Critical Notice

Torsten Wilholt, *Zahl und Wirklichkeit: Eine philosophische Untersuchung über die Anwendbarkeit der Mathematik [Number and Reality: A philosophical investigation of the applicability of mathematics].* Paderborn: Mentis, 2004. Pp. 309. ISBN 3-89785-368-x.

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1 Introduction

Torsten Wilholt's *Number and Reality* marks the most sustained treatment of philosophical problems with applied mathematics since Steiner's 1998 *The Applicability of Mathematics as a Philosophical Problem* (Steiner 1998). Unlike Steiner, though, Wilholt ties questions about applications of mathematics to debates about realist and anti-realist interpretations of mathematics generally. This leads to an argument for a new kind of limited realism about mathematics. My concerns, to be developed below, are that the limited realism he advocates depends on a number of controversial assumptions. While Wilholt's limited realism is certainly a new proposal for understanding mathematics and its applications, it remains to be seen whether it is really superior to the more traditional realist and anti-realist approaches to mathematics.

2 The Realm Problems

Framing the entire discussion are two problems for the mathematical realist which Wilholt dubs the ontological realm problem and the epistemological realm problem. Both problems involve pure and applied mathematics, although the ontological realm problem is most naturally raised in applied contexts. One example used is the unfortunate case of the stuntman Colt, the physicist Jody and her assistant Howie, the amateur metaphysician and nominalist. Colt asks Jody whether or not he should attempt a particular stunt involving driving across a gorge. After finding out the details of the situation, Jody informs Colt that there is a function f such that he will survive the crossing just in case there is no real number r in a certain interval for which $f(r) = 0$. She asks Howie whether or not such a real number exists and Howie replies that there are no real numbers, so clearly there is no real number r such that $f(r) = 0$. The obvious challenge for the nominalist here is to give some account of how these applications work so that all nominalists need not agree with Howie.

Wilholt goes on to emphasize that this situation also raises problems for the realist. First, the realist faces an ontological problem: how is it that the function's mathematical properties are relevant to Colt's proposed crossing? The realist accepts that the existence of such a real number involves abstract objects which are not physically related to Colt's actions. Given this lack of physical connection, how can the mathematical facts have any bearing on whether or not Colt will survive? The second problem is epistemological and at least in its most general form a fairly familiar objection to realism. Given the realist interpretation of mathematics, how is it that we have reliable beliefs about mathematics? Following Field's presentation, Wilholt presents the realist with the challenge of accounting for the correlation between these beliefs and the facts of the mathematical realm, despite the lack of any physical connection. Appealing to applied contexts adds a new twist to this standard epistemological objection to realism. For the realist need not only explain how mathematicians' beliefs are reliable, but also how beliefs formed in applied contexts are reliable. Some versions of realism may explain how our pure mathematical beliefs get things right, but fail to account for the role of mathematics in applications. Clearly Wilholt is only interested in a version of realism that deals with both explanatory challenges.

The targets of the first half of the book are anti-realists and those "holistic" realists who seek to resolve the realm problems solely using indispensability arguments. The two chapters on anti-realism summarize the now standard objections to the various nominalistic interpretations of mathematics. One nominalist strategy, dubbed "Evaluation instead of Interpretation", seeks to assign mathematical statements appropriate truth-values without invoking any abstract objects. A typical example is the sort of deductivism which assigns a truth-value to a mathematical claim S based on whether or not S is a logical consequence of a particular set of axioms. The thoroughgoing deductivist faces problems with pre-axiomatic mathematics, Gödel's incompleteness theorems and with giving a non-mathematical analysis of logical consequence. Wilholt argues that similar problems apply to Hellman's modal-structuralism.

The second nominalist strategy considered is fictionalism and here Hartry Field's work is carefully presented and criticized. For Wilholt, fictionalism seeks to determine which mathematical statements are acceptable without assigning truth-values. Initially the fictionalist might present mathematics as simply a game in which an acceptable statement is merely one advanced in accordance with the rules of the game. Field's program is the most sophisticated version of this approach as it not only offers a clear account of acceptability in mathematics, but also attempts to explain the applicability of mathematics in science. Wilholt notes how Field's program faces objections both due to the details of the example of classical mechanics that Field developed in *Science Without Numbers* (Field 1980), as well as regarding the scope of this strategy for the scientific theories that we actually accept, i.e. general relativity theory and quantum mechanics. In particular, Field faces all the problems regarding logical consequence that the other nominalist strategy encountered, and Wilholt argues that Field's own interpretation of metalogic is unsatisfactory.

While the discussion of nominalism draws heavily on previous objections, Wilholt presents new considerations against holistic realism. These failures offer indirect support to his own limited realism. Holistic realism is the view, advocated by Quine among others, that we have as good a reason to postulate mathematical entities as we do the theoretical physical entities of our best science. Assuming confirmational holism allows the empirical support for our scientific theories to warrant belief in all parts of the theory, including the mathematics used by the theory. Wilholt grants that holistic realism may offer a kind of solution to his epistemic realm problem, but charges that it utterly fails to deal with the ontological realm problem. On the one hand, the holistic realist must try to solve this problem by appeal to the representational role of mathematics: mathematical structures do not play a causal role in the physical world, but are invoked to represent various features of physical situations in virtue of the similarities between the situations and the structures. On the other hand, the holist's knowledge of mathematical structures results from their role in empirical science and so depends upon the prior application of these structures: "The determination of a representational relation between mathematical and empirical facts assumes that our knowledge of the mathematical facts is first determined independently. But according to holism we have it first on the basis of its successful application" (36).

This impasse is resolved in Wilholt's limited realism by identifying some parts of mathematics with properties and relations that apply directly to physical processes that we can reliably investigate. This realistic basis then warrants representational accounts of the rest of applied mathematics as well as what Wilholt calls secondary applications of the parts of mathematics which are interpreted realistically. After reviewing the details of Wilholt's positive proposal, I will turn to my concerns and comparisons with other proposals.

3 Whole and Real Numbers

Wilholt's realism identifies whole numbers with properties and real numbers with relations. He settles on specific properties and relations in order to solve the epistemic and ontological realm problems. Much of the preliminary work needed for this identification involves making the epistemic realm problem more precise by developing a plausible necessary causal condition on knowledge. After arguing against various proposals, Wilholt settles on

> (V_4) We can exhibit no reliable connection between our convictions about Fs and the F-facts unless there is an aggregate of causal processes in which at least one F plays a role in at least one causal process involving a mental state (169),

where a causal process is individuated using (KP′) (164-165), a complicated principle that demarcates causal processes in terms of the transmission of conserved physical quantities.[1] Assuming (V_4) and (KP′), Wilholt points out that if we were to identify mathematical objects with properties of causal processes then we would have a simple solution to both the epistemological and ontological realm problems. Specifically, whole numbers are identified with properties that are instantiated by aggregates of causal processes. Aggregates here are just mereological sums, but using (KP′) Wilholt argues that we can univocally assign number properties to an aggregate in virtue of its unique decomposition into parts that are causal processes. Thus, an aggregate A composed of a single causal process will have the number property 1, the aggregate that results from summing A with another single causal process will have the number property 2, etc. We can have reliable convictions about such number properties in virtue of participating in aggregates of causal processes bearing the number properties. Finally, we can explain how claims about the whole numbers are relevant to a physical situation when the numbers are instantiated by aggregates of causal processes in that situation.

[1]The specific details of Wilholt's principle arise from discussions such as Hitchcock 1995, Dowe 1995 and Salmon 1997.

A similar move results in the identification of the positive real numbers with relations tied to ratios between extensive physical magnitudes, such as lengths. Magnitudes come divided into types and a collection of magnitudes from one type is extensive when for any pair of physical objects a, b bearing magnitudes A, B of that type, there is a physical combination of those objects $a \oplus b$ bearing a magnitude of that type identifiable as $A + B$. This allows Wilholt to relate this relation of physical combination to the addition of real numbers. Thus the ratio of a's magnitude to itself is identified with the real number 1 and suppose that the ratio of B to A is 2. Then the magnitude borne by $a \oplus b$ will have the ratio of 3 to A. Likewise, $a \oplus b$ will have the ratio of 1.5 to B. So, as with the whole numbers, we can have reliable beliefs about the real numbers when we participate in causal processes involving these ratios and we can explain the relevance of claims about the real numbers to physical situations when these situations include such ratios.

Wilholt is not sure about how far this kind of realistic interpretation of mathematics can be taken, but he clearly recognizes that it cannot be used to interpret all of mathematics or even all areas of mathematics currently used in science. He argues, for example, that the objects of abstract algebra, e.g. groups, cannot be identified with properties or relations. The Klein 4-group has many different instantiations and "None of these 4-groups has, from the mathematical point of view, any special claim to be considered *the* Klein 4-group" (225). In such cases he surprisingly opts for a formalistic version of the kind of deductivism that he had rejected earlier in the book. Drawing on Curry, he presents abstract algebra and all areas of mathematics that lack a realistic interpretation as the study of formal systems. Such systems classify the sentences of an uninterpreted formal language into acceptable and unacceptable based on whether or not the sentence is formally derivable from the axioms of the system. He responds to his earlier objections by arguing that when his formalism is combined with a realistic approach to the whole and real numbers, these objections either fail to apply or are not so serious. For example, Gödel's incompleteness theorems require that all the consistent formal systems studied in reasonably strong mathematics will be incomplete, i.e. there will be sentences G in the formal language such that neither G nor $\neg G$ is derivable from the axioms. Given that he does not adopt a formalist approach to number theory and real analysis, however, this problem will not arise for those theories. Incompleteness restricted to the remaining theories is deemed an acceptable cost of solving the realm problems.

It remains to explain how Wilholt's limited realism can account for applications of the whole and real numbers that do not fit with the properties and relations he identifies them with, or indeed how he can handle any applications of his formal systems. His strategy is to contrast such 'secondary' applications with the primary applications that involve aggregates of causal processes and ratios of extensive magnitudes. Secondary applications involve representational relations between mathematical systems and the things being represented. In applying the real numbers, then, we invoke a homomorphism between the features of the physical system and an interval of the reals. This representational approach is extended to the application of formal systems by using equivalence classes of types of signs as the range of the representing function. The classes are constructed using the identity statements deducible from the axioms of the formal system. For example, if "$a * b = c$" is derivable in some formal system, then the type "$a * b$" is grouped with the type "c". Unlike the holistic realist's vicious circle reviewed above, Wilholt claims that his limited realism allows an initial degree of mathematical knowledge prior to these representational applications. He further argues that this stock of primary applications is sufficient to ground the large number of remaining secondary applications.

4 Objections

The most obvious concern with Wilholt's account of the whole numbers is that it is arbitrary to single out this particular application and to use it to isolate what properties the numbers really are. It remains unclear what is to be said in favor of his identification "from the mathematical point of view", to echo his point about the Klein 4-group. Setting this objection aside, is it even possible to identify the numbers with these properties in a way that can solve the realm problems? One concern is that we cannot identify the numbers with a family of properties unless we have a clear conception of the range of application of these properties. Wilholt explicitly denies that his number properties apply just to aggregates of causal processes and even raises the possibility that they apply to abstract aggregates, if there are any. However, the anti-realist will not be convinced that such a more general property exists based simply on interactions with its more specific instantiations. Assuming I have encountered several plants of a particular species, must I automatically countenance the property of being a plant?

Similarly, the anti-realist might grant that aggregates of causal processes have the properties Wilholt isolates, but deny that these properties have the generality that Wilholt requires. Without this generality, it becomes much less plausible to identify the numbers with these properties.

A more fundamental range of concerns arise when we compare Wilholt's account of the whole numbers with Bigelow's proposal in *The Reality of Numbers: A Physicalist's Philosophy of Mathematics* (Bigelow 1988).[2] The most significant difference between Bigelow and Wilholt is that Bigelow identifies universals (properties and relations) with physical things. Universals and individuals are distinguished by Bigelow only in virtue of their pattern of locations: universals are present in regions that are not spatio-temporally connected. Wilholt adopts a more traditional conception of universals according to which they are abstract entities that can exist even if they lack instances. At the very least Wilholt owes us an account of why his numbers cannot be identified with physical universals of the type that Bigelow adopts. If they can, then Wilholt's limited realism collapses into a thoroughgoing physicalism.

Another objection is based on Bigelow's argument against Armstrong's identification of the number n with the property of an aggregate having n-parts. On this view to say that an aggregate A of pebbles is 49 is just to say that A contains 49 pebbles. The objection is based on the assumption that this is not a simple property of A and the claim that it cannot be defined in a way that fits with Armstrong's analysis:

> In order to construct "containing 49 pebbles", we must somehow stick "49-parted" together with "being a pebble". But the first of these attaches to the whole heap, whereas the second attaches to things which result from just one particular way among many of dividing the heap into parts (Bigelow 1988, p. 41).

On Bigelow's analysis the property of containing 49 pebbles is the property of having parts, which number 49, each of which is a pebble. So 49 ends up being a *relation* among things and crucially a relation that can obtain between any kind of things, whether or not they are parts of some aggregate.

Bigelow's objection seems to apply directly to Wilholt: if 3 is a property of aggregates of causal processes, then Wilholt cannot view containing 3 causal

[2]Bigelow does not appear in Wilholt's bibliography or index, although he is mentioned in passing at p. 222fn337.

processes as a simple property. But 3 is a property of the entire aggregate, while being a causal process is a property that applies only to the parts. Bigelow's proposal that the property is best defined as having parts, which number 3, each of which is a causal process makes 3 a relation between things.

When it comes to the real numbers, Wilholt faces the analogous difficulty of explaining why ratios of extensive magnitudes are to be singled out as the positive real numbers and how it is plausible to posit such a rich array of relations. The collection of relations is so intricate, in fact, that it is hard to believe that sufficiently many ratios are instantiated in the regions of space-time that we actually experience. The typical example of an extensive magnitude, i.e. length, highlights this. While in the context of classical physics it was perhaps reasonable to posit a wide array of ratios, we no longer accept such a simple story about lengths. Indeed, physics could evolve in such a way that we might end up believing that there are no ratios of extensive magnitudes in the physical world. If this happened, then there would be no primary applications of the real numbers.

Wilholt responds to this possibility by embracing empiricism about mathematics. Thus he ends up with a conception of mathematics quite close to the holistic realism that he began by criticizing. This naturally leads to the question of whether he really can escape a version of the vicious circle objection regarding mathematics that he deployed against holistic realism. One concern would be that he must use our best scientific theories to determine which properties and relations are instantiated in the appropriate parts of the physical world. Based on what these theories tell us, it becomes plausible to identify various mathematical objects with various properties and relations. Still, if mathematics plays a central role in these very scientific theories, this identification risks being circular. As with holistic realism, we must presuppose mathematics when we use these theories. If we only know that applying mathematics is justified after using these theories, we can never use these theories to warrant confidence in our beliefs about mathematics.

Another concern relates to Wilholt's rather cavalier treatment of sets, specifically when it comes to invoking functions and equivalence classes in his account of secondary applications. Initially he presents it as open question, to be resolved in future work, whether or not set theory can be interpreted realistically in line with the whole and real numbers. Here he mentions Maddy 1992 (235fn353), although perhaps a more satisfactory treatment is Lewis' megethology (Lewis 1993). The alternative to such an interpretation of set theory is clearly the formalist proposal that Wilholt offers for abstract algebra. The problem with this is that one application of set theory is to define the functions used in a representational account of applications. Wilholt bravely recognizes this dilemma and responds by noting that it is a problem for every account of applications:

> In order for this explanation to be non-circular, either the concept of a representation-homomorphism must not be a mathematical concept or else it must involve a case of mathematical application that can be explained without appeal to a concept of representation (255).

Wilholt offers only a single paragraph to outline a proposal that the initial stages of simple type theory could provide the necessary objects for his representation functions, saying "I am optimistic that a theory of such lower types can be explained as realistic mathematics" (255), i.e. in terms of entities for which our beliefs are reliable. This is a large promissory note that Wilholt must deliver on in future work, as without it his entire approach to applied mathematics will fail.

Indeed it is precisely here that Wilholt's book comes closest to the problems presented in Pincock 2004. There I argued that the functions invoked by our best account of applications cannot be identified with sets unless we are willing to adopt modal realism. To resolve this dilemma I invoked a non-mathematical property theory to supply the needed functions. If something like this is needed to ground Wilholt's simple type theory then he faces one final problem: if we can obtain reliable beliefs about these properties then we may not have to identify the whole and real numbers with universals, as our knowledge of these properties may be sufficient for knowledge of the numbers. Of course, if we cannot obtain reliable beliefs about these properties, then no secondary applications of mathematics are possible.

5 Conclusion

In summary, Wilholt offers a new interpretation of mathematics that is partly realistic, but also largely anti-realistic and formalistic. He argues that the identification of whole numbers with certain properties and positive real numbers with certain relations provides the resources to adequately explain our knowledge of mathematics and the role of mathematics in science. While I have raised a number of objections to the specifics of his proposals, they are sure to inject new life into debates about the significance of the applicability for mathematics for the ongoing debates about realism.

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