Powers and exponents

Class content > *Modeling with mathematics* > *Math recap*

In understanding how one variable depends on another, we introduced the idea of *functional dependence*. This helps us understand not just that one variable depends on another, but how it depends on the other. This is especially important in complex situations such as biology where many variables can be involved and "which one dominates" matters.

Powers

Some of the most useful and convenient functional dependencies that we will encounter are **power laws**. This means that the variable we choose to be dependent depends on the variable we choose to be independent by some power of that variable. Thus

$$y = f(x) = x^N = x \text{ multiplied by itself N times}$$

says that "*y goes like x raised to the Nth power*".

Exponentials

We know that a quadratic function rises faster than a linear one (eventually) and a cubic rises faster than a quadratic. But there is an extremely useful function that eventually rises **faster than any power**. This is the **exponential function**. In this case, the variable is not raised to a power -- *the variable itself is in the power* that some constant is raised to. So as the variable gets bigger and bigger, the power the constant is raised to gets bigger and bigger.

We write

$$y = e^x \, .$$

Now e could be any constant, but we typically take it as a special transcendental number (that means it's decimal representation never stops and never repeats): $e = 2.712...$ This particular choice is because when we make this choice, the function e^x is its own derivative. That is, if we write $y = e^x$, then

$$\frac{dy}{dx} = y$$

[Note that **a** power law is **not** referred to as an "exponential dependence" even though the variable "has an exponent". That terminology is reserved for the case when the variable is *in* the exponent.]

In case you wondered how in the world a calculator figures out what "e" to the something is, the following result allows you to calculate the value of e raised to some number -- eventually.

$$e^x = 1 + x + x^2/(1*2) + x^3/(1*2*3) + x^4/(1*2*3*4) + ...$$

You can see how this goes on -- and it goes on forever. But for any fixed *x* you are calculating for, the denominators go up faster than powers do and so eventually start making the terms smaller and smaller so they can be dropped.

While this looks really messy and we're not going to use it, looking at it does give us two interesting messages.

1. ***It explains why the exponential function is its own derivative***.
   If you take the derivative of the power series representation of the exponential, something interesting happens. The first term vanishes, the derivative of the linear term becomes the old first term (1), the derivative of the third (quadratic) term becomes the old second (linear) term, etc. So the derivative of each term becomes the previous term in the original series. We wind up getting the same thing back that we started with. This also shows us why the denominators have the structure they do. [And with a little fancy mathematical footwork, we

can show that the exponential is the *only* function that is its own derivative.]

2. ***It shows that we can only take exponentials of pure numbers***.

   Since we know that you can't add a length and an area -- or *any* quantity that has units to its square -- the power series *only* makes sense if "*x*" is a pure number. You can't take an exponential of a quantity that has units. Whenever in science we have an exponential, it will always be the ratio of two quantities with the same units -- typically a variable and a scale for that variable. (Like a time and a rate constant.)

Logarithms

The exponential function does the interesting thing of converting sums into products. If R = e^a and S = e^b then RS = e^{a+b}. (This is the basis of a standard mathematical joke.*)

So if $f(x) = e^x$ then we have

$$f(x_1)f(x_2) = f(x_1+x_2).$$

Since multiplying is harder than adding, it's sometimes useful to go backwards from the exponential function.

Taking the exponential of *x* and setting $y = e^x$, if we are given *x* we can use our calculator (or series) to find *y*. But what if we are given *y* and want to find *x*? The answer to that is called the *natural logarithm* of *y*. So that would give us the equation:

$$y = e^{\ln(y)}$$

This shows that the natural log (ln) is the inverse function of the exponential. If we first take the natural log and then exponentiate it, we get back what we started with. It works the other way too:

$$x = \ln(e^x)$$

If we exponentiate first and then take the natural log, we get back what we started with. So the natural log function (ln) undoes the exponential function and vice versa.

We will see in the follow-ons that logarithms and exponentials are very useful in analyzing data and seeing whether something behaves like a power law.

Follow-on:

- Log-log plots

* When Noah's ark landed on Ararat, he told all the animals to go forth and multiply. The snakes said, "We can't multiply. We're adders." So Noah built them a log table. (This pre-calculator joke is based on the fact that logarithms of numbers used to be listed in large books called "log tables". Rather than doing long multiplications by hand, scientists would look up the logarithms in these tables, add them, and then look up what number had the resulting logarithm. Really! To see an image of a page from a log table, check the Wikipedia page on Math Tables. And the slide rule arranges numbers logarithmically along two lengths so that when you add the lengths corresponding to two numbers you are multiplying them.)

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