

Seventh Lecture

* Convolution Integral

* Example 2.6

* Example 2.8

Example 2.9

Unit Impulse Response only characterizes LTI systems

Commutative property of LTI systems

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

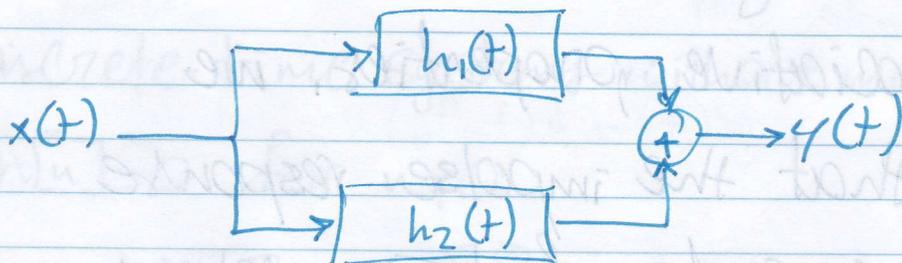
$$h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k] x[n-k]$$

Let $r = n - k$

$$h[n] * x[n] = \sum_{r=-\infty}^{\infty} h[n-r] x[r]$$
$$= x[n] * h[n]$$

Distributive Property

$$x(t) * [h_1(t) + h_2(t)]$$
$$= x(t) * h_1(t) + x(t) * h_2(t)$$



$$x(t) \rightarrow [h_1(t) + h_2(t)] \rightarrow y(t)$$

Read Example 2.10

Associative Property

$$\begin{aligned}x[n] * (h_1[n] * h_2[n]) \\ = (x[n] * h_1[n]) * h_2[n]\end{aligned}$$

* Problem 2.43 *

By using commutative and associative properties, we find that the impulse response of a cascade of LTI systems is the convolution of their individual responses

Order of cascading does
not matter

Does not apply to nonlinear
systems

$$y[n] = 4x^2[n] = (2x)^2$$

$$y[n] = 2x^2[n]$$

LTI systems with and without memory

Discrete-time systems without memory

$$h[n] = 0 \text{ for } n \neq 0$$

$$h[n] = k\delta[n]$$

$$y[n] = kx[n]$$

Example

$$h[n] = \begin{cases} 1 & , n=0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$$y[n] = x[n] + x[n-1]$$

has memory

Invertibility of LTI systems

$$h(t) * h_1(t) = \delta(t)$$

Example 2.11

$$y(t) = x(t - t_0)$$

$$h(t) = \delta(t - t_0)$$

$$x(t - t_0) = x(t) * \delta(t - t_0)$$

$$h_1(t) = \delta(t + t_0)$$

$$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

Example 2.12

$$h[n] = u[n]$$

Accumulator

$$y[n] = \sum_{k=-\infty}^n x[k] u[n-k]$$

$$= \sum_{k=-\infty}^n x[k]$$

Inverse

$$y_1[n] = x[n] - x[n-1]$$

$$h_1[n] = \delta[n] - \delta[n-1]$$

$$\begin{aligned}
 h[n] * h[n] &= u[n] * \delta[n] - u[n] * \delta[n-1] \\
 &= u[n] - u[n-1] = \delta[n]
 \end{aligned}$$

Causality for LTI systems

$$y[n] = \sum x[k] h[n-k]$$

$y[n]$ does not depend on ~~$x[k]$~~
 $x[k]$
 for ~~$k > n$~~
 $k > n$

$$\Rightarrow h[n] = 0 \quad \text{for } n < 0$$

Why?

Interpretation

Stability for LTI systems

Consider a bounded input

$$|x[n]| < B \quad \text{for all } n$$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k] x[n-k] \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]|$$

$$|y[n]| \leq B \sum_{k=-\infty}^{\infty} |h[k]| \text{ for all } n$$

If the impulse response is absolutely summable, i.e.,

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

then the system is stable.

Sufficient Condition for stability
Proof of necessity *Problem 2.49*

For continuous-time LTI systems,
Condition is:

$$\int_{-\infty}^{\infty} |h(z)| dz < \infty$$

Example 2.13

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \quad \sum_{n=-\infty}^{\infty} |\delta[n-n_0]| < \infty$$

Stable

$$\sum_{n=-\infty}^{\infty} |u[n]| < \infty \quad \sum_{n=0}^{\infty} u[n] < \infty$$

Unstable