

Fall, 2022

ME 323 – Mechanics of Materials

Lecture 6 – Axial deformation

Reading assignment: Ch.6 lecturebook



Mechanical Engineering

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Generalized Hooke's law

Generalized Hooke's law (review)

- Isotropic material: material properties are independent of the orientation of the body
- The *principle of superposition* can be used for linear elastic materials and small deformations (i.e., strains may be added linearly)

$$\sigma_x = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\epsilon_x + \nu(\epsilon_y + \epsilon_z) - (1 + \nu)\alpha\Delta T]$$

$$\sigma_y = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\epsilon_y + \nu(\epsilon_x + \epsilon_z) - (1 + \nu)\alpha\Delta T]$$

$$\sigma_z = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\epsilon_z + \nu(\epsilon_x + \epsilon_y) - (1 + \nu)\alpha\Delta T]$$

$$\tau_{xy} = G\gamma_{xy} \quad \tau_{xz} = G\gamma_{xz} \quad \tau_{yz} = G\gamma_{yz}$$

Q: Plane stress and plane strain configurations?!

Axial deformation

Axial deformation

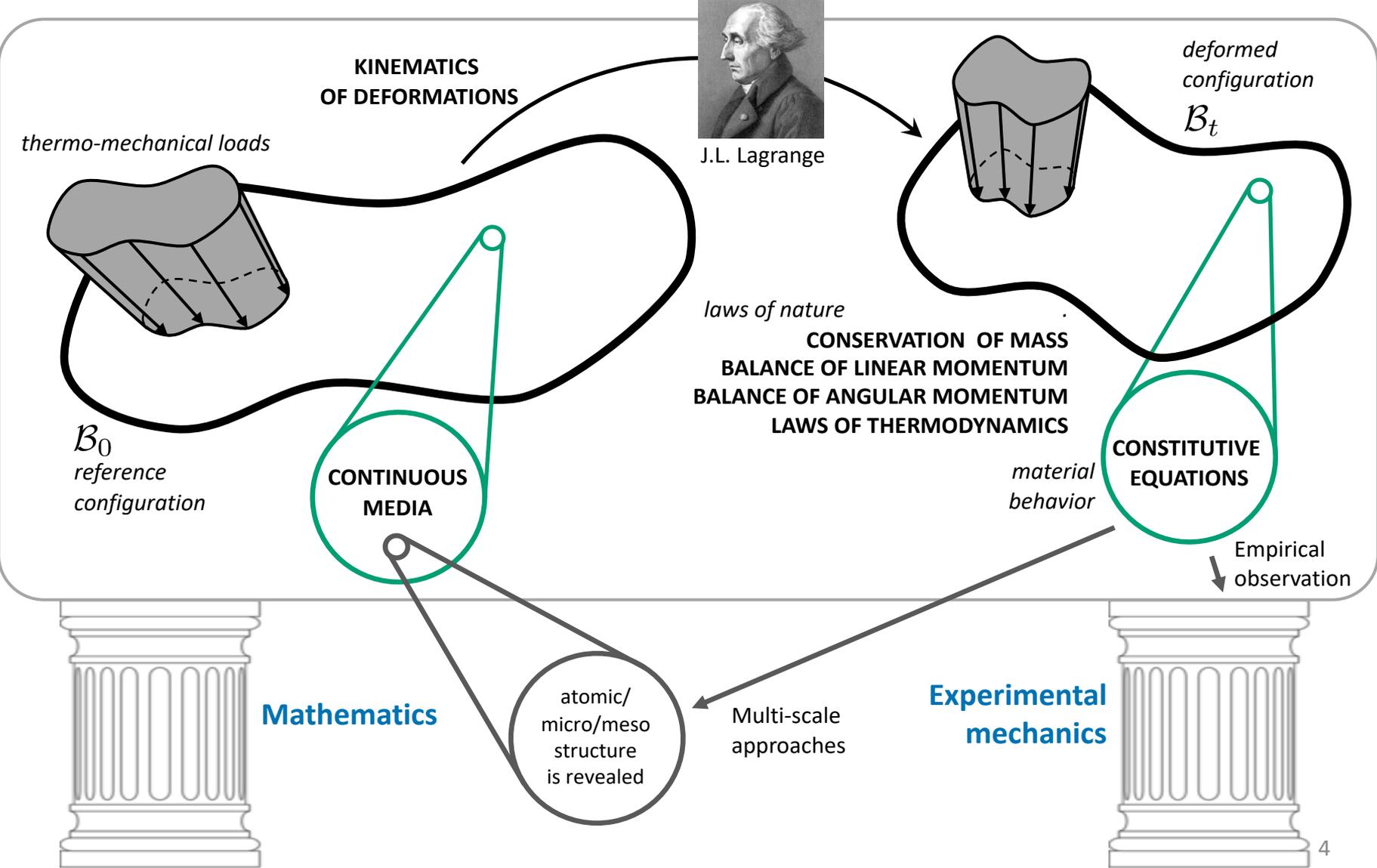
- Why axial deformations and not three-dimensional deformations?

As engineers, we will always try to identify simplifications in the problem description that will produce *idealized models* amenable to analytical solutions.

These simplifications are nothing but hypotheses that will have to be verified experimentally or with more descriptive models.

Note: [knowing the procedure for finding the analytical solution of an idealized model](#) is as important as [knowing the hypotheses behind the idealized model](#)!!

Theory of deformable bodies



Axial deformation

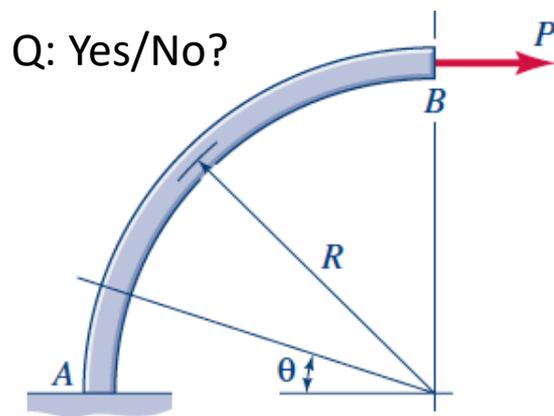
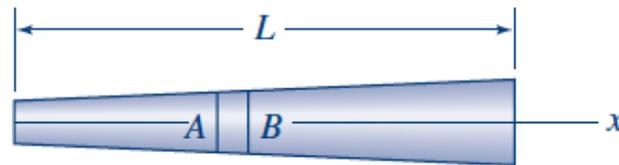
Axial deformation (@ ME 323)

- Geometry of the solid body: straight, slender member with cross section that is either constant or that changes slowly along the length of the member.
- Kinematic assumptions: cross sections, which are plane and are perpendicular to the axis before deformation, remain plane and remain perpendicular to the axis after deformation. In addition, cross sections do not rotate about the axis.
- Material behavior: isotropic linear elastic material;
small deformations.
- Equilibrium: the above assumptions reduce the problem to a one-dimensional problem!!

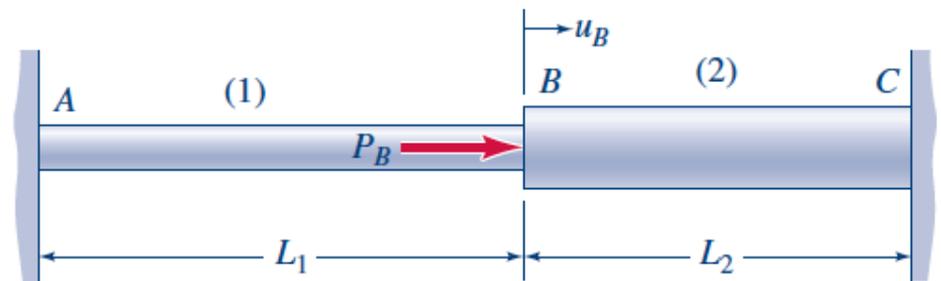
Axial deformation

Axial deformation

- Geometry of the solid body: straight, slender member with cross section that is either constant or that changes slowly along the length of the member.



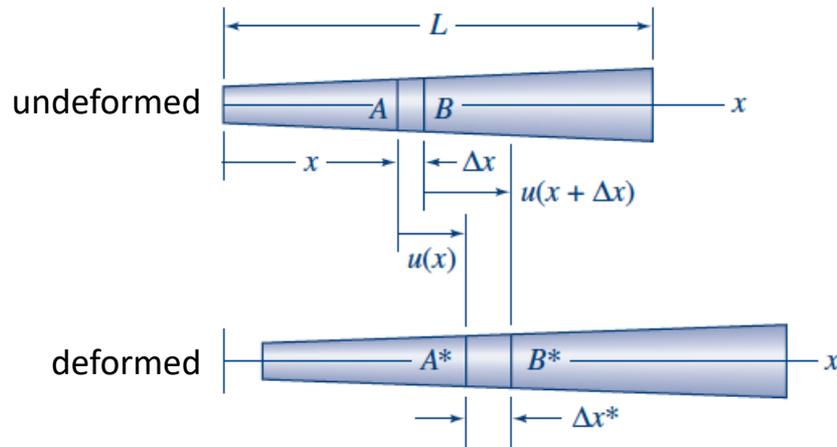
Q: Yes/No? ... "changes slowly"?!?



Axial deformation

Axial deformation

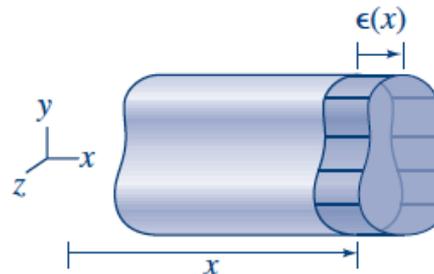
- Kinematic assumptions: cross sections, which are plane and are perpendicular to the axis before deformation, remain plane and remain perpendicular to the axis after deformation. In addition, cross sections do not rotate about the axis.



The strain-displacement relationship is ...
(recall lecture 2)

$$\epsilon_x(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta x^* - \Delta x}{\Delta x} = \frac{du(x)}{dx} = \epsilon(x)$$

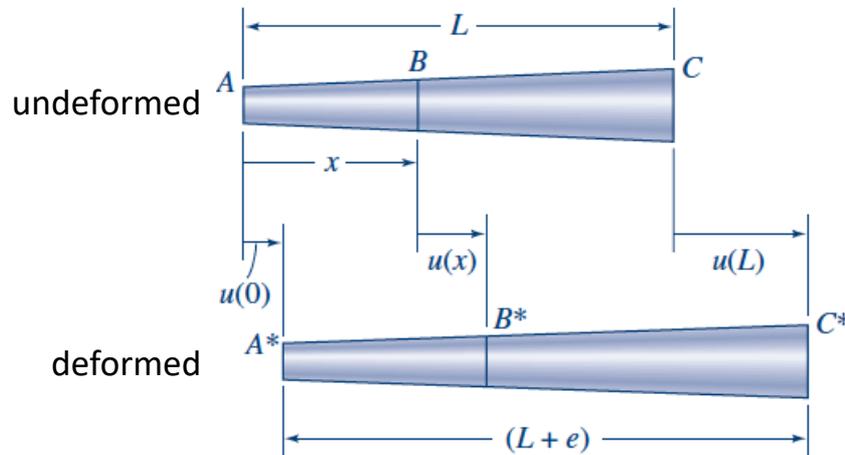
Q: Shall we go back to the three-dimension problem?



Axial deformation

Axial deformation

- Kinematic assumptions: cross sections, which are plane and are perpendicular to the axis before deformation, remain plane and remain perpendicular to the axis after deformation. In addition, cross sections do not rotate about the axis.



Q: what is the elongation of the member?

$$e = \int_0^L \epsilon(x) dx = u(L) - u(0)$$

$$\text{Sanity check: } e = \int_0^L \epsilon(x) dx = \int_0^L \frac{du(x)}{dx} dx = \int_0^L du(x) = u(L) - u(0)$$

Axial deformation

Axial deformation

- Material behavior: isotropic linear elastic material;
small deformations.

Generalized Hooke's law reduces to ...

$$\sigma(x) = E\epsilon(x)$$

Q: Can the Young's modulus depend on the axial coordinates?

$$\sigma(x) = E(x)\epsilon(x)$$

- Can the Young's modulus change across the cross-section?

If yes, would it still be a case of axial deformations?

Axial deformation

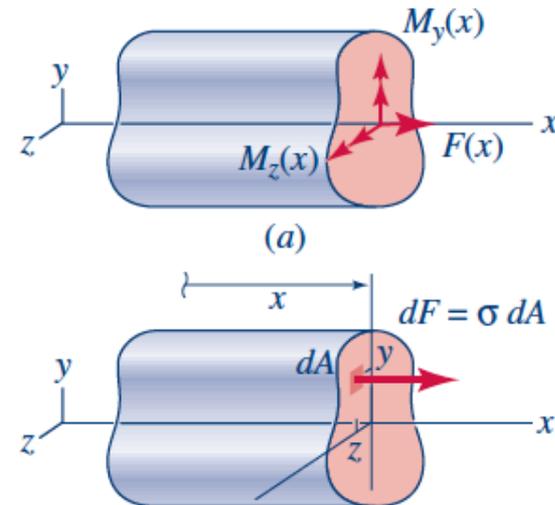
Axial deformation

- Equilibrium: the above assumptions reduce the problem to a one-dimensional problem!!
+ Resultants for homogeneous materials (recall Lecture 5)

$$F = \int_A \sigma_x dA = \sigma(x) A(x)$$

$$M_y = \int_A z \sigma_x dA = \bar{z} \sigma(x) A(x)$$

$$M_z = - \int_A y \sigma_x dA = -\bar{y} \sigma(x) A(x)$$

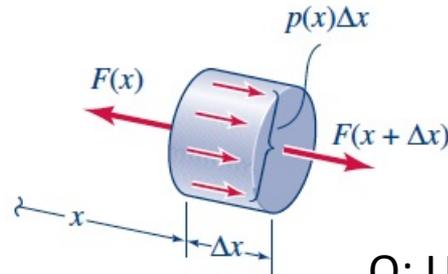
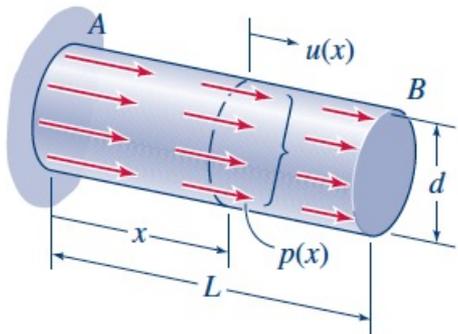


Q: Shall we use the centroid of the cross section as the axis of the member and thus as the x -axis of the idealized model? Yes!

Axial deformation

Axial deformation

- Equilibrium: the above assumptions reduce the problem to a one-dimensional problem!!
- + Let's consider an axisymmetric member loaded with axisymmetric body forces



Q: Units of the body forces $p(x)$?

Equilibrium: $F(x + \Delta x) + p(x)\Delta x - F(x) = 0$

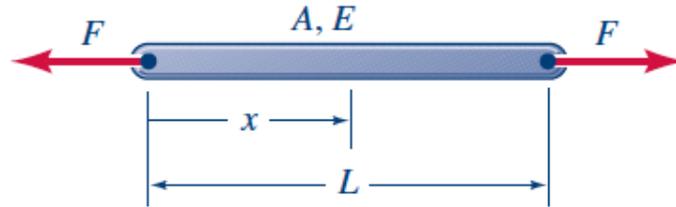
... rearrange and take the limit for a differential volume

$$\frac{dF(x)}{dx} + p(x) = 0$$

Axial deformation

Axial deformation (simplest case)

Homogeneous, constant cross section, no body forces: $F = \sigma A = E\epsilon A = \frac{EA}{L}e$



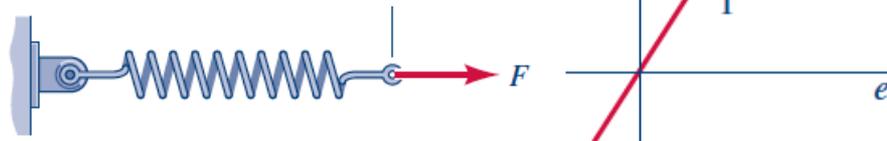
- Stiffness coefficient: force required to produce a unit elongation $F = ke$

$$k = \frac{AE}{L}$$

- Flexibility coefficient: elongation produced when a unit force is applied $e = fF$

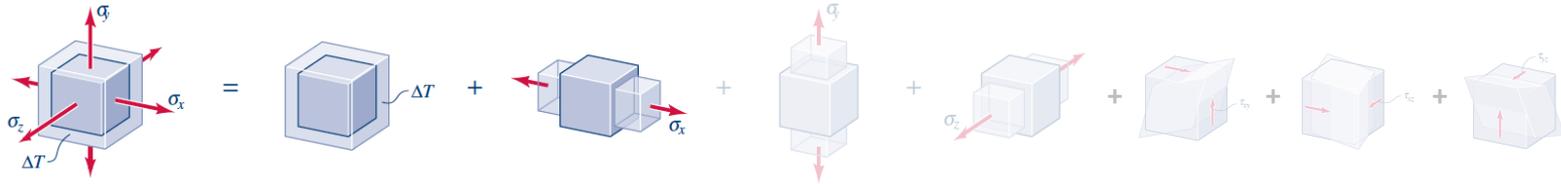
$$f = \frac{1}{k} = \frac{L}{AE}$$

Note: recall the behavior of a linear spring



Axial deformation

Axial deformation (thermal effects)



- Principle of superposition

$$\epsilon = \epsilon_{\text{total}} = \epsilon_{\text{elastic}} + \epsilon_{\text{thermal}} = \frac{\sigma}{E} + \alpha \Delta T$$

- For material properties and thermal loads that only depend on x

$$\epsilon(x) = \frac{\sigma(x)}{E(x)} + \alpha(x) \Delta T(x)$$

- Resultant force: $F(x) = \sigma(x)A(x)$
- Elongation (force-temperature-deformation)

$$e = \int_0^L \epsilon(x) dx = \int_0^L \frac{F(x)}{A(x)E(x)} dx + \int_0^L \alpha(x) \Delta T(x) dx$$

- Homogeneous, constant cross-section, thermal forces (no body forces)

$$e = \frac{FL}{AE} + \alpha L \Delta T$$

Axial deformation

Axial deformation (summary)

- Geometry of the solid body: straight, slender member with cross section that is either constant or that changes slowly along the length of the member.
- Kinematic assumptions: cross sections, which are plane and are perpendicular to the axis before deformation, remain plane and remain perpendicular to the axis after deformation. In addition, cross sections do not rotate about the axis.

Strain: $\epsilon(x) = \frac{du(x)}{dx} = \epsilon_{\text{elastic}} + \epsilon_{\text{thermal}}$

Elongation: $e = \int_0^L \epsilon(x) dx = u(L) - u(0)$

- Material behavior: isotropic linear elastic material; small deformations.

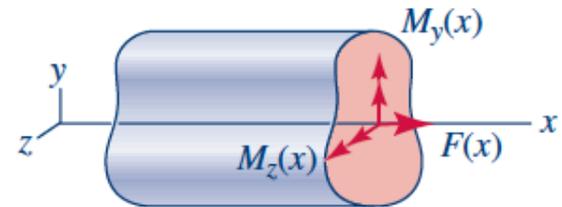
Homogeneous: $\epsilon(x) = \frac{\sigma(x)}{E} + \alpha \Delta T(x)$

- Equilibrium:

Homogeneous: $F(x) = \sigma(x)A(x)$

Homogeneous, constant cross section, no body forces, thermal load: $e = \frac{FL}{AE} + \alpha L \Delta T$

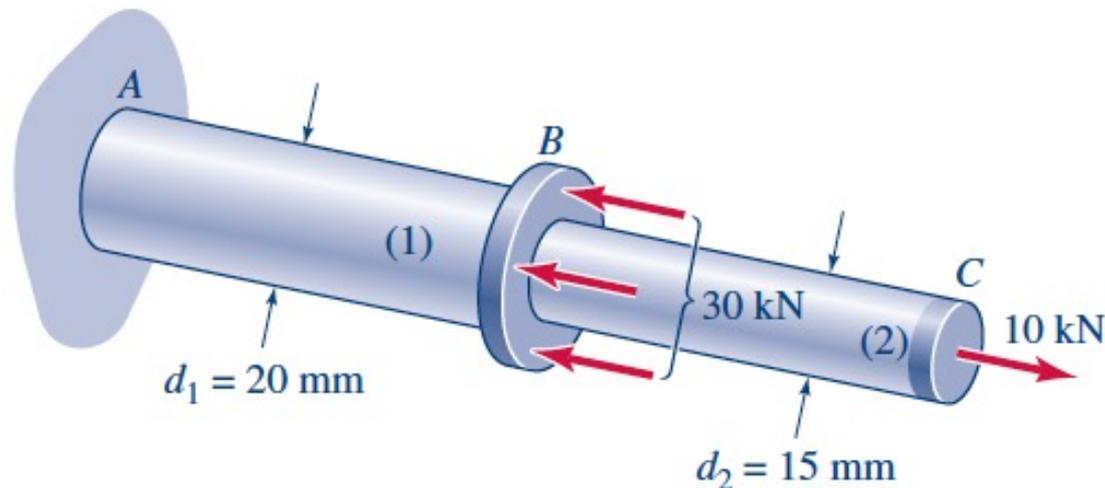
Homogeneous, loaded with body forces: $\frac{dF(x)}{dx} + p(x) = 0$



Strain energy density – Axial deformation

Example 5 (Example 3 from Lecture 2):

Two solid circular rods are welded to a plate at B to form a single rod, as shown in the figure. Consider the 30-kN force at B to be uniformly distributed around the circumference of the collar at B and the 10 kN load at C to be applied at the centroid of the end cross section. Determine the axial stress in each portion of the rod. $L_1=300\text{mm}$, $L_2=200\text{mm}$, $E_1=600\text{ GPa}$, $E_2=400\text{GPa}$.



Determine the displacement of end C.

.... easy!

Answer: $u_C = -3.6\mu\text{m}$

Axial deformation

Any questions?