

Lecture - Wednesday Dec. 2

Half-Sample Delay

$$y_e(t) = x_c(t - \Delta)$$

If input is band-limited

& Sampling rate high enough

$$Y_c(j\omega) = e^{-j\omega\Delta} X_c(j\omega)$$

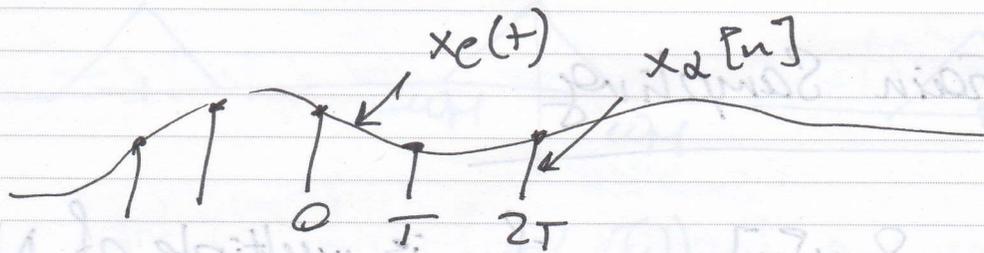
$$H_c(j\omega) = \begin{cases} e^{-j\omega\Delta}, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases}$$

$$H_d(e^{j\Omega}) = e^{-j\Omega\Delta/T}, \quad |\Omega| < \pi$$

For Δ/T an integer, the sequence

$y_d[n]$ is a delayed replica of $x_d[n]$

$\Delta/T = \frac{1}{2}$ Half-Sample Delay



$$y_c(t) = x_c(t - T/2)$$

$$y_d[n] = y_c(nT)$$

$$= x_c[(n - 1/2)T]$$

Let $x_c(t) = \frac{\sin(\pi t/T)}{\pi t}$

$$x_d[n] = x_c(nT) = \frac{1}{T} \delta[n]$$

$$y_c(t) = x_c(t - T/2) = \frac{\sin(\pi(t - T/2)/T)}{\pi(t - T/2)}$$

$$y_d[n] = y_c(nT) = \frac{\sin(\pi(n - 1/2))}{T \pi(n - 1/2)}$$

$$h_d[n] = \frac{\sin(\pi(n - 1/2))}{\pi(n - 1/2)}$$

Sampling of Discrete-Time Signals

Impulse-train Sampling

$$x_p[n] = \begin{cases} x[n] & , \text{ if } n \text{ is multiple of } N \\ 0 & \text{ otherwise} \end{cases}$$

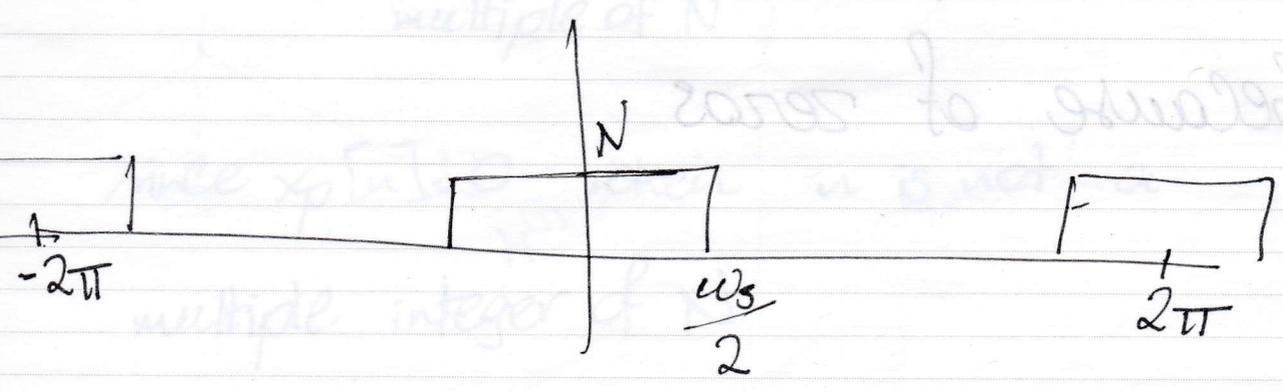
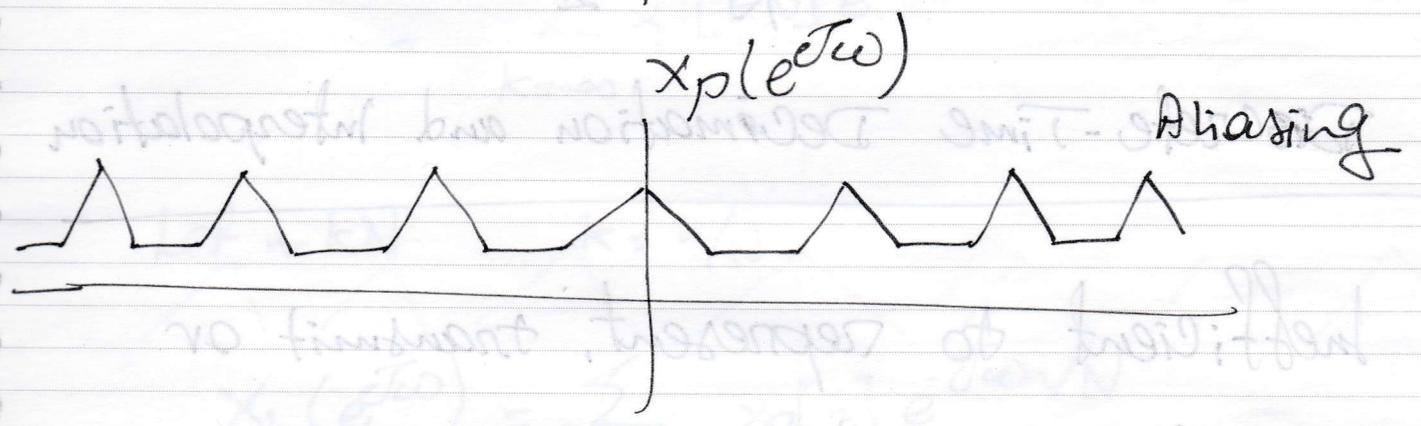
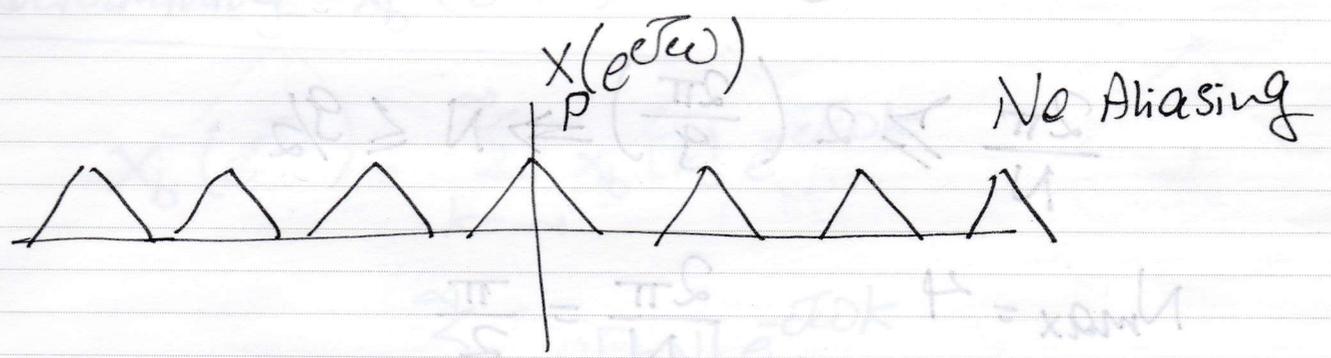
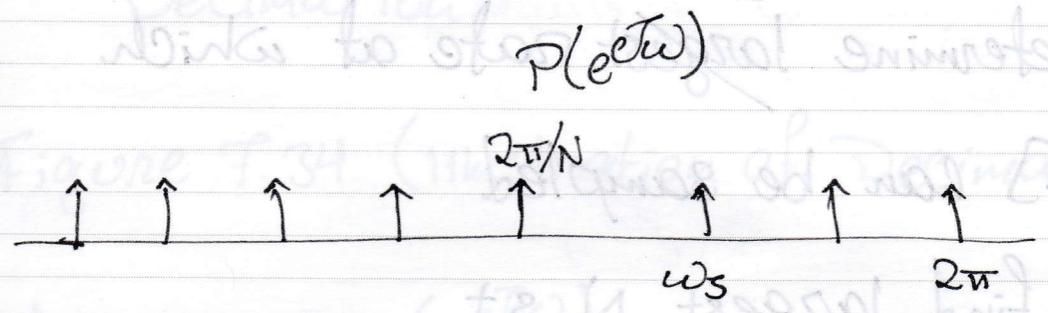
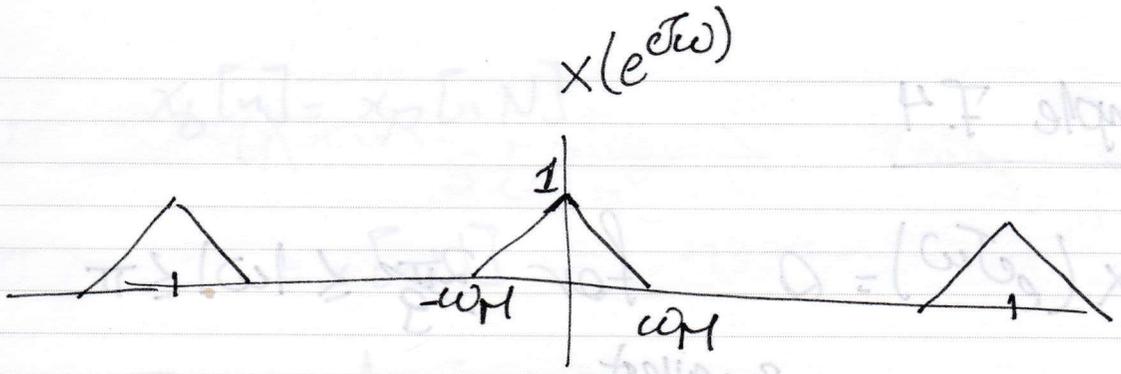
$$x_p[n] = x[n] p[n] = \sum_{k=-\infty}^{\infty} x[kN] \delta[n-kN]$$

$$X_p(e^{j\omega}) = \frac{1}{2\pi} \int_{2\pi} P(e^{j\theta}) X(e^{j(\omega-\theta)}) d\theta$$

$$P(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$\omega_s = \frac{2\pi}{N}$$

$$X_p(e^{j\omega}) = \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_s)})$$



Example 7.4

$$x(e^{j\omega}) = 0 \quad \text{for } \frac{2\pi}{9} \leq |\omega| \leq \pi$$

To determine ^{smallest} largest rate at which $x[n]$ can be sampled

We find largest N s.t.

$$\frac{2\pi}{N} \geq 2 \left(\frac{2\pi}{9} \right) \Rightarrow N \leq 9/2$$

$$N_{\max} = 4 \quad \frac{2\pi}{N} = \frac{\pi}{2}$$

Discrete-Time Decimation and Interpolation

Inefficient to represent, transmit or store the sampled sequence $x_p[n]$

Because of zeros

$$x_b[n] = x_p[nN]$$

$$= x[nN]$$

Decimation

Figure 7.34 (Illustration of Decimation)

Determining $X_b(e^{j\omega})$

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x_b[k] e^{-j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} x_p[kN] e^{-j\omega k}$$

Let $n = kN$ $k = n/N$

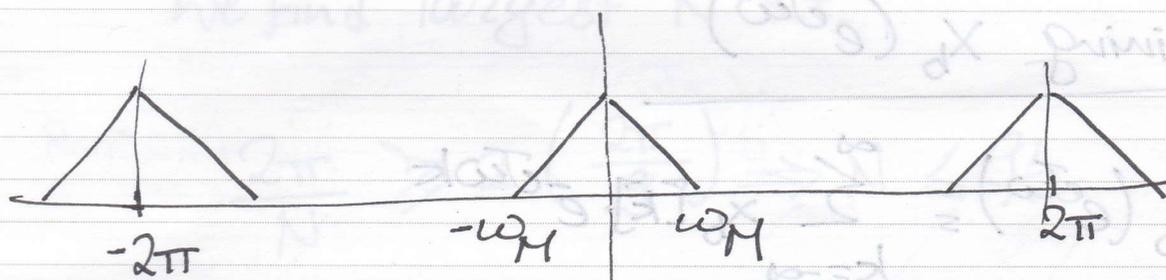
$$X_b(e^{j\omega}) = \sum_{\substack{n: \text{integer} \\ \text{multiple of } N}} x_p[n] e^{-j\omega n/N}$$

since $x_p[n] = 0$ when n is not a multiple integer of N

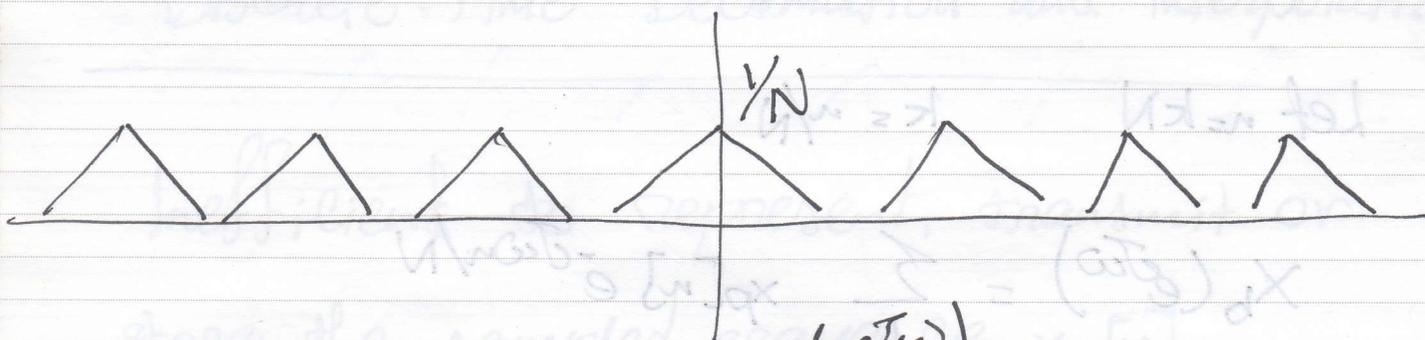
$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_p[n] e^{-j\omega n/N}$$

$$= X_p(e^{j\omega/N})$$

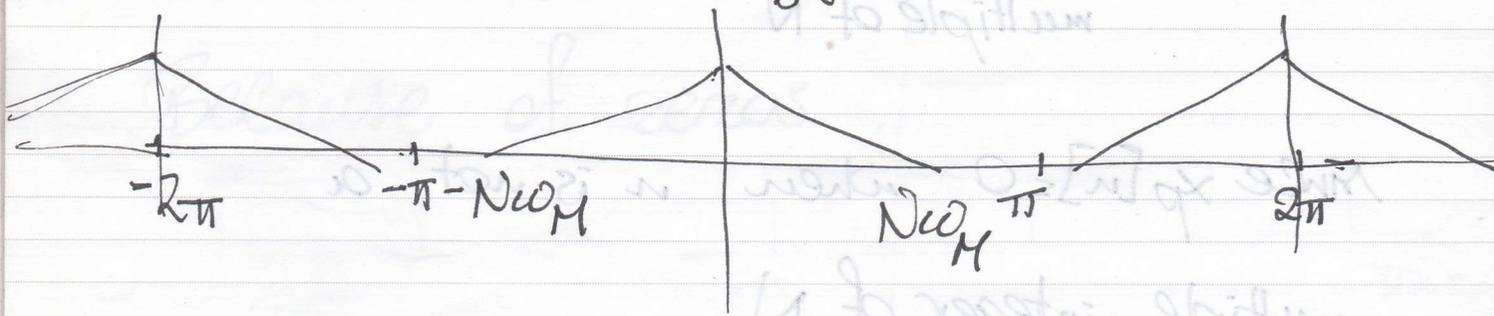
$X(e^{j\omega})$



$X_p(e^{j\omega})$



$X_b(e^{j\omega})$



If there is no aliasing then the original signal was oversampled

The process of decimation is often referred to as downsampling

80 - 89	124	A Range
70 - 79	38	B Range
60 - 69	27	C Range
50 - 59	13	D Range
40 - 49	32	Fail Range

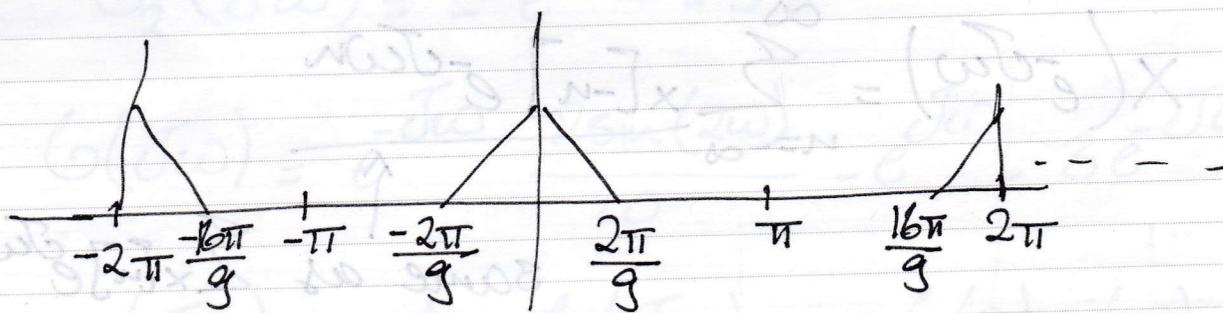
Figure 7.37

Upsampling & Interpolation

$$\text{iii) } \mathcal{E}\{x[n]\} = \frac{x[n] + x[-n]}{2} \xleftrightarrow{\mathcal{F}T} \frac{X(e^{j\omega}) + X(e^{-j\omega})}{2}$$

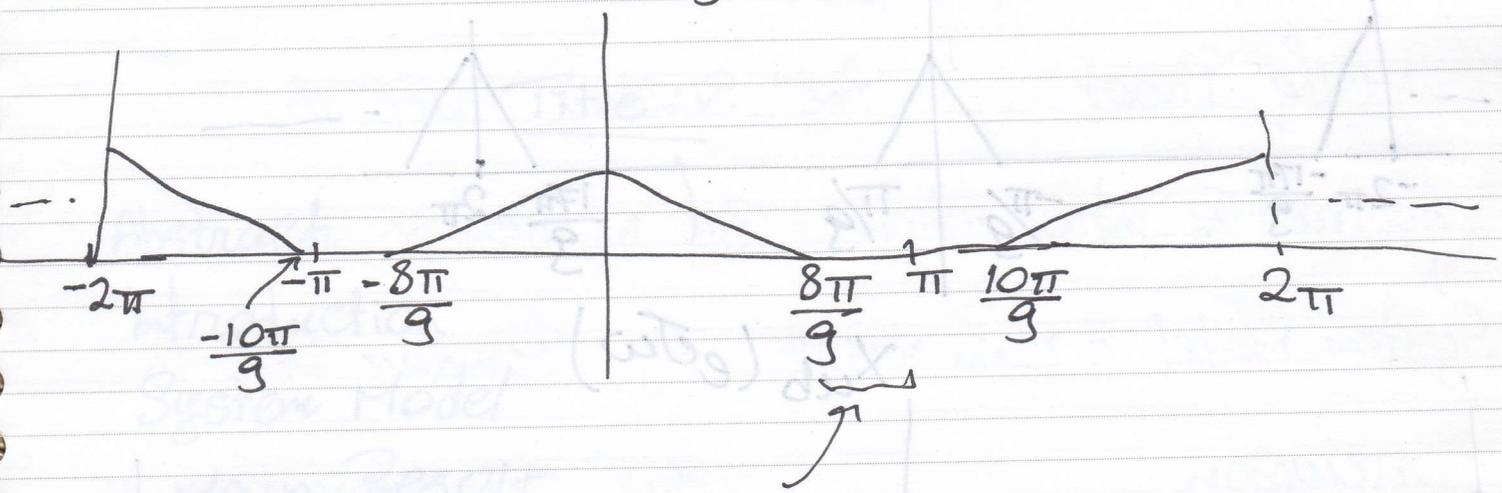
Example 7.5

The maximum possible downsampling is achieved once the non-zero portion of one period of the discrete-time spectrum has expanded to fill the entire band from $-\pi$ to π



lowest sampling rate with no aliasing = $\frac{2\pi}{4}$

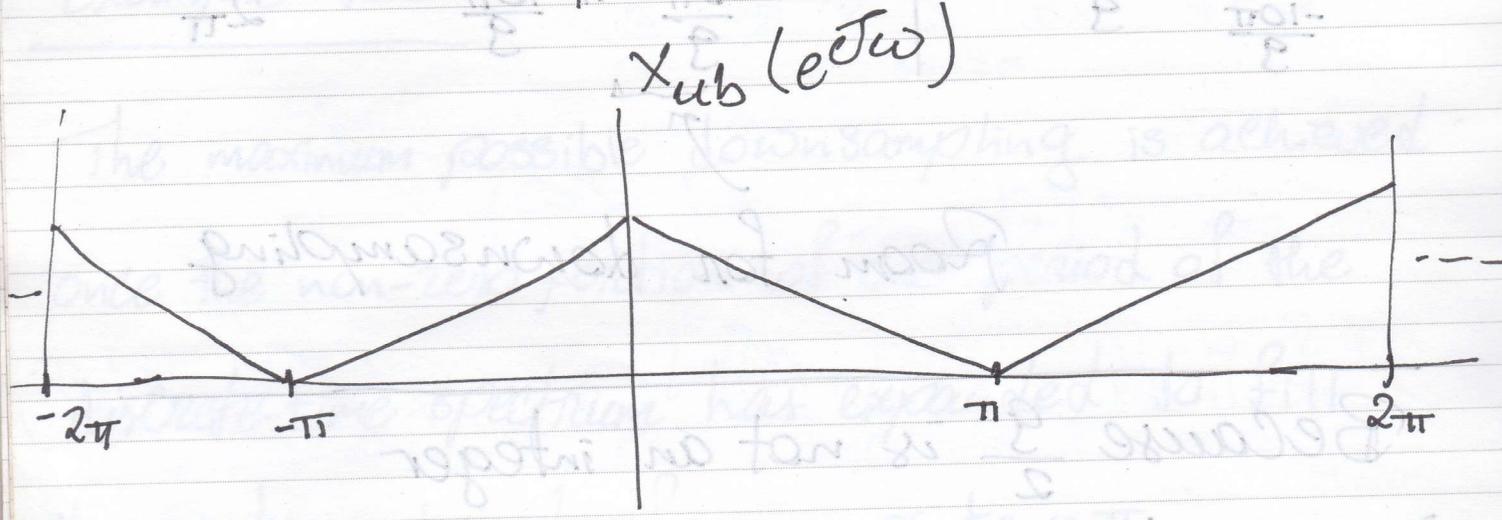
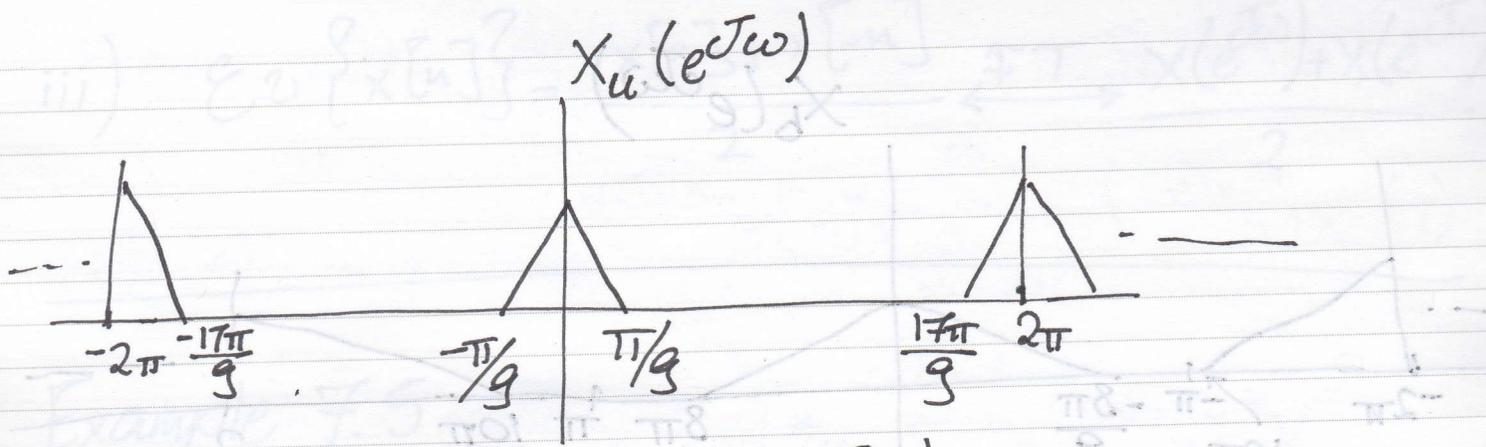
$$X_b(e^{j\omega})$$



Room for downsampling

Because $\frac{9}{2}$ is not an integer
 We cannot achieve maximum efficiency
 purely by downsampling

We first upsample by a factor of 2
 and then downsample by a factor of 9



Maximum possible (aliasing-free)

downsampling of $x_c(t)$

Assuming $x[n]$ represents unaliased samples of $x_c(t)$