A Capacitated Facility Location Model with Bidirectional Flows

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Supply chains with returned products are receiving increasing attention in the operations management community. The present paper studies a capacitated facility location model with bidirectional flows and a marginal value of time for returned products. The distribution system consists of a single supplier that provides one new product to a set of distribution centers (DCs), which then ships to the final retailers. While at the retailers’ site, products can be shipped back to the supplier for reprocessing. Each DC is capacitated and handles stocks of new and/or returned products. The model is a nonlinear mixed-integer program that optimizes DC location and allocation between retailers and DCs. We show that it can be converted to a conic quadratic program that can be efficiently solved. Some valid inequalities are added to the program to improve computational efficiency. We conclude by reporting numerical experiments that reveal some interesting properties of the model.

Keywords: capacitated facility location model; conic quadratic programming; valid inequalities; closed-loop supply chain

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1. Introduction
In the increasingly competitive global manufacturing environment, the success of a corporation depends on its ability to favorably manage its supply chains. A supply chain includes all of the components necessary to design, fabricate, distribute, sell, support, use, and recycle (or dispose of) a product. Competitive and regulatory pressures present new challenges in supply chain management. Consequently, green supply chains, reverse supply chains, closed-loop supply chains, and sustainable supply chains are getting more attention. In particular, supply chain managers are interested in economically handling returned products by reusing them to obtain numerous financial benefits (Blackburn et al. 2004).

Companies face time and cost trade-offs in the implementation of integrated supply chains. Time-varying prices of returned products, especially for time-sensitive and short life-cycle products, complicate the problem. For example, consumer electronics products such as PCs can lose their value at rates of 1% per week (Guide and Van Wassenhove 2009). In response, a company might prefer to shorten the flow time of returned products in a reverse supply chain. This strategy results in more profits from the salvage value of returned products by reentering them into the market as quickly as possible. Meanwhile, batch processing of products to benefit from economies of scale is also a recommended strategy but requires a slower flow-time supply chain. Thus, the time versus cost trade-off enters into planning.

We consider a three-tiered supply network: one supplier, some distribution centers (DCs) with capacity limitation, and retailers. The product can simultaneously flow in two directions. The forward direction is the flow of the retailer’s order of the product from the DC. In turn, the DCs get replenished from the supplier based on the specified inventory policy. The reverse direction is the flow of returned products from the retailers to the corresponding DCs and then back to the supplier to be reprocessed. Note that the DCs can hold stocks of both new and returned products. Figure 1 illustrates the structure of the three-tiered supply network.
The main contributions of this paper can be summarized as follows:

1. It is the first model to our knowledge in closed-loop supply chain design to jointly consider capacitated DCs, stochastic demands of new and returned products, risk pooling to buffer random demands, savings from colocating of forward and reverse flows in the same DC, and value loss related to inventory and transportation times.

2. It employs a novel and powerful solution technique, the conic integer programming approach, that is convenient for the model presented and many others with similar nonlinear optimization forms.

3. The convex hull of the feasible solutions is explored to add valid inequalities that improve computational efficiency.

4. From our computational studies, we obtain interesting managerial insights. For example, we show that the more time sensitive the returned product is, the less costly it is to retrieve salvage value. We also show the effects of value loss related to inventory and transportation times, where a smaller or larger optimal number of opened DCs is recommended depending on whether the dominant factor is time spent in inventory or in transportation.

The remainder of the paper is organized as follows. In §2, we present a literature review on the integrated forward/reverse network design and the capacitated facility location problem. Section 3 develops a nonlinear mixed-integer programming formulation of the supply chain. In §4, the model is converted into a conic quadratic mixed-integer program to be solved efficiently. Subsequently, some valid inequalities are developed to improve the computational efficiency of the branch-and-cut algorithm and the quality of the solution. Next, in §5, we explore the behavior of the supply chain under this optimization strategy through computational experiments on real data. In the last section, we conclude and discuss future research avenues.

2. Literature Review

The integrated forward/reverse supply chain network design is an emerging research topic. The interested reader can refer to the works of Guide and Van Wassenhove (2009) and Akçalı, Cetinkaya, and Üster (2009) for comprehensive reviews on closed-loop supply chains. We now briefly review some of the most relevant papers.

Sahyouni, Savaskan, and Daskin (2007) developed three generic uncapacitated, integrated closed-loop supply chain design models that minimize fixed locating and transportation costs, whereas Lu and Bostel (2007) presented a two-level uncapacitated location problem with three types of facilities that minimize fixed setup costs and transportation costs. Ko and Evans (2007) proposed a genetic algorithm-based heuristic to solve a multiperiod, two-echelon, multicommodity, capacitated facility location model. Üster et al. (2007) and Easwaran and Üster (2009, 2010) studied multiproduct closed-loop supply chain network design problems. Their objective was to locate collection centers and finite-capacity manufacturing facilities while coordinating the forward and reverse flows in the network so as to minimize the processing, transportation, and fixed location costs. Pishvaee, Farahani, and Dullaert (2010) developed a bi-objective mixed-integer program to minimize the total costs and maximize the responsiveness of an integrated forward/reverse logistics network.
network. Fishvae, Rabbani, and Torabi (2011) proposed a closed-loop supply chain network design model in a robust optimization framework. Our work employs conic programming (Atamtürk, Berenguer, and Shen 2012) as the solution approach for our closed-loop supply chain network design model. Further, a significant difference between our model and previous work is the integration of capacitated facility location and inventory decisions in which facilities can accept forward or returned products with the possibility of pooling forward inventory for different retailer sites.

We have seen models in the literature that integrate location decisions with other types of decisions in supply chains, such as transportation decisions, robustness, and reliability considerateness. Daskin (1995); Langevin and Riopel (2005); and Melo, Nickel, and Saldanha-da-Gama (2009) provide good surveys on this subject. We are particularly interested in models that combine location with inventory decisions. Shen, Coullard, and Daskin (2003) and Daskin, Coullard, and Shen (2002) studied the impact of inventory costs on location decisions in a stochastic demand environment. Their model incorporates nonlinear working inventory costs and nonlinear safety-stock inventory costs. Shen, Coullard, and Daskin applied a column generation technique, whereas Daskin, Coullard, and Shen used Lagrangian relaxation to solve this joint location-inventory model. During the last decade, scholars have studied different versions of this problem by extending it to capacitated warehouses (Ozsen, Coullard, and Daskin 2008), customer service considerations (Shen and Daskin 2005), multicommodities (Shen 2005), supply uncertainty (Qi and Shen 2007), profit maximization (Shen 2006), disruptions (Qi, Shen, and Snyder 2010), etc. The present paper can be considered a novel extension of the integrated supply chain design problem since it is the first to our knowledge to integrate reverse flows into a joint location-inventory model in which warehouses are assumed to be capacitated. For a summary of publications that study integrated supply chain design problems, we refer to Shen (2007).

Capacity restrictions in the facility location problem are a natural extension of the original problem and play a critical role. The capacitated facility location problem (CFLP) and its variants are well studied in the literature. For a review, please refer to Mirchandani and Francis (1990). We note that most solution algorithms for capacitated facility location problems are adaptations of algorithms for uncapacitated problems. Therefore, heuristics such as Lagrangian relaxation-based algorithms (Holmberg, Rönqvist, and Yuan 1999; Daskin, Coullard, and Shen 2002; Langevin and Riopel 2005; Lu and Bostel 2007; Sahyouni, Savaskan, and Daskin 2007; Ozsen, Coullard, and Daskin 2008; Liu, Zhou, and Zhang 2010); Benders’ decomposition-based solution approaches (Üster et al. 2007; Easwaran and Üster 2009, 2010); and metaheuristics such as genetic algorithms (Ko and Evans 2007), tabu searches (Easwaran and Üster 2009), and memetic algorithms (Fishvæe, Farahani, and Dullaert 2010) are used extensively.

Both uncapacitated and capacitated supply chain design solution algorithms share the same problem: the subproblems generated are still intractable or only near-optimal solutions can be obtained. Therefore, some scholars have studied the cutting-plane method, which explores the polyhedral convex hull of the feasible solutions and constructs valid inequalities to combine with the branch-and-bound or branch-and-cut algorithm. Normally, it can dramatically improve the efficiency of these algorithms. The cutting-plane method for solving uncapacitated facility location problems has been studied since Cornejols, Fisher, and Nemhauser (1977). At the end of the 1980s, it was extended to solve the capacitated facility location problem (Leung and Magnanti 1989).

Valid inequalities have been used in the literature to improve the efficiency and quality of the models’ solutions. Some of the most used valid inequalities are clique inequalities (Leung and Magnanti 1989), odd-cycle inequalities (Leung and Magnanti 1989; Klose 2000), submodular inequalities (Aardal, Pochet, and Wolsey 1995; Klose 2000), (k, I, S, l) inequalities (Aardal, Pochet, and Wolsey 1995), flow cover inequalities (Aardal, Pochet, and Wolsey 1995; Aardal 1998; Klose 2000), knapsack cover inequalities, effective capacity inequalities, combinatorial inequalities (Aardal, Pochet, and Wolsey 1995; Aardal 1998), single-depot inequalities (Aardal 1998), lifted cover inequalities (Klose 2000), and extended polymatroid inequalities (Atamtürk, Berenguer, and Shen 2012). Some of them are facets for the capacitated facility location problems. Other valid inequalities are incorporated into Lagrangian relaxation-based methods to tighten the feasible region for capacitated facility location problems (Klose 2000; Miranda and Garrido 2008). The present article adds extended polymatroid inequalities to tighten the feasible region of a conic quadratic mixed-integer program.

3. Problem Formulation

There are three types of distribution centers in the network: forward DCs (new products), reverse DCs (returned products), and joint DCs (both new and returned products). We determine the DC locations among potential sites and the assignment of retailers to the DCs. The objective is to minimize the fixed charges of locating the distribution centers, working inventory costs, transportation costs, and the value loss of returned products.

To benefit from the risk pooling strategy, inventories are not kept at the retailers’ sites but at the DCs. The DCs can fill retailer demand and can store returned
products temporarily. An approximation to the \((Q, R)\) model with Type I service (Özsen, Coullard, and Daskin 2008) is used for managing the stock of new products. The inventory policy followed by the return products is an approximation of the economic order quantity (EOQ) since the EOQ formula will not always provide the optimal quantity (the system is capacitated).

To exploit economies of scale in transportation costs, returned products will be shipped back to the supplier for reprocessing after a predetermined quantity at the DCs is reached. At the same time, getting returned products back to the market quickly will bring more profit. Blackburn et al. (2004) investigated reverse supply chains for commercial returns (in particular, products returned by customers for any reason within 90 days of sale). In a real-world example, for $1,000 worth of product returns, nearly half the product value (> 45%) is lost in the return process by waiting for the product to be reprocessed. Indeed, a returned consumer product could wait in excess of 3.5 months before it is sent to disposition. Thus, we analyze the trade-off between efficiency and responsive costs when designing a forward/reverse supply chain network.

Before proposing the model, some important assumptions are followed. First of all, customer demands are Poisson distributed. Thus, variances of daily demand and returns are identical to the means of daily demand \((\mu_i^f)\) and returns \((\mu_i^r)\), respectively, for each retailer \(i\). Further, demands at the retailers are uncorrelated over time and across retailers. Demand of returned products is independent from the new product’s demand. The model also assumes that there is sufficient transportation capacity but controls capacity at each DC. Tables 1–3 define the variables and parameters.

In summary, model (3) is

\[
\min_{X, Y} \sum_{i \in I} \left( \sum_{j \in J} f_i^j X_{ij}^f + \sum_{i \in I} \beta_i x d_{ij} \mu_i^f Y_{ij}^f \right) + \theta h z_d \sum_{i \in I} \beta_i x d_{ij} \mu_i^f Y_{ij}^f + W I_i^f (D_i^f, Q_i^f) \right) \\
+ \sum_{i \in I} \left( \sum_{j \in J} f_i^j X_{ij}^r + \sum_{i \in I} \beta_i x d_{ij} \mu_i^r Y_{ij}^r + W I_i^r (D_i^r, Q_i^r) \right) \\
- \sum_{i \in I} S_i^c X_i^c + W \sum_{i \in I} R (Y_i^f, Q_i^r) \right) \tag{1}
\]

s.t. \(\sum_{j \in J} Y_{ij}^f = 1, \sum_{j \in J} Y_{ij}^r = 1 \quad \forall i \in I,\)

\(Y_{ij}^f \leq X_{ij}^f, \quad Y_{ij}^r \leq X_{ij}^r \quad \forall i \in I, \forall j \in J,\)

\(X_{ij}^c \leq X_{ij}^f, \quad X_{ij}^c \leq X_{ij}^r \quad \forall i \in I, \forall j \in J,\)

\(Q_i^f + z_d \sum_{i \in I} \beta_i x d_{ij} Y_{ij}^f + L_i \sum_{i \in I} \beta_i x d_{ij} Y_{ij}^r + Q_i^r \leq C_i \quad \forall j \in J,\)

\(Q_i^f, Q_i^r \geq 0 \quad \forall j \in J,\)

where

\[
W I_i^f (D_i^f, Q_i^f) = \begin{cases} 
F_i^f D_i^f + \beta_i x d_{ij} Q_i^f + \theta h z_d & \text{if } Q_i^f > 0, \\
0 & \text{if } Q_i^f = 0,
\end{cases}
\]
The objective function (1) consists of four parts: cost of forward flows, cost of reverse flows, savings from colocation of forward and reverse DCs, and the time value of returned products.

The first part sums the costs of handling new products including the fixed charge of locating forward DCs, the DC-to-retailer shipping costs, the safety stock costs to ensure customer satisfaction, and the working inventory cost. The working inventory cost of new products is formulated as Equation (9), which is the sum of the fixed costs for handling orders, the DC-to-supplier shipping costs, and the average order holding costs per year. The detailed explanation of Equation (9) can be found in Shen, Coullard, and Daskin (2003).

The second part of the objective contains the costs of the reverse flows. Except for the safety stock costs, it has the same cost components as the first part of the objective. The working inventory cost of returned products is represented as Equation (10). We suppose that the return rates of used products are constant and do not change over time.

The third part of the objective represents the fixed costs saved by the colocation of forward and reverse DCs at the same site. Note that, normally, the cost saving must be less than the minimum of the fixed charges of forward and reverse DCs. We therefore assume that $S^F = \min\{f^F, f^R\}$ (Sahyouni, Savaskan, and Daskin 2007).

The fourth part concerns the time value of returned products. $R(Y^R, Q^R)$ is the total average value loss of returned product per year, and it is related to the returned product’s marginal value of time. Derivation of the formula of $R(Y^R, Q^R)$ is given at the end of this section.

Constraints (2) ensure that each retailer is served by exactly one DC. Constraints (3) state that a retailer can only be assigned to an open DC. Constraints (4) stipulate that if a DC is assigned to serve both forward and reverse flows (i.e., a joint DC), then it acts as not only a forward DC but a reverse DC as well. Consequently, cost savings occur. Note that forward (reverse) DCs refer to stand-alone forward (reverse) DCs and forward (reverse) facilities at joint DCs throughout the rest of this paper. Constraints (5) are the capacity restrictions of each DC $j$ (further described in the next paragraph). Constraints (6) are nonnegative constraints. Constraints (7) and (8) are standard integrality constraints.

Ozsen, Coullard, and Daskin (2008) pointed out that the capacity of a DC must withstand the worst-case scenario because the amount of space the warehouse needs is proportional to peak inventory. In particular, this happens when there is no demand for new products and no shipment of returned products during the replenishment lead time. Thus, the capacity constraints can be formulated as

$$Q^F_j + x_{j} z_{j} L_j \sum_{i} \mu_{i} Y_{i}^R \leq C_j \quad \forall j \in J,$$

where $I^r$ is the set of retailers served by DC $j$. The first and fourth terms represent the order quantity of new products and returned products, respectively. The second term is the safety stock under the assumption of normal demands and covers stock-outs that occur with a probability of $\alpha$ or less. Note that $z_{j}$ is a standard normal deviate such that $P(z \leq z_{j}) = \alpha$. The third term is the average demand during lead times. We could tighten the right-hand side of this capacity constraint by multiplying $C_j$ by $(X^F_j + X^R_j - X^C_j)$ for the cases in which $X^C_j \geq X^F_j + X^R_j - 1$. This alternative formulation does not provide significantly better results for our experiments.

### 3.1. Derivation of the Average Value Loss of Returned Product

We start the derivation by defining the average value loss of returned product associated with inventory times at DC $j$ to build lot $Q^R_j$ per year for a linear decay rate ($R_{inv}(Q^R)$). An exponential decay rate could also be used but, as shown in Appendix A, a linear decay rate is a good enough approximation.

Based on Blackburn et al. (2004), the daily marginal value of time, denoted by $\gamma$, can be represented by the

$$\text{Value of returned product ($\$\)}$$

Figure 2  
Time Value of Product Returns

Source: Blackburn et al. (2004).
The average value loss of returned product associated with inventory times at DC $j$, denoted by $R_{av}(Q^R_j)$, can be defined in a similar way as we calculate the inventory holding cost of returned products in a lot-sizing problem. Since, on average, there will be $\chi Q^R_j/2$ returned product of inventory on hand per year, the average value loss of returned products per year is defined as

$$ R_{av}(Q^R_j) = \frac{\gamma V}{2} Q^R_j, $$

where $V$ is the initial price of returned products.

To completely define the average value loss of returned product, we need to add the value loss related to transportation times of returned products to the average value loss function associated to building up $Q_j$ at DC $j$. For a linear decay rate, we define the average amount of returned product in transportation per year associated to DC $j$ as $\sum_{i \in I} (d^R_{ij} + a^R_{ij})/k \chi \mu_i^R Y^{R\prime}_{ij}$, where $k$ is the daily transportation cost per unit of returned product, and so $(d^R_{ij} + a^R_{ij})/k$ represents the time spent by a returned product in transportation from retailer $i$ to DC $j$ and from DC $j$ back to the supplier. Thus, we define the total average value loss per year as

$$ R(Y^R_j, Q^R_j) = R_{av}(Q^R_j) + R_t(Y^R_j) $$

$$ = \frac{\gamma V}{2} (\chi Q^R_j + \sum_{i \in I} (d^R_{ij} + a^R_{ij})/k) \chi \mu_i^R Y^{R\prime}_{ij}. $$

(12)

4. Model Properties and Reformulation

The problem is modeled as a nonlinear mixed-integer program; in general, its optimal solutions are hard to find in a reasonable amount of time. However, we note that our model could be identified as a novel version of the family of joint location-inventory models that was introduced for the first time by Shen, Coullard, and Daskin (2003). From Atamtürk, Berenguier, and Shen (2012), we can define an equivalent conic quadratic mixed-integer program that will be directly solved using commercial optimization packages. Further, Atamtürk, Berenguier, and Shen (2012) suggested that some cuts can be beneficial valid inequalities for joint location-inventory models. In the current work, we show how the polymatroid cuts are beneficial for our specific model.

**DEFINITION 1.** A conic quadratic mixed-integer program (CQMIP) is an optimization problem of the form:

$$ \min \ c^T x $$

**s.t.** $Ax + b \leq d^T x + e$,  \quad $i = 1, \ldots, p$,  

where $x \in \mathbb{Z}^n \times \mathbb{R}^m$, $c \in \mathbb{R}^{n+m}$, $A \in \mathbb{R}^{p \times (n+m)}$, $b \in \mathbb{R}^n$, $d \in \mathbb{R}^{p \times (n+m)}$, $e \in \mathbb{R}^n$, $\| \cdot \|_2$ is the Euclidean norm, and all parameters are rational.

The following proposition provides an equivalent CQMIP formulation of problem (3).

**PROPOSITION 1.** Problem (3) is equivalent to the following (CQMIP):

$$ \min_{x, y, \nu} Z = \sum_{i \in I} \left\{ f^T x^T + \theta h z_n \omega_j + \sum_{i \in I} \beta(d^R_{ij} + a^R_{ij}) \chi \mu_i^R Y^{R\prime}_{ij} \right\} $$

$$ + \sum_{i \in I} \left\{ f^T x^R + \sum_{i \in I} \beta(d^R_{ij} + a^R_{ij}) \chi \mu_i^R Y^{R\prime}_{ij} \right\} $$

$$ + \frac{W y V x + \theta h \nu_j}{2} - \sum_{i \in I} S^C X^C $$

**s.t.** $\sum_{i \in I} Y^R_{ij} = 1$,  $\sum_{i \in I} Y^{R\prime}_{ij} = 1$  \quad $\forall i \in I$,  

$$ Y^R_{ij} \leq X^T_{ij} \leq X^{R\prime}_{ij} \quad \forall i \in I, \forall j \in J, $$

$$ X^T_{ij} \leq X^R_{ij} \leq X^{R\prime}_{ij} \quad \forall j \in J, $$

$$ \omega_j^2 \geq L_j \sum_{i \in I} \mu_i^T (Y^T_{ij})^2 \quad \forall j \in J, $$

$$ \frac{1}{2} (u^T_j + Q^R_{ij}) \geq H_j \chi \sum_{i \in I} \mu_i^R (Y^R_{ij})^2 + \frac{3}{2} (Q^R_{ij})^2 + \frac{1}{2} u^2_j \quad \forall j \in J, $$

$$ \frac{1}{2} (v^T_j + Q^R_{ij}) \geq H_j \chi \sum_{i \in I} \mu_i^R (Y^R_{ij})^2 + \frac{3}{2} (Q^R_{ij})^2 + \frac{1}{2} v^2_j \quad \forall j \in J, $$

$$ Q^T_{ij} + z_n \omega_j + L_j \sum_{i \in I} \mu_i^T Y^T_{ij} + Q^{R\prime}_{ij} \leq C_j \quad \forall j \in J, $$

$$ \omega_j, u_j, v_j, Q^T_{ij}, Q^{R\prime}_{ij} \geq 0 \quad \forall j \in J, $$

$$ X^T_{ij}, X^R_{ij}, X^{R\prime}_{ij}, Y^T_{ij}, Y^{R\prime}_{ij} \in \{0, 1\} \quad \forall i \in I, \forall j \in J, $$

where $\bar{\beta} = \beta + W y V x / h$, $H_j^T = 2(F^T_{ij} + \beta g^T_{ij})/(\theta h)$ and $H_j^R = 2(F^R_{ij} + \beta g^R_{ij})/(W y V x + \theta h)$.

**PROOF.** A conic transformation is employed to linearize the objective of problem (3) in order to convert it into a CQMIP model. First, three sets of auxiliary variables, $\omega_j, u_j,$ and $v_j,$ are introduced, which satisfy the following inequalities:

$$ \omega_j \geq \sqrt{L_j} \sum_{i \in I} \mu_i^T Y^T_{ij}, $$

$$ \frac{\theta h}{2} u_j \geq (F^T_{ij} + \beta g^T_{ij}) Q^T_{ij} + \frac{\theta h}{2} Q^{R\prime}_{ij}, $$

$$ \left( \frac{W y V x}{2} + \frac{\theta h}{2} \right) v_j \geq (F^R_{ij} + \beta g^R_{ij}) Q^R_{ij} + \left( \frac{W y V x}{2} + \frac{\theta h}{2} \right) Q^R_{ij}. $$

(24)
Recall that $\gamma_{ij} = Y_{ij}$ if $Y_{ij}$ is a binary variable, so we transform the previous inequalities as follows:

$$
\omega_i^2 \geq \sum_{i \in I} \mu_i^2 (Y_{ij}^2)^2,
$$

$$
\frac{1}{2} (u_j + Q_j^2) \geq \frac{(E^2 + \beta g_j^2)}{\theta h} \sum_{i \in I} \mu_i^2 (Y_{ij}^2)^2 + \frac{1}{2} Q_j^2,
$$

$$
\frac{1}{2} (v_j + Q_j^2) \geq \frac{(E^2 + \beta g_j^2)}{W_i V_j + \theta h} \sum_{i \in I} \mu_i^2 (Y_{ij}^2)^2 + \frac{1}{2} Q_j^2.
$$

Then the objective of problem (3) is reformulated as

$$
\sum_{j \in J} \left\{ f_j^T X_j + \theta h z_w \omega_j + \sum_{i \in I} \beta (d_{ij} + a_{ij}^2) \chi \mu_i^2 Y_{ij}^2 + \frac{\theta h}{2} u_j\right\}
$$

$$
+ \sum_{j \in J} \left\{ f_j^T X_j + \sum_{i \in I} \beta (d_{ij} + a_{ij}^2) \chi \mu_i^2 Y_{ij}^2 + \frac{W_{ij} V_j + \theta h}{2} v_j \right\}
$$

$$
- \sum_{j \in J} \gamma_j^2 X_j.
$$

The set of capacity constraints (5) is linearized by substituting the nonlinear term by $z_w \omega_j$, obtaining the set of constraints (19).

The rest of the constraints of problem (3) remain untransformed because they are linear. □

### 4.1. Extremal Extended Polymatroid Inequalities

Utilizing submodularity, the conic quadratic constraints (16)–(18) lead to a class of valid inequalities that can improve the performance of the solution algorithm. Before presenting the results, some definitions are introduced. To simplify the notation, we drop the superscripts $F$ and $R$ in this subsection.

**Definition 2.** A set function $f : 2^N \rightarrow R$ is submodular if $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$ for all $S, T \subseteq N$.

**Definition 3 (Schröjer 2003).** The polyhedron associated with the submodular function $f$ on $N$,

$$
EP_f := \left\{ \pi \in R^N \mid \pi(S) \leq f(S) \text{ for each } S \subseteq N \right\},
$$

is called the extended polymatroid associated with $f$ if $f(\emptyset) = 0$, where $\pi(S) = \sum_{i \in S} \pi_i$.

**Definition 4** (Atamtürk and Narayanan 2008). The inequalities associated with the extended polymatroid of $f$, $\pi x \leq w$, $\pi \in EP_f$, are called extended polymatroid inequalities. When the inequalities are defined by the extreme points of the extended polymatroid $EP_f$, they are called extremal extended polymatroid inequalities of $Q_f$.

**Proposition 2.** Let $\bar{\ell}_f$ denote the lower convex envelope of the sets of solutions that satisfy constraints (16); i.e.,

$$
\bar{\ell}_f = \text{conv} \left\{ (Y_{ij}, \omega_j) \in [0, 1]^{|I|} \times R : \omega_j \geq f(S) \right\},
$$

$$
\sqrt{L_i} \sum_{i \in S} \mu_i \forall S \subseteq I.
$$

Then, the inequality $\sum_{i \in S} \pi_i Y_{ij} \leq \omega_i$ is valid for $\bar{\ell}_f$, where $\pi_i = \sqrt{L_i} \sum_{i \in S} \mu_i - \sqrt{L_i} \sum_{i \in S \setminus i} \mu_i \in EP_f$, $S = \{i \mid Y_{ij} = 1\}$, and $S(i) = \{(1), (2), \ldots, (i)\}$, $1 \leq i \leq |I|$ for some permutation.

This valid inequality is an extremal extended polymatroid inequality.

**Proof.** See Appendix B. □

A similar result can be derived for the set of constraints (17) and (18).

**Proposition 3.** Let $\ell_u$ denote the lower convex envelope of the sets of solutions that satisfy constraints (17) and (18); i.e.,

$$
\ell_u = \text{conv} \left\{ (Y_{ij}, u_j, Q_j) \in [0, 1]^{|I|} \times R : \frac{1}{2} (u_j + Q_j)^2 \right\},
$$

$$
\geq H \sum_{i \in I} \mu_i (Y_{ij})^2 + \frac{3}{2} (Q_j)^2 + \frac{1}{2} u_j^2.
$$

Then, $\sum_{i \in I} \pi_i Y_{ij} \leq u_j + Q_j$ is a valid inequality for $\ell_u$, where $\pi_i = \sqrt{8 H} \sum_{i \in S} \mu_i - \sqrt{8 H} \sum_{i \in S \setminus i} \mu_i$, $S = \{i \mid Y_{ij} = 1\}$, and $S(i) = \{(1), (2), \ldots, (i)\}$, $1 \leq i \leq |I|$ for some permutation. This valid inequality is an extremal extended polymatroid inequality of $\bar{\ell}_u = \text{conv} \left\{ (Y_{ij}, \bar{u}_j) \in [0, 1]^{|I|} \times R : \frac{1}{2} \bar{u}_j^2 \geq 4\sum_{i \in I} \mu_i (Y_{ij})^2 \right\}$.

**Proof.** See Appendix B. □

To find these valid inequalities, we introduce the concept of separation problem.

**Definition 5.** The separation problem associated with a combinatorial optimization problem is as follows: Given $x^* \in R^n$, is $x^* \in \text{conv}(X)$? If not, find an inequality $\pi x \leq \pi_0$ satisfied by all points in $X$ but violated by the point $x^*$.

The separation problem for the extremal extended polymatroid inequality can be computed by a greedy algorithm described in Edmonds (1970) and Atamtürk and Narayanan (2008).

The greedy algorithm will find valid extremal extended polymatroid inequalities of the types described in Propositions 2 and 3. We will add them to our formulation to speed up the solution process.

### 5. Computational Experiments

In this section, we perform computational experiments to test the model and check how the addition of valid inequalities can speed up the computation. The parameter values and descriptions of this model are listed in Table 4. We start by varying the inventory and transportation weights (see Table 5) without adding valid inequalities. The subsequent analysis studies the effect of the valid inequalities over a range of different DC capacity values (see Tables 6–8). The second subsection is devoted exclusively to evaluating the impact of the DC capacities (see Table 9). We continue by studying...
their experiments on the following modifications of these data sets (e.g., Daskin, Coullard, and Shen 2002; Shen, Coullard, and Daskin 2003; Atamtürk, Berenguer, and Shen 2012). For the first data set, the mean demand of new products is obtained by dividing the first group of demand data by 100, and the fixed forward facility location costs are obtained by dividing the facility location costs by 100. For the second data set, the mean demand is obtained by dividing the first group of demand data by 1,000, and the fixed forward facility location costs are obtained as in the first data set. The mean quantity of returned products is calculated by multiplying the return rate with the second group demand data from each data set. The return rate is identical among all retailers. The fixed reverse facility location costs are equal to the fixed forward facility location costs. Because of space limitations and the fact that we observe the same trends and insights when changing the value of the weight W, we assign it a unique weight value through all experiments. Each retailer location is also a candidate DC location. The cost savings of joint DCs are set to 0.2\min(f_{f_1}^r, f_{f_2}^r) in all experiments. The capacities of all DCs are equal to each other for the same experiment.

We do not directly employ these data to the computational experiments in Tables 6–8, which focus on computational performance. In these tables we report the average of 10 random instances per row. In turn, each instance is generated by adding noise to some of the main parameters defined previously. In particular, we multiply the values of the mean demand, standard deviation, and fixed costs by (1 + \epsilon), where \epsilon is drawn from Uniform[−0.1, 0.1].

### Table 4 Parameters of the Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>h</td>
<td>1</td>
<td>Inventory holding cost per unit of products per year for each DC</td>
</tr>
<tr>
<td>\alpha</td>
<td>97.5%</td>
<td>Service level</td>
</tr>
<tr>
<td>z</td>
<td>1.96</td>
<td>Standard normal deviate such that ( P(z \leq z) = \alpha )</td>
</tr>
<tr>
<td>F_j</td>
<td>10</td>
<td>Fixed order costs</td>
</tr>
<tr>
<td>g_j</td>
<td>10</td>
<td>Fixed transportation costs between the DCs and supplier</td>
</tr>
<tr>
<td>a_j, d_j</td>
<td>5</td>
<td>Per-unit shipment costs between the DCs and supplier</td>
</tr>
<tr>
<td>L_j</td>
<td>1</td>
<td>Load time in days</td>
</tr>
<tr>
<td>\chi</td>
<td>1</td>
<td>Number of days worked in a year</td>
</tr>
<tr>
<td>W</td>
<td>1</td>
<td>Weight factor associated with loss in value of returned products</td>
</tr>
<tr>
<td>\gamma</td>
<td>10%</td>
<td>Marginal value of time of returned products</td>
</tr>
<tr>
<td>V</td>
<td>49</td>
<td>Initial price of returned products</td>
</tr>
<tr>
<td>k</td>
<td>100</td>
<td>Daily transportation cost per unit returned product</td>
</tr>
</tbody>
</table>

### Table 5 Performance of the Model Without Adding Valid Inequalities (Time Limit = 3,600 s)

<table>
<thead>
<tr>
<th>No. of cities</th>
<th>\theta</th>
<th>\beta</th>
<th>Total cost</th>
<th>DCs_f</th>
<th>DCs_g</th>
<th>DCs^c</th>
<th>CPU time (s)</th>
<th>Gap (%)</th>
<th>Nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>49</td>
<td>0.10</td>
<td>0.0010</td>
<td>122,663.26</td>
<td>10</td>
<td>2</td>
<td>2</td>
<td>371.99</td>
<td>0.0092</td>
<td>27,136</td>
</tr>
<tr>
<td>49</td>
<td>0.10</td>
<td>0.0020</td>
<td>161,920.12</td>
<td>10</td>
<td>2</td>
<td>2</td>
<td>11.33</td>
<td>0.0094</td>
<td>1,930</td>
</tr>
<tr>
<td>49</td>
<td>0.10</td>
<td>0.0030</td>
<td>192,657.53</td>
<td>10</td>
<td>2</td>
<td>2</td>
<td>1.82</td>
<td>0.0039</td>
<td>46</td>
</tr>
<tr>
<td>49</td>
<td>0.10</td>
<td>0.0040</td>
<td>216,988.24</td>
<td>15</td>
<td>3</td>
<td>3</td>
<td>1.24</td>
<td>0.0036</td>
<td>40</td>
</tr>
<tr>
<td>49</td>
<td>0.10</td>
<td>0.0050</td>
<td>239,955.55</td>
<td>16</td>
<td>3</td>
<td>3</td>
<td>1.20</td>
<td>0.0000</td>
<td>33</td>
</tr>
<tr>
<td>49</td>
<td>0.20</td>
<td>0.0020</td>
<td>163,068.68</td>
<td>10</td>
<td>2</td>
<td>2</td>
<td>26.23</td>
<td>0.0097</td>
<td>4,366</td>
</tr>
<tr>
<td>49</td>
<td>0.50</td>
<td>0.0050</td>
<td>244,652.01</td>
<td>15</td>
<td>3</td>
<td>3</td>
<td>1.51</td>
<td>0.0075</td>
<td>40</td>
</tr>
<tr>
<td>49</td>
<td>1.00</td>
<td>0.0050</td>
<td>249,012.60</td>
<td>15</td>
<td>3</td>
<td>3</td>
<td>1.12</td>
<td>0.0032</td>
<td>28</td>
</tr>
<tr>
<td>49</td>
<td>2.00</td>
<td>0.0050</td>
<td>256,340.58</td>
<td>15</td>
<td>3</td>
<td>3</td>
<td>0.99</td>
<td>0.0047</td>
<td>35</td>
</tr>
<tr>
<td>49</td>
<td>5.00</td>
<td>0.0050</td>
<td>274,761.25</td>
<td>15</td>
<td>3</td>
<td>3</td>
<td>1.20</td>
<td>0.0091</td>
<td>81</td>
</tr>
<tr>
<td>88</td>
<td>0.10</td>
<td>0.0010</td>
<td>24,545.54</td>
<td>10</td>
<td>9</td>
<td>9</td>
<td>9.98</td>
<td>0.0000</td>
<td>121</td>
</tr>
<tr>
<td>88</td>
<td>0.10</td>
<td>0.0020</td>
<td>33,869.69</td>
<td>12</td>
<td>10</td>
<td>10</td>
<td>8.38</td>
<td>0.0054</td>
<td>148</td>
</tr>
<tr>
<td>88</td>
<td>0.10</td>
<td>0.0030</td>
<td>41,693.04</td>
<td>15</td>
<td>11</td>
<td>11</td>
<td>9.35</td>
<td>0.0058</td>
<td>146</td>
</tr>
<tr>
<td>88</td>
<td>0.10</td>
<td>0.0040</td>
<td>47,569.63</td>
<td>22</td>
<td>11</td>
<td>11</td>
<td>3.96</td>
<td>0.0087</td>
<td>45</td>
</tr>
<tr>
<td>88</td>
<td>0.10</td>
<td>0.0050</td>
<td>52,248.90</td>
<td>23</td>
<td>15</td>
<td>14</td>
<td>3.05</td>
<td>0.0024</td>
<td>21</td>
</tr>
<tr>
<td>88</td>
<td>0.20</td>
<td>0.0020</td>
<td>34,542.68</td>
<td>11</td>
<td>10</td>
<td>10</td>
<td>9.24</td>
<td>0.0098</td>
<td>296</td>
</tr>
<tr>
<td>88</td>
<td>0.50</td>
<td>0.0050</td>
<td>55,286.35</td>
<td>22</td>
<td>14</td>
<td>14</td>
<td>6.99</td>
<td>0.0099</td>
<td>321</td>
</tr>
<tr>
<td>88</td>
<td>1.00</td>
<td>0.0050</td>
<td>57,923.66</td>
<td>22</td>
<td>13</td>
<td>13</td>
<td>18.16</td>
<td>0.0100</td>
<td>1,500</td>
</tr>
<tr>
<td>88^*</td>
<td>2.00</td>
<td>0.0050</td>
<td>62,222.82</td>
<td>21</td>
<td>13</td>
<td>13</td>
<td>136.71</td>
<td>0.0100</td>
<td>11,338</td>
</tr>
<tr>
<td>88^*</td>
<td>5.00</td>
<td>0.0050</td>
<td>72,004.29</td>
<td>17</td>
<td>12</td>
<td>12</td>
<td>3,601.80</td>
<td>0.0792</td>
<td>126,865</td>
</tr>
</tbody>
</table>

*For related computational experiments, see Tables 6 and 7.
Table 6  Comparison Between the Instances With and Without the Valid Equalities (88-City Data Set, \( W = 1, \theta = 2, \beta = 0.005 \), Time Limits = 3,600 s)

<table>
<thead>
<tr>
<th></th>
<th>CPLEX</th>
<th>CPLEX + CUTS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CPU time (s) Nodes (%)</td>
<td>CPU time (s) Nodes (%)</td>
</tr>
<tr>
<td>10,000</td>
<td>123.98</td>
<td>8,548 0.0100</td>
</tr>
<tr>
<td>9,000</td>
<td>130.12</td>
<td>9,431 0.0100</td>
</tr>
<tr>
<td>8,000</td>
<td>758.97</td>
<td>53,770 0.0100</td>
</tr>
<tr>
<td>7,900</td>
<td>624.83</td>
<td>48,511 0.0100</td>
</tr>
<tr>
<td>7,800</td>
<td>378.02</td>
<td>30,033 0.0100</td>
</tr>
<tr>
<td>7,700</td>
<td>152.83</td>
<td>11,338 0.0100</td>
</tr>
</tbody>
</table>

Table 7  Comparison Between the Instances With and Without the Valid Equalities (88-City Data Set, \( W = 1, \theta = 5, \beta = 0.005 \), Time Limits = 3,600 s)

<table>
<thead>
<tr>
<th></th>
<th>CPLEX</th>
<th>CPLEX + CUTS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CPU time (s) Nodes (%)</td>
<td>CPU time (s) Nodes (%)</td>
</tr>
<tr>
<td>10,000</td>
<td>3,602.39</td>
<td>123,846 0.2067</td>
</tr>
<tr>
<td>9,000</td>
<td>3,602.02</td>
<td>123,682 0.1269</td>
</tr>
<tr>
<td>8,000</td>
<td>3,602.19</td>
<td>118,435 0.2454</td>
</tr>
<tr>
<td>7,900</td>
<td>— 3 —</td>
<td>1,381.11</td>
</tr>
<tr>
<td>7,800</td>
<td>— 6 —</td>
<td>785.52</td>
</tr>
<tr>
<td>7,700</td>
<td>3,601.80</td>
<td>118,520 0.0855</td>
</tr>
</tbody>
</table>

Table 8  Comparison Between the Instances With and Without the Valid Equalities (88-City Data Set, \( W = 1, \theta = 10, \beta = 0.005 \), Time Limits = 3,600 s)

<table>
<thead>
<tr>
<th></th>
<th>CPLEX</th>
<th>CPLEX + CUTS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CPU time (s) Nodes (%)</td>
<td>CPU time (s) Nodes (%)</td>
</tr>
<tr>
<td>10,000</td>
<td>3,601.61</td>
<td>42,141 0.8330</td>
</tr>
<tr>
<td>9,000</td>
<td>3,601.54</td>
<td>41,329 0.9515</td>
</tr>
<tr>
<td>8,000</td>
<td>3,601.44</td>
<td>51,072 0.9322</td>
</tr>
<tr>
<td>7,900</td>
<td>3,601.51</td>
<td>47,565 0.9614</td>
</tr>
<tr>
<td>7,800</td>
<td>3,601.63</td>
<td>44,235 0.8668</td>
</tr>
<tr>
<td>7,700</td>
<td>— 4 —</td>
<td>3,601.51</td>
</tr>
</tbody>
</table>

Note. The em dash means we did not find any feasible solution within the time limits.

The computational experiments are conducted on a Hewlett-Packard 380 G7 server running the CentOS5.4 operating system. We used the MIQCP solver of CPLEX 12.1, which solves CQMIP relaxations at the nodes of the branch-and-bound tree, with CPLEX heuristics turned off.

5.1. Computational Performance of the Algorithm

In this section, we confirm the validity of the model and the efficiency of the algorithm. In Table 5, we report the results of numerical experiments carried out over different values of \( \theta \) and \( \beta \) (in the second and third columns, respectively). Note that the quantity of returned products is less than the demand of new products as defined in both data sets. The capacities of the DCs are set to 31,000 and 7,700 for the 49-city and 88-city data sets, respectively. They are 1,05 times the maximum daily demand of new products. Total costs are listed in the fourth column. DC usage, determined from Table 5, we observe the following.

- Total costs increase when weight factors (\( \theta \) or \( \beta \)) increase.
- More forward/reverse DCs are opened if unit transportation cost is expensive (larger \( \beta \)). In contrast,
Table 12  Impact of Returned Products’ Marginal Time Value (Return Rate = 60%, k = ∞)

<table>
<thead>
<tr>
<th>Exp. no.</th>
<th>γ (%)</th>
<th>Total cost</th>
<th>Cost $^a$</th>
<th>Cost $^b$</th>
<th>DCs $^c$</th>
<th>DCs $^d$</th>
<th>DCs $^e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>54,755</td>
<td>32,033</td>
<td>25,124</td>
<td>23</td>
<td>20</td>
<td>19</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>56,489</td>
<td>32,033</td>
<td>25,521</td>
<td>23</td>
<td>17</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>58,439</td>
<td>32,033</td>
<td>26,360</td>
<td>23</td>
<td>16</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>50</td>
<td>59,751</td>
<td>32,033</td>
<td>26,940</td>
<td>23</td>
<td>13</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>70</td>
<td>60,745</td>
<td>32,033</td>
<td>27,438</td>
<td>23</td>
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<td>12</td>
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<tr>
<td>6</td>
<td>90</td>
<td>61,805</td>
<td>32,033</td>
<td>27,863</td>
<td>23</td>
<td>13</td>
<td>12</td>
</tr>
</tbody>
</table>

Table 13  Impact of Returned Products’ Marginal Time Value (Return Rate = 100%, k = ∞)

<table>
<thead>
<tr>
<th>Exp. no.</th>
<th>γ (%)</th>
<th>Total cost</th>
<th>Cost $^a$</th>
<th>Cost $^b$</th>
<th>DCs $^c$</th>
<th>DCs $^d$</th>
<th>DCs $^e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>60,957</td>
<td>32,412</td>
<td>31,708</td>
<td>24</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>63,351</td>
<td>32,387</td>
<td>32,639</td>
<td>24</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>66,230</td>
<td>32,376</td>
<td>33,983</td>
<td>24</td>
<td>24</td>
<td>24</td>
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<tr>
<td>4</td>
<td>50</td>
<td>68,176</td>
<td>32,370</td>
<td>34,876</td>
<td>23</td>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>5</td>
<td>70</td>
<td>69,722</td>
<td>32,349</td>
<td>35,616</td>
<td>23</td>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>6</td>
<td>90</td>
<td>71,054</td>
<td>32,330</td>
<td>36,320</td>
<td>23</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

some forward/reverse DCs are closed because the holding cost becomes expensive (larger $\theta$).

- In most cases, joint DCs are preferred because of cost savings, but we can find some cases in which the numbers of reverse DCs and joint DCs are different because of capacity restrictions.
- Computational times have an increasing trend when we increase the value of $\theta$, but they decrease when $\beta$ increases.

Finally, we present Tables 6–8 to confirm the computational benefits of the extremal extended polymatroid inequalities. In these tables, “C” reports the capacities of the DCs, “CPU time” reports the average running time, and “Nodes” reports the average number of nodes in branch-and-bound tree. The last three columns report the number of valid inequalities added to the root node of the corresponding branch-and-bound tree; “$W$,” “$U$,” and “$V$” are the average number of extremal extended polymatroid inequalities generated based on constraints (17), (18), and (16), respectively. Note that the valid inequalities contribute to finding the optimal solution in less time. For these experiments, the optimality gap is set to CPLEX’s default optimal gap at 0.01%. Even in instances where the optimal solutions cannot be found in the time limit specified, adding the valid inequalities improves the quality of
the solutions by providing a solution closer to the optimal (smaller “Gap (%)”).

5.2. The Impact of DC Capacity
In this section, we study the effect of DC capacity on both the number of open DCs and the operations at the DCs, as illustrated in Table 9. From this table, we see that if we tighten the capacities, the number of forward DCs increases and more DCs are utilized at capacity.

The impact of DC capacity on order sizes at joint DCs is summarized in the following property.

PROPERTY 1. Given a joint DC $j$, 
(a) If capacity at DC $j$ is binding, the optimal order quantities of new products ($Q^F_j$) and returned products ($Q^R_j$) are either • strictly less than the corresponding EOQ quantities or • equal to the corresponding EOQ quantities.
(b) If capacity at DC $j$ is not binding, the optimal order quantities are the corresponding EOQ quantities:

\[ Q^F_j = Q^F_{j,EOQ} = \sqrt{\frac{2(F^F_j + \beta_j^F D^F_j)}{\theta h}}, \]

\[ Q^R_j = Q^R_{j,EOQ} = \sqrt{\frac{2(F^R_j + \beta_j^R D^R_j)}{\theta h + W^j V^j \gamma}}, \]

PROOF. See Appendix B. □

5.3. The Impact of Returned Products’ Marginal Value of Time
Returned products’ marginal value of time is associated with the degree of time sensitivity of the product’s price. If we consider it along with DC capacity, it leads to distinct characterizations of DC location decisions. Table 11 shows the effects of $\gamma$ with $k = 500$. To emphasize the impacts of returned products’ inventory, we also report some results with different return rates and $k = \infty$, which simulates the cases when transportation times are ignored (see Tables 12 and 13). The marginal value of time in the experiments, $\gamma$, is set to 1%, 10%, 30%, 50%, 70%, and 90%. The columns labeled “Cost $^F$” and “Cost $^R$” list the costs associated with the forward and reverse flows, respectively. Table 10 lists the parameters of the experiments. From the experiments, we find that $\gamma$ has a different effect on the results with and without the considerations of loss values associated with transportation times. We find the following.

1. Fewer reverse DCs are needed for highly time-sensitive returned products (higher $\gamma$) when transportation times are neglected. Intuitively, the storage time of time-sensitive returned products has been reduced in order to retrieve more salvage value from them. This implies more shipments of smaller quantity, so the storage space needed decreases. In contrast, more reverse DCs are built for higher $\gamma$ if transportation times are part of the value loss function. This is because having more DCs reduces transportation times.

2. Returned products impact the forward DCs in number and location. While more forward DCs are constructed in the case of nonnegligible transportation times (see Table 11) if the product is more time sensitive, fewer forward DCs are constructed in the case of negligible transportation times and highly time-sensitive returned products (see Table 13). Even if the number of forward DCs is identical, the locations of some forward DCs are different. For instance, in Table 12, a forward DC is opened in Atlanta in experiment 2, whereas a Charlotte DC is constructed in experiment 3 and an Atlanta one is closed. Similar results are also observed in reverse DCs. However, the number of the stand-alone forward DCs increases for highly time-sensitive returned products (the difference between the DCs $^F$ and DCs $^C$ columns) in Tables 12 and 13 to offset for the reduction of forward product capacity created by the drop in the number of joint DCs. Similarly, the number of stand-alone forward DCs decreases for higher $\gamma$ in Table 11 to offset for the increment of capacity created by the increase in the number of joint DCs.

3. Reverse flows impact forward flows’ decisions not only on facility location but also on inventory management. In Table 13, it is interesting to note that the forward flow costs slightly diminish for highly time-sensitive returned products. This shows an opposite trend with the total costs and reverse flow costs.

Figure 3 aims to describe the trade-off between working inventory costs and value loss of returned products associated with inventory times. As shown, the working inventory cost decreases while the loss in value increases in the range of $(0, Q^R_{j,EOQ})$. From the proof of Property 2 (see Appendix B), $Q^F_j$ refers to the optimal shipment quantity of returned products when considering the loss in value. Once taking the loss in value of returned products into account, $Q^R_{j,EOQ}$ must be less than $Q^R_j$. This leads to different decisions because
of different priorities/preferences of the decision makers. It is therefore interesting to find the corresponding non-inferior solutions.

We vary the weight factor $W$ and plot the trade-off curves corresponding to working inventory cost and value loss of returned products (see Figures 4 and 5). Table 14 summarizes the parameters used in these experiments.

As shown in Figure 4, we find that more DCs are constructed if we try to reduce the loss in value of returned products (in other words, we try to retrieve more salvage value from returned products). However, Figure 5 shows the opposite results if we ignore the transportation times between the retailers, the DCs, and the supplier. As already observed in Tables 11 and 12, this implies that time in transportation and time in inventory of returned products have opposed influences on supply chain design decisions.

Figures 6 and 7 report the trade-off curves between working inventory cost and loss in value of returned products with different $\gamma$ (i.e., returned products with different marginal values of time). These figures show that, given a fixed change of loss in value, the change in working inventory costs of time-sensitive returned products (higher $\gamma$) is smaller than that of time-insensitive returned products (lower $\gamma$). Therefore—and confirming our intuition—it makes more sense to salvage the value of time-sensitive returned products. We also note that the working inventory costs in Figure 6 are larger than those in Figure 7 (likewise in Figures 4 and 5). This is because of the influence of transportation times in the value loss function of the model behind Figure 6. Since there is a major influence of transportation times, more DCs are opened, which implies that smaller amounts of returned product will be shipped to each opened DC ($Q^R_j$). Figure 3 shows that smaller $Q^R_j$ leads to larger working inventory costs.

### 6. Conclusions and Future Research

This paper studies the capacitated facility location problem with bidirectional flows, which is starting to receive much attention in the literature. This model minimizes the fixed location costs, the working inventory, and the transportation costs. Moreover, we consider the loss in value of returned products when making location decisions. We transform the model into a conic quadratic mixed-integer program. The model can be solved efficiently in most cases by using CPLEX. Some valid inequalities are added to improve the efficiency of the branch-and-cut algorithm and the quality of the solutions.

We perform an extensive computational study and observe the following interesting results:

1. Extremal extended polymatroid inequalities are computationally beneficial for the formulation presented.

2. DC capacity has an impact on facility location decisions and inventory operations. The optimal order quantities of new or returned products at a joint DC are at the EOQ level if DC capacity is nonbinding and below the EOQ level if capacity is binding.

3. Marginal value of time of returned products impacts the location and inventory decisions not only of reverse facilities but also of forward facilities. If the transportation times of returned products are negligible, fewer DCs are constructed for highly time-sensitive returned products. However, the reverse effect occurs when transportation times become longer.

4. To retrieve more salvage value from returned products, it is necessary to tolerate higher working inventory costs of returned products. However, for a fixed change of loss in value, the respective working inventory cost increment will be smaller for time-sensitive products than for time-insensitive products. In addition, retrieving more salvage value might result in a smaller or larger optimal number of opened DCs, depending on whether inventory or transportation times of returned products have a major influence. Thus, it is helpful to consider carefully the salvage value of time-sensitive returned products when making location decisions.

We suggest some avenues of further research. First, this model can naturally be extended to incorporate multiple products (Shen 2005). Second, it would be interesting to explore the impact of pricing on the design of the integrated network with returned products (Shen 2006). Finally, other decisions, such as the vehicle routing decisions for collecting returned products (e.g., Berger, Coullard, and Daskin 2007; Shen and Qi 2007), can be integrated into the capacitated facility location problem.

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### Appendix A. The Linear Decay Rate Is a Good Enough Approximation to Define $R_{inv}(Q^R_j)$

We can define the average value loss of returned product associated with inventory times ($R_{inv}(Q^R_j)$) to tightly approximate...
the well-studied case of returned products with exponential price decay function. In this case,

\[ \text{Price}(t) = V e^{-\gamma t}, \]

where \( t \) denotes the time the unit is held at DC. This case is used extensively in the literature to investigate the loss in value of returned products (Guide et al. 2006; Blackburn and Scudder 2009).

The salvage value of returned products (with batch size equal to \( Q_R^k \)) can be found by integrating over \([0, Q_R^k]\),

\[ \int_0^{Q_R^k} \text{Price} \left( \frac{Q_R^k - q \chi Q_R^k}{Q_R^k} \right) dq. \]

Then, the loss in value per batch equals

\[ VQ_R^k - \int_0^{Q_R^k} \text{Price} \left( \frac{Q_R^k - q \chi Q_R^k}{Q_R^k} \right) dq. \]

Subsequently, \( R_{inv}(Q_R^k) \) can be determined by multiplying the number of shipments to the supplier per year, \( D_R^k/Q_R^k \),

\[ R_{inv}(Q_R^k) = \frac{D_R^k}{Q_R^k} \left( VQ_R^k - \int_0^{Q_R^k} \text{Price} \left( \frac{Q_R^k - q \chi Q_R^k}{Q_R^k} \right) dq \right) \]

\[ = VD_R^k \frac{\chi \gamma Q_R^k}{Q_R^k} \left( 1 - e^{-\chi \gamma Q_R^k/D_R^k} \right). \]  

(A1)

It is intractable to find the optimal solution of the model because of the complexity of the resulting \( R_{inv}(Q_R^k) \). Therefore, we replace \( e^{-\chi \gamma Q_R^k/D_R^k} \) with its second-order Taylor-series expansion:

\[ e^{-\chi \gamma Q_R^k/D_R^k} \approx 1 - \frac{\chi \gamma Q_R^k}{D_R^k} + \frac{\chi^2 \gamma^2 (Q_R^k)^2}{2(D_R^k)^2}. \]

Then, Equation (A1) can be approximated as follows:

\[ R_{inv}(Q_R^k) \approx \hat{R}_{inv}(Q_R^k) = \frac{V \chi \gamma}{2} Q_R^k. \]  

(A2)

Let us define \( R_{ij}^k(Q_R^k) = WI_j^k(D_R^k, Q_R^k) + W \cdot R_{inv}(Q_R^k) \), which represents the sum of working inventory costs and value loss of returned products associated with inventory. We next present the following property regarding this expression that is studied experimentally in §5.3.

**Property 2.** \( R_{ij}^k(Q_R^k) \) is unimodal in \( Q_R^k \).

**Proof.** See Appendix B. \( \square \)

From the approximation in (A2), we have that

\[ R_{ij}^k(Q_R^k) \approx \hat{R}_{ij}^k(Q_R^k) = WI_j^k(D_R^k, Q_R^k) + W \cdot \hat{R}_{inv}(Q_R^k). \]

To examine the accuracy of the approximation, we define

\[ \text{ERR}(Q_R^k) = R_{ij}^k(Q_R^k) - \hat{R}_{ij}^k(Q_R^k) = W(R_{inv}(Q_R^k) - \hat{R}_{inv}(Q_R^k)), \]

which measures the error between \( R_{ij}^k \) and its approximation, \( \hat{R}_{ij}^k \).

**Property 3.** \( \text{ERR}(Q_R^k) \) is a concave function of \( Q_R^k \).

**Proof.** See Appendix B. \( \square \)

Owing to Property 3, the quantities of returned products with respect to the maximal error, denoted by \( Q_{R_{\text{max}}}^k \), can be determined by using search algorithms such as the golden section method or bisection method. Since the optimal shipment quantity of returned products with capacity constraints must be less than that without consideration of capacity constraints, we also calculate the latter one, denoted by \( Q_{R_{\text{max}}}^k \).

Because of Property 2, search algorithms are employed as well. Then \( Q_R^k = \min(Q_{R_{\text{max}}}^k, Q_{R_{\text{max}}}^k) \) is used to examine the accuracy of the approximation.

We perform 2,000,000 numerical experiments with the parameters drawn uniformly from the range given in Table A.1. The values of \( Q_{R_{\text{max}}}^k \) and \( Q_{R_{\text{max}}}^k \) are found by the bisection method. The results are summarized in Tables A.2 and A.3, in which we normalize the error by \( R_{ij}^k \), \( \text{error} = |\text{ERR}/R_{ij}^k| \), and by \( R_{ij}^k \), \( \text{error} = |\text{ERR}/R_{ij}^k| \), respectively. The results show that \( \hat{R}_{ij}^k \) is a tight approximation of \( R_{ij}^k \).

Note that Equation (A2) has the same form as Equation (11). Therefore, we can adopt Equation (11) to approximate the loss in value of returned products associated with inventory times.

**Appendix B. Proofs**

Proof of Property 1. The shipment quantities of new and returned products can be obtained exogenously by solving
the following program, whose set of solutions we denote by $W_i(D_i^f, D_i^R)$:

$$\min \left\{ \frac{D_i^f}{Q_i^f} + \beta(Q_i^f + a_i^f Q_i^f) \frac{D_i^R}{Q_i^R} + \theta \frac{h Q_i^R}{2} + \theta h z \sqrt{L_i^f + \frac{D_i^R}{\lambda}} \right\}$$

$$+ \frac{D_i^R}{Q_i^R} + \beta(Q_i^R + a_i^R Q_i^R) \frac{D_i^R}{Q_i^R} + \frac{2h + W_i V X}{2} \frac{D_i^R}{Q_i^R}$$

s.t. $Q_i^f + z \sqrt{L_i^f + \frac{D_i^f}{\lambda}} + \frac{D_i^f}{\lambda} + Q_i^R \leq C_i$,

$Q_i^f, Q_i^R, \lambda \geq 0.$

Because the problem is convex, we apply Karush-Kuhn-Tucker conditions and obtain the following equations:

$$- \left( \frac{(F_i^f + \beta g_i^f)D_i^f}{Q_i^f} \right)^2 + \frac{h \theta}{2} + \lambda_i = 0,$$

$$- \left( \frac{(F_i^R + \beta g_i^R)D_i^R}{Q_i^R} \right)^2 + \frac{h \theta + W_iV X}{2} + \lambda_i = 0,$$

$$\lambda \left( Q_i^f + \frac{D_i^f}{\lambda} + z \sqrt{L_i^f + \frac{D_i^f}{\lambda}} + Q_i^R - C_i \right) = 0,$$

$Q_i^f, Q_i^R, \lambda \geq 0.$

where $\lambda_i$ is a nonnegative Lagrangian multiplier. If the capacity constraint is strictly satisfied, then $\lambda_i = 0$, and the shipment quantities of new and returned products can be determined by their economic order quantities. That is,

$$Q_i^f = \sqrt{\frac{2(F_i^f + \beta g_i^f)D_i^f}{h \theta}},$$

$$Q_i^R = \sqrt{\frac{2(F_i^R + \beta g_i^R)D_i^R}{h \theta + W_iV X}}.$$

If the capacity constraint is binding, then $\lambda_i \geq 0$, and the shipment quantities of new and returned products are

$$Q_i^f = \sqrt{\frac{2(F_i^f + \beta g_i^f)D_i^f}{h \theta + 2\lambda_i}},$$

$$Q_i^R = \sqrt{\frac{2(F_i^R + \beta g_i^R)D_i^R}{h \theta + W_iV X + 2\lambda_i}}.$$

Note that in both cases, the shipment quantities are less than the respective economic order quantities if $\lambda_i > 0$ or are equal to the economic order quantities if $\lambda_i = 0.$

**Proof of Proposition 2.** Let $f(S) = \sqrt{\ell_i \mu(S)}$, where $\mu(S) = \sum_{t \in S} \mu_t$, is a submodular function because of its concavity based on the following result studied in Nemhauser and Wolsey (1999) and Shen, Coullard, and Daskin (2003).

A set function $f: 2^N \to \mathbb{R}$ defined by $f(S) = g(a(S))$, where $g(\cdot)$ is concave and $a(S)$ is the sum of the components of $a \in R_N^+$ on $S \subseteq N$, is submodular.

Hence, $\pi_i = \sqrt{\ell_i \mu(S_i)} - \sqrt{\ell_i \mu(S_{i-1})}$ is an extreme point of the extended polymatroid $EP_i$ based on Edmonds (1970).

That is, $\pi_i \in EP_i$. Therefore, $\pi(S) \leq f(S) \leq \omega_j$, which completes the proof. □

**Proof of Proposition 3.** Let $\tilde{u}_i = u_i + Q_i$. From constraints (17) and (23), we obtain the following relaxed form of constraints (17):

$$\tilde{u}_i^2 \geq \frac{4(F_i^f + \beta g_i^f)D_i}{h \theta} + 3(Q_i)^2 + u_i^2$$

$$= \frac{4(F_i^f + \beta g_i^f)D_i}{h \theta} + 3(Q_i)^2 + \left( \frac{2(F_i^f + \beta g_i^f)D_i}{h \theta} \frac{Q_i}{Q_i^f} + Q_i \right)^2.$$  \hspace{1cm} (B1)

Taking the derivative of the right-hand side of the aforementioned inequality with respect to $Q_i$, we obtain

$$6Q_i + 2 \left( -1 + \frac{2D_i(F_i^f + \beta g_i^f)}{h \theta Q_i^f} \right) \left( \frac{Q_i + 2D_i(F_i^f + \beta g_i^f)}{h \theta Q_i^f} \right) = 0.$$

Solving for $Q_i$, we obtain $Q_i = \sqrt{(F_i^f + \beta g_i^f)D_i/(h \theta)}$. Substituting this into inequality (B1), we obtain the following relaxed constraint:

$$\tilde{u}_i^2 \geq \frac{4(F_i^f + \beta g_i^f)D_i}{h \theta} + 3(Q_i)^2 + u_i^2$$

$$\geq \frac{4(F_i^f + \beta g_i^f)D_i}{h \theta} + 12 \frac{D_i(F_i^f + \beta g_i^f)}{h \theta} D_i$$

$$\geq \frac{16(F_i^f + \beta g_i^f)D_i}{h \theta} D_i.$$

According to Proposition 2, we can get a valid inequality (that is also an extremal extended polymatroid inequality), $\sum_{i \in \mathcal{I}} \pi_i Y_{ij} \leq u_j + Q_j$, for the lower convex envelope of the relaxed constraint; that is,

$$e_u = \operatorname{conv} \left\{ (Y_i, \tilde{u}_i) \in [0, 1]^{|\mathcal{I}|} \times R : \frac{1}{2} \tilde{u}_i^2 \geq 4H_j \sum_{i \in \mathcal{I}} \mu_i(Y_{ij})^2 \right\},$$

where $\pi_i = \sqrt{8H_j \sum_{t \in \mathcal{I}(i-1)} \mu_t} - \sqrt{8H_j \sum_{t \in \mathcal{I}(i-1)} \mu_t}$.

Note that the suggested inequality is also valid for $e_u$ of constraints (17), where

$$e_u = \operatorname{conv} \left\{ (Y_i, u_i, Q_i) \in [0, 1]^{|\mathcal{I}|} \times R^2 : \frac{1}{2}(u_i + Q_i)^2 \geq H_j \sum_{i \in \mathcal{I}} \mu_i(Y_{ij})^2 + \frac{1}{2} u_i^2 \right\}.$$  \hspace{1cm} (B2)

The same proof can be derived for constraints (18) and (24). □

**Proof of Property 2.** Let $R_i^f$ be the sum of a concave function ($R_{rv}(Q_i^f)$) and a convex function ($W_i^f(D_i^f, Q_i^f)$). The second-order derivative of $R_i^f$ with respect to $Q_i^f$ is

$$\frac{d^2 R_i^f}{d Q_i^f} = \frac{1}{\chi Y(Q_i^f)^2} \left\{ -2D_i^f [W_i^f + (F_i^f + \beta g_i^f)Y] 

+ e^{-(x_i Y_i^f)^2} Y_i^f V W [2(D_i^f)^2 + 2D_i^f Q_i^f Y_i^f + (Q_i^f)^2 Y_i^f] \right\}. \quad (B2)$$
We cannot determine whether or not it is positive. As such, $R^I\{x\}$ is neither convex nor concave.

Let $Q^R\{x\}$ denote $Q^R$ such that
\[
\frac{\partial R^I\{x\}}{\partial Q^R\{x\}} = \frac{1}{2\chi (Q^R)^3} \left[ -2e^{-\chi D^R (Q^R)^2} D^R J\gamma (Q^R)^2 V W (D^R + Q^R) \gamma \\
+ 2(D^R)^2 V W - 2D^R (F^R + \beta^R) \chi \gamma \\
+ h(Q^R)\chi \gamma \theta \right] = 0.
\]

Substituting $Q^R\{x\}$ into Equation (B2), we find
\[
\frac{\partial^2 R^I\{x\}}{\partial (Q^R)^2} = \frac{1}{\chi (Q^R)^3} \left[ h(Q^R)\chi \gamma \theta + e^{-\chi D^R (Q^R)^2} V W (Q^R) \gamma \theta \right] \geq 0.
\]

It shows that $R^I\{x\}$ is unimodal in $Q^R$ and $Q^R\{x\}$ is global minimum. □

Proof of Property 3. Taking the first- and second-order derivative of $R_{inv} (Q^R)$ with respect to $Q^R$, we can show that $R_{inv} (Q^R)$ is an increasing and concave function of $Q^R$:
\[
\frac{dR_{inv} (Q^R)}{dQ^R} = V (D^R)^2 e^{-\chi D^R (Q^R)^2} \\
\left[ \frac{(Q^R) \gamma \chi}{D^R} \left( e^{\chi D^R (Q^R)^2} - 1 - \frac{Q^R \gamma \chi}{D^R} \right) > 0, \right.
\]
\[
\frac{d^2 R_{inv} (Q^R)}{d(Q^R)^2} = Ve^{-\chi D^R (Q^R)^2} \\
\left( \frac{(Q^R)^3 \gamma \chi}{D^R} \right)
\left[ -2(D^R)^2 (e^{-\chi D^R (Q^R)^2} - 1 - e^{\chi D^R (Q^R)^2}) \\
+ 2D^R (Q^R)^2 \gamma \chi + (Q^R)^2 \gamma \chi \right] \\
= \frac{2(D^R)^2 V e^{-\chi D^R (Q^R)^2}}{(Q^R)^3 \gamma \chi} \\
\left[ e^{-\chi D^R (Q^R)^2} + 1 + \frac{Q^R \gamma \chi}{D^R} + \frac{(Q^R)^2 \gamma \chi \chi^2}{2(D^R)^2} \right] < 0,
\]
and $R_{inv} (Q^R)$ is linear. Therefore, $ERR (Q^R)$ is a concave function of $Q^R$. □

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