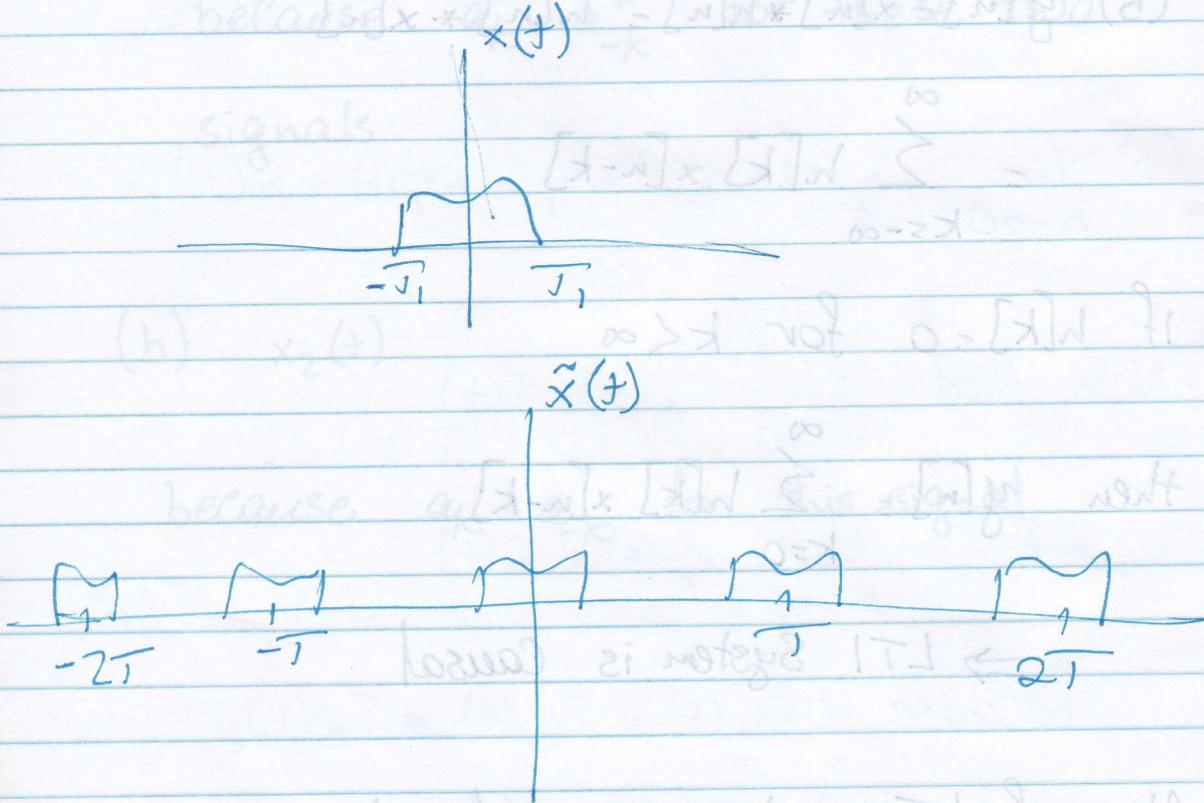


Lecture 18 - Wed. Oct. 13

As the period increases to infinity,

Fourier series becomes an integral



new. Labeled as multiple  $|T|$  if  $\omega_0$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{j k \omega_0 t}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \hat{x}(t) e^{-j k \omega_0 t} dt$$

$$= \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-j\omega_0 t} dt$$

Define  $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$a_k = \frac{1}{T} X(j\omega_0)$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(j\omega_0) e^{jk\omega_0 t}$$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(j\omega_0) e^{jk\omega_0 t} \omega_0 \quad (1)$$

As  $T \rightarrow \infty$ ,  $\tilde{x}(t) \rightarrow x(t)$

$\omega_0 \rightarrow 0$ , R.H.S. of (1) becomes  
integral

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$a_k = \frac{1}{T} \int_{-\infty}^{s+j\omega} \tilde{x}(t) e^{-jk\omega t} dt$$

$$= \frac{1}{T} \int_{-\infty}^{\omega} x(t) e^{-jk\omega t} dt$$

$$= \frac{1}{T} X(j\omega) \Big|_{\omega = k\omega_0}$$

\*Problem 4.3F

## Convergence of the Fourier Transform

We considered only signals of finite

duration

Let

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(j\omega) e^{j\omega t} d\omega$$

when is  $x(t) = \hat{x}(t)$ ?

If  $\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$  L4 shows

and  $e(t) = \hat{x}(t) - x(t)$

then  $\int_{-\infty}^{\infty} |e(t)|^2 dt = 0$

Dirichlet conditions:

Guarantee that  $x(t) = \hat{x}(t)$  except at a

discontinuity

1.  $\int_{-\infty}^{\infty} |x(t)| dt < \infty$  L4 shows

2. Finite number of maxima and minima  
in any finite interval

3. Finite number of discontinuities in  
any finite interval. Each of the

discontinuities is finite.

### Example 4.1

$$x(t) = e^{-at} u(t), a > 0$$

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= -\frac{1}{a+j\omega} e^{-(a+j\omega)t} \Big|_0^{\infty}$$

$$X(j\omega) = \frac{1}{a+j\omega}, a > 0$$

### Example 4.2

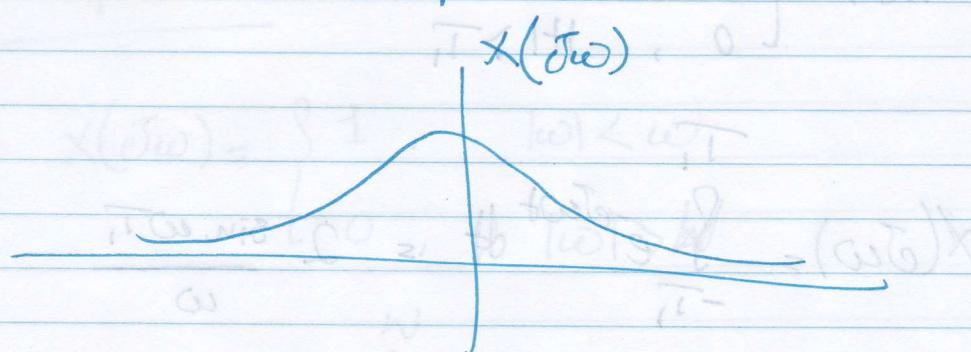
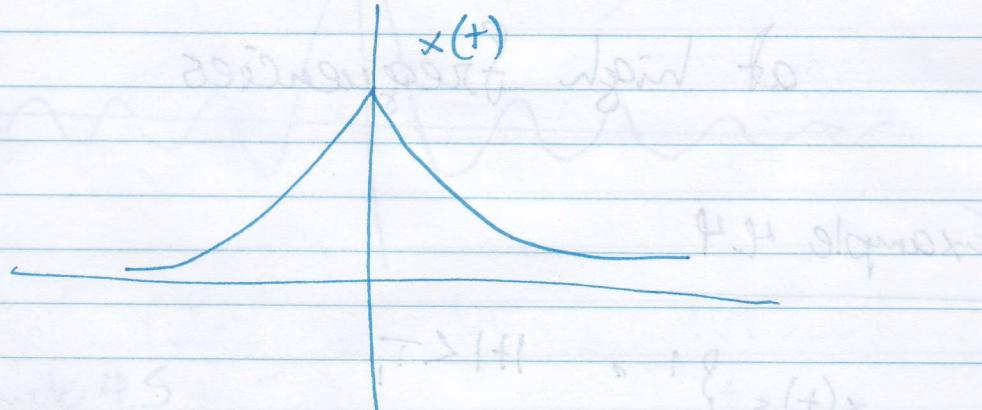
$$x(t) = e^{\alpha t} u(t), \alpha > 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} e^{\alpha t} e^{-j\omega t} dt$$

$$= \int_{-\infty}^0 e^{at} e^{-\omega t} dt + \int_0^{\infty} e^{-at} e^{-\omega t} dt$$

$$= \frac{1}{a - \omega} + \frac{1}{a + \omega}$$

$$= \frac{2a}{a^2 + \omega^2}$$



High Frequency Components  
are diminishing

### Example 4.3

Example 4.1

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Does not diminish

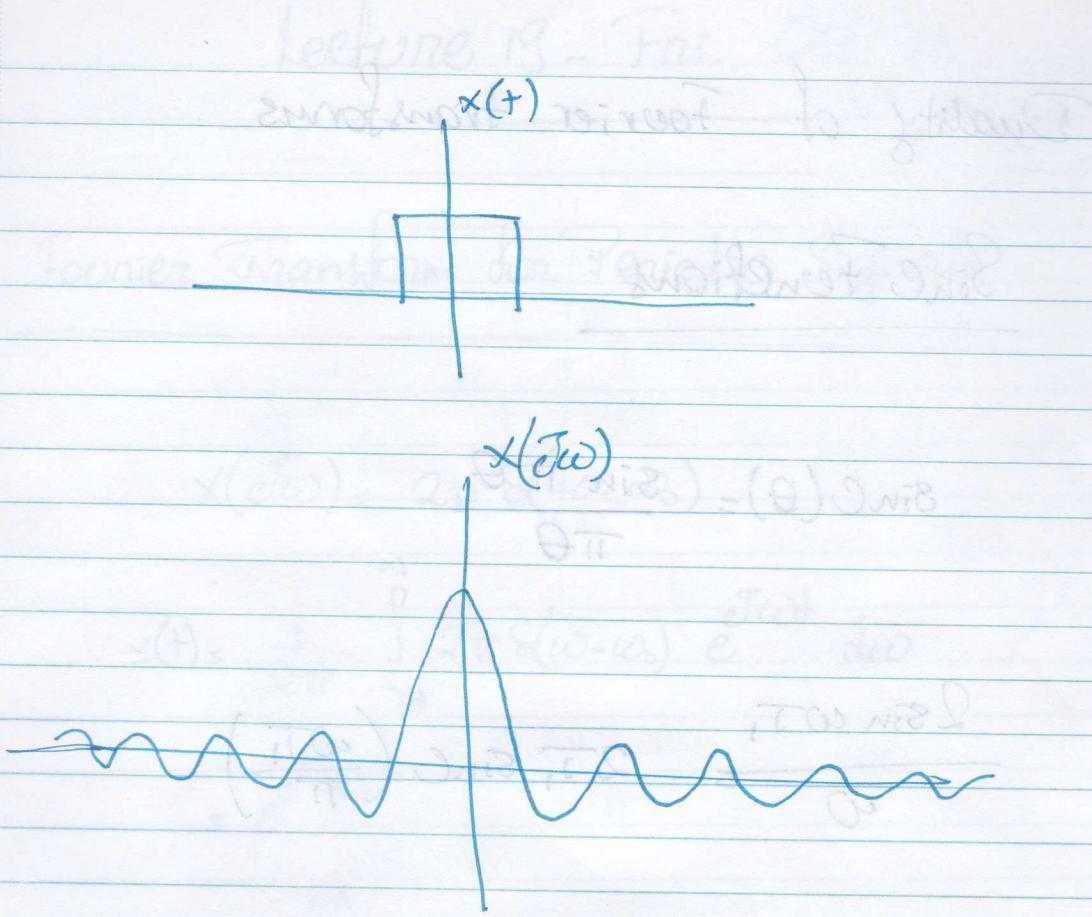
at high frequencies

### Example 4.4

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

Example 4.2

$$X(j\omega) = \frac{1}{T_1} \int_{-T_1}^{T_1} e^{-j\omega t} dt = 2 \frac{\sin \omega T_1}{\omega}$$



### Example 4.5

$$X(j\omega) = \begin{cases} 1, & |\omega| < \omega \\ 0, & |\omega| > \omega \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\omega}^{\omega} e^{j\omega t} d\omega = \frac{\sin \omega t}{\pi t}$$

## Duality of Fourier transforms

### Sinc Functions

$$\text{sinc}(\theta) = \frac{\sin \pi \theta}{\pi \theta}$$

$$\frac{2 \sin \omega t_1}{\omega} = 2 T_1 \text{sinc} \left( \frac{\omega t_1}{\pi} \right)$$

$$\frac{\sin \omega t}{\omega} = \frac{\omega}{\pi} \text{sinc} \left( \frac{\omega t}{\pi} \right)$$