Strain analysis

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Plates vs. continuum

- Most plates are rigid at the <1 mm/yr level => until now we have studied a purely discontinuous approach where plates are individual rigid entities.
- But:
  - Some plates are affected by GIA deformation
  - Some plates are affected by broad tectonic deformation
  - Some plate boundaries are broad
- Need for a “continuous” approach = strain
Strain - 1 D case

- Solid rod in a 1-dimensional coordinate system \((0,x)\). Let us take two points A \((x)\) and B \((x+dx)\), very close to each other.

- Apply a traction \(P\) in the \(x\)-direction at the end of the rod. The rod will elongate, points A and B will be displaced.

- Assume infinitesimal displacements. The amount of displacement depends on the position of the point along the rod: the closer to the place where the force is applied, the larger the displacement will be.

- Therefore, B will be displaced slightly more than A. The displacement at A is noted \(u(x)\), the displacement at B is noted \(u(x+\delta x)\).
Strain - 1D case

• Because the displacement $\delta x$ is infinitesimal, we can write:

$$u(x + dx) = u(x) + \frac{\partial u}{\partial x} \, dx$$

and:

$$A'B' = dx + \left(u(x) + \frac{\partial u}{\partial x} \, dx\right) - u(x) = dx + \frac{\partial u}{\partial x} \, dx$$

• Strain is defined as [change of length] / [original length]. Therefore:

$$e_x = \frac{\Delta L}{L} = \frac{A'B' - AB}{AB} = \frac{\left(dx + \frac{\partial u}{\partial x} \, dx\right) - dx}{\frac{\partial u}{\partial x}} = \frac{\partial u}{\partial x}$$

• Hence, infinitesimal strain is the spatial gradient of the displacement
**2D case: velocity gradient tensor**

- **Assuming:**
  - Local 2-dimensional Cartesian frame
  - Infinitesimal displacements
- **Velocity** (as a function of position $X$) can be expanded in a series:

  \[
  v(X + \delta X) = v(X) + \frac{\partial v}{\partial X} \delta X
  \]

- With the two components:

  \[
  \begin{align*}
  v_x (X + \delta X) &= v_x (X) + \frac{\partial v_x}{\partial x} \delta x + \frac{\partial v_x}{\partial y} \delta y \\
  v_y (X + \delta X) &= v_y (X) + \frac{\partial v_y}{\partial x} \delta x + \frac{\partial v_y}{\partial y} \delta y
  \end{align*}
  \]

- Therefore, one can write:

  \[
  v(X + \delta X) = v(X) + \nabla V \cdot \delta X
  \]

  with

  \[
  \nabla V = \begin{bmatrix}
  \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} \\
  \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y}
  \end{bmatrix} = \text{velocity gradient tensor}
  \]
Velocity gradient tensor

- Given 2 sites \((i\) and \(j\)) separated by (small) distance \(\delta X\), equation:
  \[
  \frac{v(X + \delta X)}{v_j} = \frac{v(X) - \nabla V \delta X}{v_i}
  \]

- Becomes: \(v_i - v_j = v_{ij} = \nabla V \delta X\)

- We can write it for 3 sites as:
  \[
  V_{ij} = AF
  \]

  with:
  \[
  V_{ij} = \begin{bmatrix}
  v_{x_{12}} \\
v_{y_{12}} \\
v_{x_{13}} \\
v_{y_{13}}
  \end{bmatrix},
  A = \begin{bmatrix}
x_{12} & y_{12} & 0 & 0 \\
0 & 0 & x_{12} & y_{12} \\
x_{13} & y_{13} & 0 & 0 \\
0 & 0 & x_{13} & y_{13}
  \end{bmatrix},
  F = \begin{bmatrix}
  \frac{\partial v_x}{\partial x} \\
  \frac{\partial v_x}{\partial y} \\
  \frac{\partial v_y}{\partial x} \\
  \frac{\partial v_y}{\partial y}
  \end{bmatrix}
  \]

  (assuming point 1 is reference)

  Observables = Velocity differences
  Model matrix = Position differences
  Unknown = Velocity gradients

- Which can be solved using least-squares:
  \[
  F = \left( \frac{A^T C_V^{-1} A}{C_F^{-1}} \right)^{-1} A^T C_V^{-1} V_{ij}
  \]
Velocity gradient tensor: covariance

- Observables are velocity differences. The covariance associated with $v_{ij}$ can then be found using:

$$
\begin{bmatrix}
  v_{xij} \\
  v_{yij}
\end{bmatrix} =
\begin{bmatrix}
  -1 & 0 & 1 & 0 \\
  0 & -1 & 0 & 1
\end{bmatrix} A
\begin{bmatrix}
  v_{xi} \\
  v_{yi} \\
  v_{xj} \\
  v_{yi}
\end{bmatrix}
$$

- One usually knows the covariance matrix $C_{ind}$ of the individual site velocities -- the variance propagation law then gives:

$$C_{v_{ij}} = A C_{ind} A^T$$

- $C_{v_{ij}}$ is then propagated in the least squares estimation (see previous slide) to get the covariance matrix of the unknowns $C_F$. 

Velocity gradient tensor: covariance of unknowns

- Given that: \( C_{V_{ij}} = A.C_{ind}.A^T \)

- One can write the variance on the east (for instance) component of the velocity difference as:
  \[
  \sigma_{e_{ij}}^2 = \sigma_{e_i}^2 + \sigma_{e_j}^2 - 2\sigma_{e_ie_j}
  \]

- Therefore, if the covariance terms are large enough, velocities can be non-significant while relative velocities (and velocity gradient tensor) are…
Strain and rotation rates

- Displacements actually combine strain and a rigid-body rotation.
- Tensor theory states that any second-rank tensor can be decomposed into a symmetric and an antisymmetric tensor. One can therefore write:

\[
\nabla V = \frac{1}{2} \left[ \nabla V + \nabla V^T \right] + \frac{1}{2} \left[ \nabla V - \nabla V^T \right] \\
\Leftrightarrow \nabla V = \begin{bmatrix}
\frac{\partial v_x}{\partial x} & \frac{1}{2} \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \\
\frac{1}{2} \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) & \frac{\partial v_y}{\partial y}
\end{bmatrix}
\begin{bmatrix}
0 & \frac{1}{2} \left( \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) \\
\frac{1}{2} \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) & 0
\end{bmatrix}
\]

Symmetric tensor \hspace{1cm} Antisymmetric tensor
Strain and rotation rates

• In other words:

\[ \nabla V = \begin{bmatrix} e_{xx} & e_{xy} \\ e_{xy} & e_{yy} \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \]

\[ E \text{ strain rate tensor} \quad W \text{ rigid rotation} \]

• With:

\[ E = \frac{1}{2} [\nabla V + \nabla V^T] \]
\[ W = \frac{1}{2} [\nabla V - \nabla V^T] \]
\[ e_{xx} = \frac{\partial v_x}{\partial x}, e_{yy} = \frac{\partial v_y}{\partial y}, e_{xy} = e_{yx} = \frac{1}{2} \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \]
\[ \omega = \frac{1}{2} \left( \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) \]
Estimating strain rate

• Given 2 sites \((i\) and \(j\)), one can write the velocity difference (w.r.t. site \(i\)) as a function of the site position (w.r.t. site \(I\)) as:

\[
v_{ij} = \nabla V \delta X
\]

\[
\Leftrightarrow v_{ij} = (E + W) \delta X
\]

\[
\Leftrightarrow \begin{cases} 
  v_{x_{ij}} = e_{xx} \delta x + e_{xy} \delta y + \omega \delta y \\
  v_{y_{ij}} = e_{xy} \delta x + e_{yy} \delta y - \omega \delta x 
\end{cases}
\]

• For 3 sites as, this can be written as:

\[
V_{ij} = A \mathbf{S}
\]

with:

\[
V_{ij} = \begin{bmatrix} v_{x_{12}} \\ v_{y_{12}} \\ v_{x_{13}} \\ v_{y_{13}} \end{bmatrix}, \quad A = \begin{bmatrix} x_{12} & y_{12} & 0 & y_{12} \\ 0 & x_{12} & y_{12} & -x_{12} \\ x_{13} & y_{13} & 0 & y_{13} \\ 0 & y_{13} & y_{13} & -x_{13} \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} e_{xx} \\ e_{xy} \\ e_{yy} \\ \omega \end{bmatrix}
\]

Observables = velocity differences
Model matrix = Position differences

• Which can be solved using least-squares:

\[
\mathbf{S} = \left( A^{T} C_{V}^{-1} A \right)^{-1} A^{T} C_{V}^{-1} V_{ij}
\]
Estimating strain: A better approach

- First, estimate velocity gradient components (and covariance) using least squares (see previous slides).

- Then, recall that:

\[
\begin{align*}
    e_{xx} &= \frac{\partial v_x}{\partial x}, \\
    e_{yy} &= \frac{\partial v_y}{\partial y}, \\
    e_{xy} &= e_{yx} = \frac{1}{2} \left( \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right), \\
    \omega &= \frac{1}{2} \left( \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right)
\end{align*}
\]

- Therefore, strain and rotation rate components can be found using the following transformation matrix:

\[
\begin{pmatrix}
    e_{xx} \\
    e_{xy} \\
    e_{yx} \\
    \omega
\end{pmatrix} = 
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.5 & -0.5 & 0 \end{pmatrix} 
\begin{pmatrix}
    \frac{\partial v_x}{\partial x} \\
    \frac{\partial v_x}{\partial y} \\
    \frac{\partial v_y}{\partial y} \\
    \frac{\partial v_y}{\partial x}
\end{pmatrix}
\]

- Propagate velocity gradient covariance to strain and rotation rate covariance using:

\[
C_{E,W} = AC_{VV}A^T
\]
Representing strain

- Tensor = independent of the coordinate system = retains its properties independently from the ref. system.
- Find reference system where:
  - Shear strain \((e_{xy})\) is zero
  - Two other components are maximal
- Equivalent to diagonalize \(E\):
  - Eigenvectors = principal axes of strain rate tensor
  - Eigenvalues = principal strain rates
Principal strains

- Principal values of $E$:
  
  $\det(E - \lambda I) = 0$

  $\Rightarrow \det \begin{bmatrix} e_{xx} - \lambda & e_{xy} \\ e_{xy} & e_{yy} - \lambda \end{bmatrix} = 0$

  $\Rightarrow (e_{xx} - \lambda)(e_{yy} - \lambda) - e_{xy}e_{xy} = 0$

  $\Rightarrow \lambda = \frac{e_{xx} + e_{yy}}{2} \pm \sqrt{\left(\frac{e_{xx} - e_{yy}}{2}\right)^2 + e_{xy}^2}$

- Principal directions:
  
  - Use rotation matrix $A$:
    
    $A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

  - Expand $AEAT^T$ and write that shear component = 0 to find:

    $e_{12} = (e_{xx} - e_{yy})\sin \theta \cos \theta + e_{xy}(\cos^2 \theta - \sin^2 \theta) = 0$

    $\Rightarrow \tan(2\theta) = \frac{2e_{xy}}{e_{xx} - e_{yy}}$

    \[\sin \theta \cos \theta = \frac{1}{2}\sin(2\theta)\]

    \[\cos^2 \theta - \sin^2 \theta = \cos(2\theta)\]
Principal strains

• Principal strains:
  – One maximal, one minimal
  – By convention, extension is taken positive

\[ e_1, e_2 = \frac{e_{xx} + e_{yy}}{2} \pm \sqrt{\left(\frac{e_{xx} - e_{yy}}{2}\right)^2 + e_{xy}^2} \]

• Principal angle (direction of \( e_1 \)):

\[ \tan(2\theta) = \frac{2e_{xy}}{e_{xx} - e_{yy}} \]
Back transformation

- Rotate principal strain rate tensor by angle $-\theta$ using rotation matrix $A$:

$$A = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

- Recall that tensor rotation is given by:

$$AE_p^{}A^T = \begin{pmatrix} e_1 \cos^2 \theta + e_2 \sin^2 \theta & -e_1 \sin \theta \cos \theta + e_2 \sin \theta \cos \theta \\ -e_1 \sin \theta \cos \theta + e_2 \sin \theta \cos \theta & e_1 \sin^2 \theta + e_2 \cos^2 \theta \end{pmatrix}$$

- Which gives:

$$\begin{cases} 
e_{xx} = e_1 \cos^2 \theta + e_2 \sin^2 \theta = \frac{e_1 + e_2}{2} + \frac{e_1 - e_2}{2} \cos(2\theta) \\
e_{yy} = e_1 \sin^2 \theta + e_2 \cos^2 \theta = \frac{e_1 + e_2}{2} - \frac{e_1 - e_2}{2} \cos(2\theta) \\
e_{xy} = -e_1 \sin \theta \cos \theta + e_2 \sin \theta \cos \theta = -\frac{e_1 - e_2}{2} \sin(2\theta) \\
\end{cases}$$

$$\sin \theta \cos \theta = \frac{1}{2} \sin(2\theta)$$

$$\cos^2 \theta - \sin^2 \theta = \cos(2\theta)$$
Shear strain

• Shear strain (from previous transformation) is maximal when $\sin(2\theta)=1$, giving:

$$e_{xy} = -\frac{e_1-e_2}{2}\sin(2\theta)$$

$$\Rightarrow e_{xy,\text{max}} = \frac{e_1-e_2}{2} = \sqrt{\left(\frac{e_{xx}-e_{yy}}{2}\right)^2 + e_{xy}^2}$$

• Recall the expression for $e_{12}$ (which we set to zero to find the principal angle $\theta$):

$$e_{12} = (e_{xx} - e_{yy})\sin\theta\cos\theta + e_{xy}(\cos^2\theta - \sin^2\theta)$$

$$= \frac{1}{2}(e_{xx} - e_{yy})\sin(2\theta) + e_{xy}\cos(2\theta)$$

• Angle at which shear strain is maximal is obtained by differentiating w.r.t. $\theta$:

$$e_{12} = \max \Rightarrow \frac{de_{12}}{d\theta} = 0$$

$$\frac{1}{2}(e_{yy} - e_{xx})\cos(2\theta_s) - e_{xy}\sin(2\theta_s) = 0$$

$$\Rightarrow \tan(2\theta_s) = \frac{e_{yy} - e_{xx}}{2e_{xy}}$$
Shear strain

- Maximum shear strain: 
  \[ e_{xy,\text{max}} = \sqrt{\left(\frac{e_{xx} - e_{yy}}{2}\right)^2 + e_{xy}^2} \]

- Maximum shear angle: 
  \[ \tan(2\theta_S) = \frac{e_{yy} - e_{xx}}{2e_{xy}} \]
  \[ \theta_S = \theta \pm 45^\circ \]

Shear strain tensor is the **average** of two strains, i.e., 
\[ \varepsilon_{xy} = (\partial\gamma/\partial x + \partial\nu/\partial y)/2 = \varepsilon_{yx} \]

Engineer shear strain is the **total** shear strain, i.e., 
\[ \gamma'_{xy} = \partial\gamma/\partial x + \partial\nu/\partial y \]
Example 1: pure extension

- Strain rate tensor:
  - $e_{xx} = 0.00$ ppb/yr
  - $e_{xy} = -0.00$ ppb/yr
  - $e_{yy} = 177.08$ ppb/yr

- Principal strains:
  - $e_1 = 177.08$ ppb/yr (most extensional)
  - $e_2 = 0.00$ ppb/yr (most compressional)
  - $\theta = 90.00$ ($e_1$, CW from north)

- Rotation:
  - $\omega = 0.00$ deg/My
Example 2: simple shear

- Strain rate tensor:
  - $e_{xx} = 0.00$ ppb/yr
  - $e_{xy} = 88.54$ ppb/yr
  - $e_{yy} = -0.00$ ppb/yr

- Principal strains:
  - $e_1 = 88.54$ ppb/yr (most extensional)
  - $e_2 = -88.54$ ppb/yr (most compressional)
  - $\theta = -45.00$ (clockwise from north)

- Rotation:
  - $\omega = 0.09$ deg/My
Strain invariants

- As any second rank tensor, strain rate tensor has 3 invariants = quantities that remain unchanged regardless of the reference system
- First invariant = tensor trace
  \[ I_E = tr(E) = e_{xx} + e_{yy} = e_1 + e_2 \]
- Second invariant = sometimes used to represent strain “magnitude”
  \[ II_E = \frac{1}{2} \left( tr\left( E^2 \right) - tr(EE) \right) = e_{xx} e_{yy} - e_{xy}^2 = e_1 e_2 \]
- Third invariant = determinant, same as second invariant in case of 2x2 symmetric tensor
  \[ III_E = \det(E) = e_{xx} e_{yy} - e_{xy}^2 = e_1 e_2 \]
Example: current strain rates in Asia

- The $3 \times 10^{-9}$ yr$^{-1}$ line coincides with the 95% significance level.
- A large part of Asia shows strain rates that are not significant at the 95% confidence level and are lower than $3 \times 10^{-9}$ yr$^{-1}$. 
In practice

• Discretize space
• Using polygons with vertices corresponding to data points
• For instance: Delaunay triangulation:
  – Circumcircle of every triangle does not contain any other point of the triangulation
  – Minimize “sliver” triangles
• Using arbitrary polygons: requires interpolation
• Calculate strain within each polygon

WARNING: assumes homogeneous and continuous strain within each polygon…
For instance

GPS velocities and principal strain rates in Central Asia
WARNING!

- Result depends on the size of the elements used to discretize the domain...
- Example:
  - Generate a random 2D velocity field (e.g. case of non significant residual velocities showing only noise)
  - Triangulate with smaller elements in the middle of the domain
  - Calculate strain
  - Result will be apparently larger strain rates where elements are smaller...