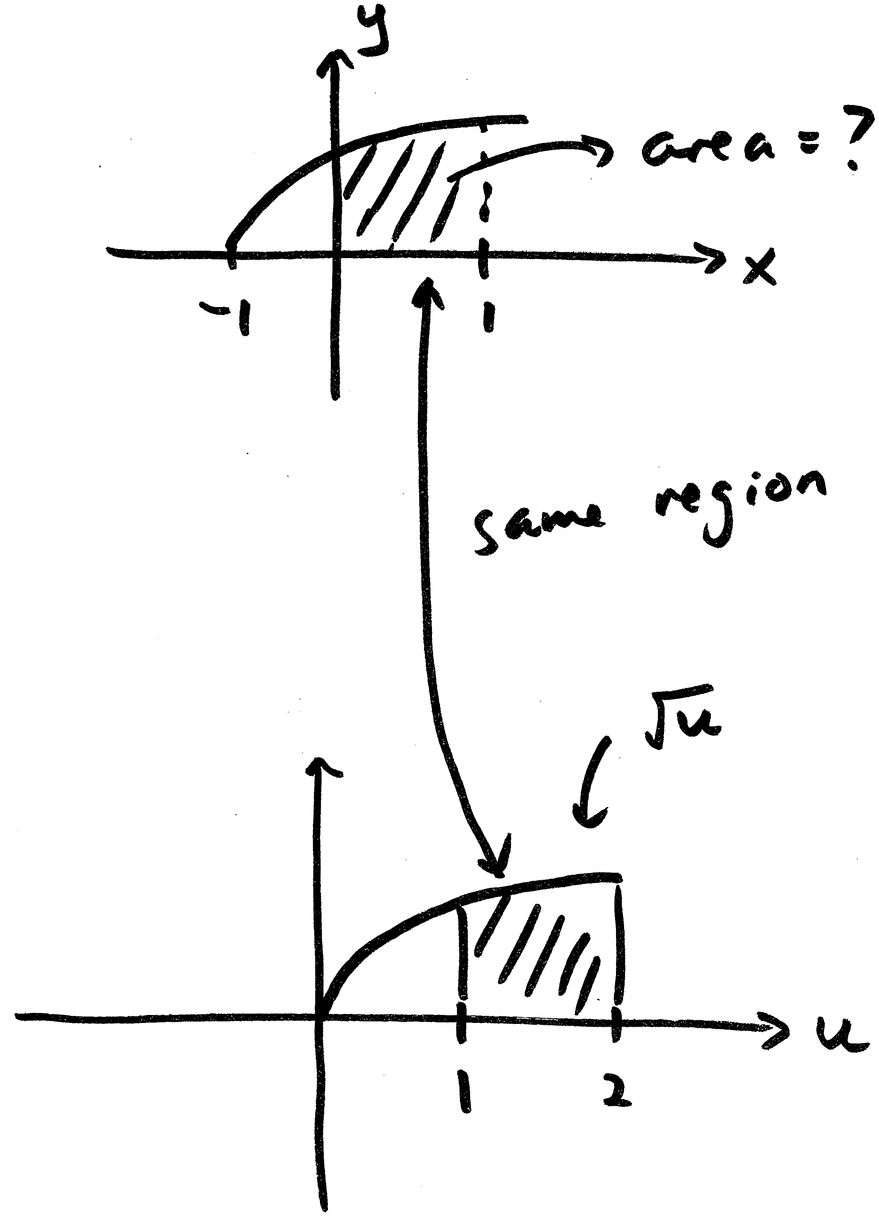
$u = 1 + x$

$du = dx$

$x = 1 \longrightarrow u = 2$

$x = 0 \longrightarrow u = 1$

$$= \int_1^2 \sqrt{u}\,du$$



here, we shifted the location of y-axis

→ re-centered axes → NOT true in ALL cases

⇒ ~~re-n~~ ~~worke~~ work w/ ~~the~~ new axes