Gravitational potential

- Let's assume:
  - A particle of unit mass moving freely
  - A body of mass $M$
- The particle is attracted by $M$ and moves toward it by a small quantity $dr$.
- This displacement is the result of work $W$ exerted by the gravitational field generated by $M$:
  \[
  W = Fdr = m\ a\ dr = a\ dr
  \]
  \[
  \Rightarrow W = G\frac{M}{r^2}\ dr
  \]
- The potential $U$ of mass $M$ is the amount of work necessary to bring the particle from infinity to a given distance $r$:
  \[
  U = \int_{\infty}^{r} G\frac{M}{r^2} dr = GM\int_{\infty}^{r} \frac{1}{r^2} dr = GM\left[\frac{1}{\infty} - \frac{1}{r}\right]
  \]
  \[
  \Rightarrow U = -\frac{GM}{r}
  \]
- At distance $r$, the gravitational potential generated by $M$ is $U$
Gravitational potential of the Earth

- Earth’s gravitational acceleration $g$ exerts a work to move a unit mass particle from $U$ to $U+dU$ (spherical homogeneous non-rotating Earth):

\[
U = -\frac{GM}{r} \implies \frac{dU}{dr} = \frac{GM_E}{r^2} \implies dU = -gdr
\]

\[
\implies g = -\frac{dU}{dr} = -\text{grad}(U)
\]

(think in terms of energy)
Gravitational potential of the Earth

- Gravity potential of the Earth = - gradient of gravity
- Since gravity = gravitational attraction + centrifugal acceleration + flattening, therefore:

\[ g = - \nabla U \]

\[ U = -\frac{GM}{R} + \frac{GMa^2}{2R^3} J_2(3\sin^2\Phi - 1) - \frac{1}{2}\omega^2R^2\cos^2\Phi \]
Equipotential surfaces

- = surfaces on which the potential is constant
- \( U = \) constant, recall that: \( dU = -g \, dr \)
  \( \Rightarrow \) \( dU = \) zero on equipotential surfaces
  \( \Rightarrow \) \( g \) not necessarily constant on equipotential surfaces
- Non-rotating homogeneous Earth:
  - Recall that: \( U = GM_E / r \)
  - Therefore, \( U = \) constant \( \Rightarrow \) \( r = \) constant \( \Rightarrow \) equipotential surfaces = spheres centered on \( M_E \)
- Practical use of equipotential surfaces:
  - Definition of the vertical = direction of the gravity field = perpendicular to equipotential surfaces
  - Equipotential surfaces = define the horizontal
The geoid

• There is an infinity of equipotential surfaces
• There is a particular surface on the Earth that is “easy” to locate: the mean sea level
• **The Geoid = the particular equipotential surface that coincides with the mean sea level**
• This is totally arbitrary.
• But is makes sense because the oceans are made of water (!): the surface of a fluid in equilibrium must follow an equipotential.
The Geoid = The figure of the Earth

- Over the oceans, the geoid is the ocean surface (assuming no currents, waves, etc)
- Over the continents, the geoid is not the topographic surface (its location can be calculated from gravity measurements)
- Geoid “undulations” are caused by the distribution of mass in the Earth
The Ellipsoid

- First evidence that the Earth is round: Erathostene (275-195 B.C.)
- First hypothesis that the Earth’s is flattened at the poles: Newton
- First measurement of the Earth’s flattening at the poles: Clairaut (1736) and Bouguer (1743)
- **The shape of the Earth can be mathematically represented as an ellipsoid** defined by:
  - Semi-major axis = equatorial radius = \(a\)
  - Semi-minor axis = polar radius = \(c\)
  - Flattening (the relationship between equatorial and polar radius): \(f = (a-c)/a\)
  - Eccentricity: \(e^2 = 2f-f^2\)

Comparison between the WGS-84 ellipsoid and a sphere of identical volume

- \(a = 6378.136\, \text{km}\)
- \(c = 6356.751\, \text{km}\)
- \(R = 6371.000\, \text{km}\)
The reference ellipsoid

- Many different reference ellipsoids have been defined and are in use!
- **Reference ellipsoid = the ellipsoid that best fits the geoid.**
- Totally arbitrary, but practical
- Reference ellipsoid = WGS-84
- Geoid undulations = differences, in meters, between the geoid reference ellipsoid (= geoid “height”).

### Selected Reference Ellipsoids

<table>
<thead>
<tr>
<th>Ellipsoid</th>
<th>Semi-Major Axis (meters)</th>
<th>1/Flattening</th>
</tr>
</thead>
<tbody>
<tr>
<td>Airy 1830</td>
<td>6377563.396</td>
<td>299.3249646</td>
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<tr>
<td>Bessel 1841</td>
<td>6377397.155</td>
<td>299.1528128</td>
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<tr>
<td>Clarke 1866</td>
<td>6378206.4</td>
<td>294.9785982</td>
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<tr>
<td>Clarke 1880</td>
<td>6378249.145</td>
<td>293.465</td>
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<tr>
<td>Everest 1830</td>
<td>6377276.345</td>
<td>300.8017</td>
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<tr>
<td>Fischer 1960 (Mercury)</td>
<td>6378166.0</td>
<td>298.3</td>
</tr>
<tr>
<td>Fischer 1968</td>
<td>6378150.0</td>
<td>298.3</td>
</tr>
<tr>
<td>GRS 1967</td>
<td>6378160.0</td>
<td>298.24167427</td>
</tr>
<tr>
<td>GRS 1975</td>
<td>6378140.0</td>
<td>298.257</td>
</tr>
<tr>
<td>GRS 1980</td>
<td>6378137.0</td>
<td>298.257222101</td>
</tr>
<tr>
<td>Hough 1956</td>
<td>6378270.0</td>
<td>297.0</td>
</tr>
<tr>
<td>International</td>
<td>6378388.0</td>
<td>297.0</td>
</tr>
<tr>
<td>Krassovsky 1940</td>
<td>6378245.0</td>
<td>298.3</td>
</tr>
<tr>
<td>South American 1969</td>
<td>6378160.0</td>
<td>298.25</td>
</tr>
<tr>
<td>WGS 60</td>
<td>6378165.0</td>
<td>298.3</td>
</tr>
<tr>
<td>WGS 66</td>
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<td>WGS 72</td>
<td>6378135.0</td>
<td>298.26</td>
</tr>
<tr>
<td>WGS 84</td>
<td>6378137.0</td>
<td>298.257223563</td>
</tr>
</tbody>
</table>
Gravity on the reference ellipsoid

- It can be shown (Clairaut, ~1740) that the (theoretical) value of gravity on the rotating reference ellipsoid is:

\[ g = g_o \left( 1 + k_1 \sin^2 \Phi - k_2 \sin^2 2\Phi \right) \]

- \( g_o \) = gravity at the equator
- \( k_1 \) and \( k_2 \) = constant that depend on the shape and rotation of the Earth
- \( g_o, k_1, \) and \( k_2 \) are estimated from actual measurements. For GRS-1967:
  - \( g_o = 978 \, 031.846 \) mGals
  - \( k_1 = 0.005 \, 302 \, 4 \)
  - \( k_2 = 0.000 \, 005 \, 8 \)

- = normal gravity formula

- According to this formula:
  - \( g \) depends only on latitude, no longitude dependence.
  - \( g \) does not vary linearly with latitude.
  - \( g \) increases/decreases when latitude increases?

- **This formula assumes an homogeneous Earth:** heterogeneities \( \Rightarrow \) deviations from this formula \( \Rightarrow \) gravity **anomalies**
Let’s be clear...

- Geoid = the equipotential surface of the Earth’s gravity field that best fits (in a least squares sense) the mean sea level
  - Potential is constant on the geoid
  - Gravity is not constant on the geoid
- Reference Ellipsoid = the ellipsoid that best fits the geoid
- Geoid = the (actual) figure of the Earth
- Ellipsoid = the (theoretical) shape of the Earth
Geoid anomalies

Geoid anomalies, or undulations = differences, in meters, between the geoid reference ellipsoid (= geoid “height”):

- Excess of mass ⇒ geoid (ocean surface) goes up (geoid = eqp surface ⇒ must remain perpendicular to the gravity field direction!)
- Deficit of mass ⇒ geoid (ocean surface) goes down
Geoid anomalies

• Contribution to gravity of hot rising mantle:
  – Elevated surface over hot spot => positive anomaly (mass excess due to extra topography)
  – Hot, buoyant (less dense) mantle material => negative anomaly (mass deficit)
  – Positive > negative => overall positive anomaly

• Contribution of cold sinking lithospheric material:
  – Depressed topography => negative anomaly
  – Cold, dense material enters mantle => positive anomaly
  – Negative > positive => overall negative anomaly
Global geoid undulations
The geoid of the Conterminous United States

Geoid undulations (≡ heights) for the conterminous United States and surrounding areas ([http://www.ngs.noaa.gov/GEOID](http://www.ngs.noaa.gov/GEOID)).

Geoid heights range from a low of -51.6 meters in the Atlantic (magenta) to a high of -7.2 meters (red) in the Rockies.
Example: The geoid of the Conterminous United States

- The geoid high, seen as a red spot, is located over the "Yellowstone hot spot". While high elevations do contribute to a portion of the geoid signal here, some geophysicists feel that the geoid high is evidence of a thermal mantle plume.

- The isolated ridge (visible more from the light/shadowing than from a different color) seen running from Minnesota to Iowa is associated with the "mid-continent gravity high". This gravity high (and subsequently, the geoid high) is the result of dense masses which lie close to the surface. These masses lie in an old rift in the North American plate.

- The brightly lit slope in the geoid, off of the east coast, is the effect of a 4000 meter (~13,000 feet) drop in the bathymetry, demarking the edge of the continental shelf, a passive continental margin that formed some 120 million years ago.

- The few noticeably large bumps in the Pacific ocean are caused by massive seamounts up to 3000 meters (~10,000 feet) in height above the sea floor.

- Details of the topographic anomalies of the Western Rockies can be seen superimposed upon this anomaly, although with much less magnitude.

- The Great San Joaquin Valley of California, formed through the tectonics of the earlier subduction of the Pacific plate by North America is outlined in detail in the Geoid. Comparison with this feature can be made with those smaller yet similar Geoid lows to the north in Oregon and Washington state.

- At the very top of the figure on the right can be seen the outline of the most recently formed feature of Geoid of North America. This is the postglacial Geoid low caused by the depression of the continent under the ice load from the last Ice Age some 20,000 years ago. Because of the viscous nature of the Earth's Mantle this feature will slowly disappear until the end of the next Ice Age when the process will repeat itself again.

- From: http://www.ngs.noaa.gov/GEOID
A very practical use of the geoid

- GPS positioning → height above or below a reference ellipsoid (WGS-84) = **ellipsoidal height**.
- Topographic maps, markers, etc.: height above or below **mean sea level = orthometric height**.
- Transformation from ellipsoidal height to orthometric height requires to know the geoid height.
- In the conterminous US, this transformation implies a correction ranging from 51.6 m (Atlantic) to 7.2 m (Rocky Mountains)

\[ H = h + N \]

$h$ = ellipsoidal height
$N$ = geoid height
$H$ = orthometric height
= height above mean sea level
"h = H + N"

h (Ellipsoid Height) = Distance along ellipsoid normal (Q to P)
N (Geoid Height) = Distance along ellipsoid normal (Q to P₀)
H (Orthometric Height) = Distance along Plumb line (P₀ to P)
What have we learned?

- The gravitational field generated by a mass $M$ is associated to a potential.
- There is a particular equipotential surface called the geoid, defined to best fit the mean sea level.
- The geoid undulations are expressed in meters above the reference ellipsoid.
- The reference ellipsoid is a mathematical representation of the shape of the Earth.
- The geoid undulations reflect rock density variations, topography, mantle processes, etc.