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# Derivation and application conditions of the one-dimensional heat equation

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# Outline

- Background and terminology
- Derivation conditions
- Application conditions
- Tentative conclusions

- Besides the indispensability argument, what is left for the philosopher to worry about when it comes to the application of mathematics in science?
- Example: which mathematical truths concerning the real numbers play a role in using real numbers to represent temperature?
- "temperature and other scalar fields used in physics are assumed to be continuous, and this guarantees that if point $x$ has temperature $\psi(x)$ and point $z$ has temperature $\psi(z)$ and $r$ is a real number between $\psi(x)$ and $\psi(z)$, then there will be a point $y$ spatio-temporally between $x$ and $z$ such that $\psi(y) = r$" (Field 1980, 57).

- Response: not *all* mathematical properties transfer to temperatures.
- There is no least real number but there is a lowest temperature.
- Case study: For $u(x,t)$ representing the temperature of point $x$ at time $t$, we can derive the partial differential equation (Boyce and DiPrima 1986, 514-584):

$$\alpha^2 u_{xx} = u_t \tag{1}$$

where $\alpha^2 = \kappa/\rho s$, $\kappa$ is the thermal conductivity of the material, $\rho$ its density and $s$ the specific heat of the material. Throughout subscripts indicate partial differentiation with respect to that variable, e.g. $u_t = \frac{\partial}{\partial t} u(x,t)$.

- Derivation conditions: under what conditions are scientists warranted in adding this equation to their scientific theory of heat?
- Application conditions: under what conditions are scientists warranted in using this equation to describe a particular physical system?
- Conclusions:
  1. There are two attitudes that we can take to (1), only one of which is warranted by the evidence that scientists typically have available.
  2. Even in this simple case, the derivation and application conditions diverge.
  3. This account can give insight into some interpretative questions about the role of mathematics in the sciences.

To add (1) to their theories scientists appeal to two experimentally determined laws:

- Newton's law of cooling: the amount of heat per unit of time that passes from warmer plate 2 to cooler plate 1 is

$$H = \frac{\Delta Q}{\Delta t} = \frac{\kappa A |T_2 - T_1|}{d} \tag{2}$$

  where $T_2$ and $T_1$ are the respective temperatures of the plates, $A$ is their area, $d$ their distance from each other and $\kappa$ is the thermal conductivity of the material.
- A claim about the average change in temperature: $\Delta u$ increases in proportion to $Q\Delta t$, but is inversely proportional to $s\Delta m = s\rho A \Delta x$, where $\Delta m$ is the mass of the element:

$$\Delta u = \frac{Q \Delta t}{s \rho A \Delta x} \tag{3}$$

# Step 1

Use (2) to determine the net heat flow $Q$ for our element:

$$Q = \kappa A(u_x(x_0 + \Delta x, t) - u_x(x_0, t)) \tag{4}$$

Deriving (4) from (2) requires taking $d$ in (2) to 0 once for each boundary of the element.

# Step 2

Assume that there is some position $x_0 + \theta \Delta x$ $(0 < \theta < 1)$ where the actual change in temperature equals the average change in temperature in the entire element:

$$Q\Delta t = [u(x_0 + \theta\Delta x, t + \Delta t) - u(x_0 + \theta\Delta x, t)][s\rho A\Delta x] \qquad (5)$$

# Step 3

Multiply both sides of (4) by $\Delta t$ and identify the right-hand side with the right-hand side of (5):

$$\kappa A(u_x(x_0 + \Delta x, t) - u_x(x_0, t))\Delta t = \\ [u(x_0 + \theta\Delta x, t + \Delta t) - u(x_0 + \theta\Delta x, t)][s\rho A\Delta x] \quad (6)$$

Dividing both sides by $\Delta x \Delta t$, taking the limit of $\Delta x \to 0$, $\Delta t \to 0$ and rearranging gives (1).

Two derivation conditions:

- (D1) Various limits are well defined.
- (D2) Given the average change in temperature $\Delta u$ in the bar element, there is some point in the bar $x$ whose temperature change is identical to $\Delta u$.

Two attitudes are possible here:

1. Physical attitude: Throughout the derivation we are talking directly about physical systems and physical magnitudes.
2. Mathematical attitude: Throughout the derivation we are talking directly about mathematical entities and their properties and only indirectly about physical systems.

# A Choice

- The physical attitude would ground Field's claim about temperatures, but such an attitude is not warranted by the experimental evidence available. No iron bar has elements corresponding to the elements assumed in the derivation.
- The mathematical attitude allows the derivation to go through as there is no doubt about its mathematical correctness. For it to succeed, though, we must explain how it involves indirect claims about physical systems and what these claims are.

# A Proposal

- The derivation is directly about a wholly mathematical model $M_1$ and indirectly about a physical system $P$ when (i) there is a mathematical model $M_2$ that *matches* $P$ and (ii) there is an *acceptable* mathematical transformation from $M_1$ to $M_2$.
- Matching & Resolution: A mathematical model's degree of resolution will be *perfect* when the constituents and processes in the model match precisely the constituents and processes of the physical situation modeled (mapping account). A model will *lack* (*have excess*) resolution if its constituents and processes are of a *larger* (*smaller*) scale than the constituents and processes of the physical situation modeled. E.g. an actual discrete iron bar vs. the continuous bar.

# A Proposal (cont.)

- When is a transformation acceptable?
- Batterman 2002: In asymptotic reasoning we construct 'minimal' models which lack resolution but that nevertheless represent the physical situation. This occurs when the relevant features of the model correspond to the relevant features of the situation. E.g. the buckling load for the strut in the minimal model vs. the buckling load for the actual strut.
- A generalization: Batterman has isolated just one type of acceptable transformation between a matching model and a model that lacks resolution. Others exist, as do acceptable transformations between matching models and models with excess resolution.

# The Heat Equation

- Our derivation involves a model with excess resolution. We believe that adding details to it does not change its temperature dynamics captured by our experimental laws (2), (3).
- To prove: For reasonable temperature and time intervals, the distribution of temperature magnitudes for the two mathematical models agree.
- Objection: This is "lazy optimism" (Wilson 2000).

- A different set of conditions must be satisfied for (1) to be used to describe a particular physical system. We review perhaps the simplest case.
- For boundary conditions

$$\begin{aligned} u(0,t) &= 0, \\ u(l,t) &= 0, t > 0 \end{aligned} \tag{7}$$

with initial conditions

$$\begin{aligned} u(x,0) &= f(x), 0 \le x \le l \\ f(x) &= \sum_{n=1}^{\infty} b_n \sin(n\pi x/l) \end{aligned} \tag{8}$$

every solution to (1) is of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp(-n^2\pi^2\alpha^2 t/l^2) \sin(n\pi x/l) \tag{9}$$

# Step 1

- We use the separation of variables technique:

$$u(x,t) = X(x)T(t) \tag{10}$$

- Substituting (10) into (1) and rearranging, we see that both sides of

$$[X''/X] = [1/\alpha^2][T'/T] \tag{11}$$

  must be equal to some constant, call it $\sigma$.
- Rearranging the resulting two equations leads to

$$X'' - \sigma X = 0 \tag{12}$$

$$T' - \alpha^2 \sigma T = 0 \tag{13}$$

## Step 2

- We replace real $\sigma$ by complex $-\lambda^2$ and use the boundary conditions to show that nontrivial solutions require that $\sigma = -n^2\pi^2/l^2$ and

$$X(x) = \sin(n\pi x/l), n = 1, 2, \ldots \tag{14}$$

- Similar considerations require that

$$T(t) = \exp(-n^2\pi^2\alpha^2 t/l^2), n = 1, 2, \ldots \tag{15}$$

- But we know that any linear combination of the product of (14) and (15) will be a solution to (1) satisfying (7), so we can write the general solution as (9).

# Two application conditions

- (A1) Separation of variables is appropriate, i.e. there are functions $X$ and $T$ such that $u(x,t) = X(x)T(t)$.
- (A2) Given a function of a real variable, there are additional parallel functions of a complex variable.

Neither condition is reasonable given the physical attitude. The mathematical attitude is required to make scientific practice intelligible.

# First conclusion

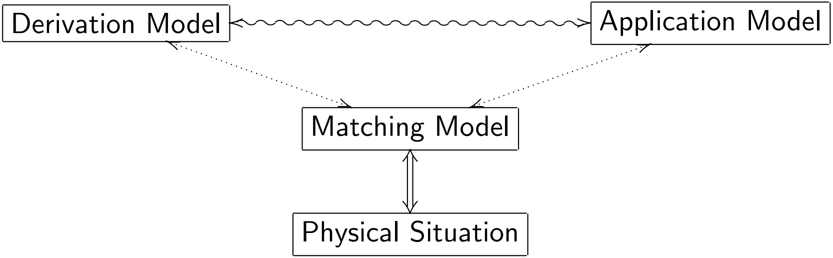
There are two attitudes that we can take to (1), only one of which is warranted by the evidence that scientists typically have available.

- Taking the physical attitude requires adopting the axioms needed for a Field-style nominalistic version of our physical theories.
- Adopting the mathematical attitude allows us to reject these axioms and suggests a role for mathematics in science: we use mathematics when we are unable to construct a matching model.

# Second conclusion

Even in this simple case, the derivation and application conditions diverge.

- Even when (1) is part of our physical theory we need additional application conditions to be satisfied to solve it.
- The same equation can have associated with it different, conflicting application conditions. It seems possible that the application and derivation conditions could also conflict, requiring three distinct models:



# Third conclusion

This account can give insight into some interpretative questions about the role of mathematics in the sciences.

- If the use of mathematics is tied to a lack of understanding of the matching model, then we can see why highly mathematical theories pose the greatest interpretative challenges.
- Moreover, by investigating the appropriate transformations between the matching, derivation and application models, we can make progress on some of these debates.

Batterman, R. W. (2002).
*The Devil in the Details: Asymptotic Reasoning in Explanation, Reduction, and Emergence*.
Oxford University Press.

Boyce, W. E. and R. C. DiPrima (1986).
*Elementary Differential Equations and Boundary Value Problems* (Fourth ed.).
John Wiley & Sons.

Field, H. (1980).
*Science Without Numbers: A Defence of Nominalism*.
Princeton University Press.

Wilson, M. (2000).
The unreasonable uncooperativeness of mathematics in the natural sciences.
*The Monist 83*, 296–314.