Fall, 2022

ME 323 – Mechanics of Materials

Lecture 31 – Transformation of stress

Reading assignment: Ch.13 lecturebook

PURDUE UNIVERSITY | Mechanical Engineering

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Thin wall pressure vessels – Review

Cylindrical body with hemispherical end caps

$$\sigma_a = \frac{pr}{2t}$$ axial stress in the cylinder

$$\sigma_h = \frac{pr}{t}$$ Hoop stress in the cylinder

$$\sigma_s = \frac{pr}{2t}$$ normal stress in the sphere

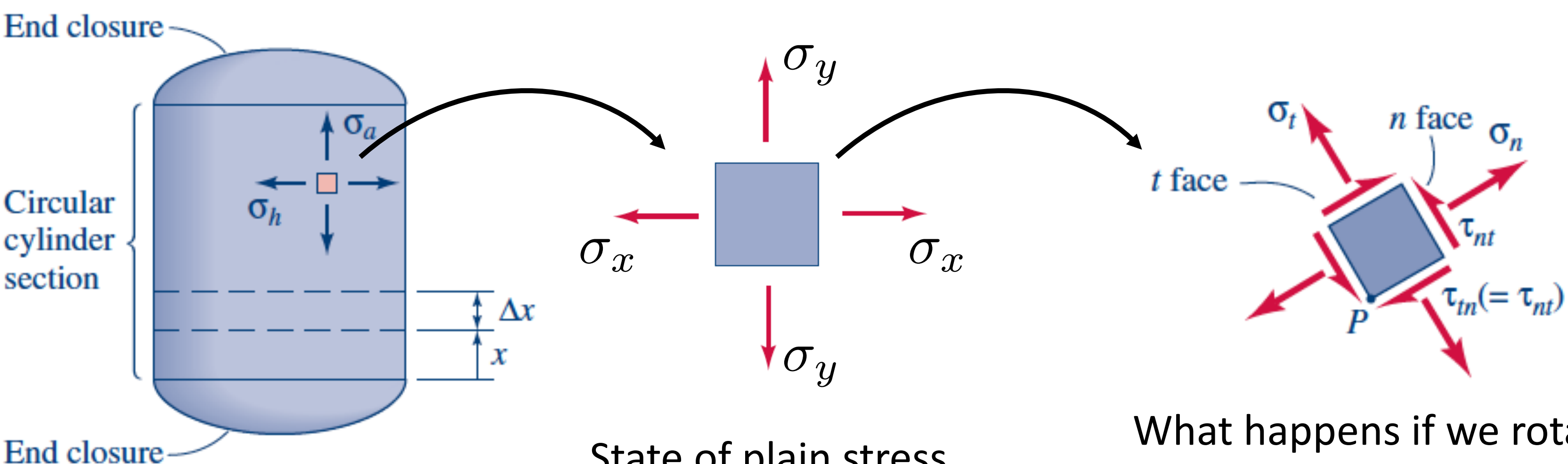
If the maximum (or allowable) tensile stress of the material is σ_{allow} .

What is the thickness of the body and the caps?

Thin wall pressure vessels – Mohr's circle

Problem 65:

Determine the stresses in a cylindrical pressure vessel 2.5 m in diameter and 10 mm thick. The internal pressure is 1200 kPa.



State of plain stress

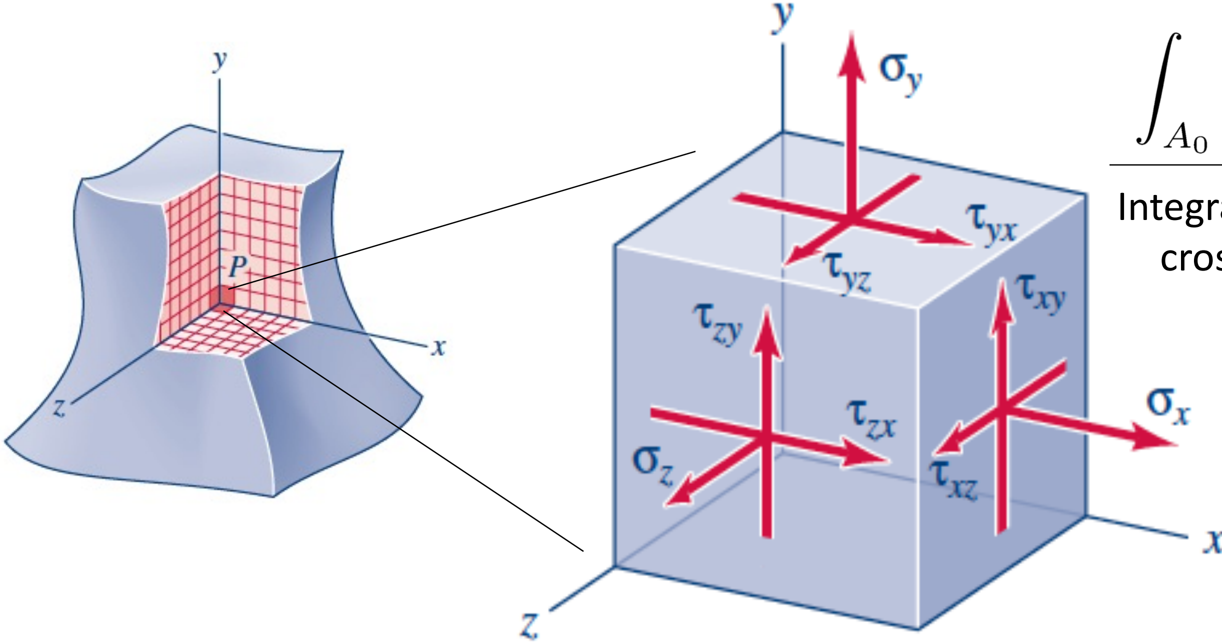
$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

What happens if we rotate the stress elements?

$$\tau_{nt} \neq 0$$

Transformation of stress

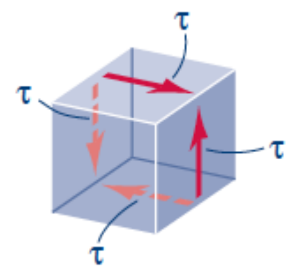
Three-dimensional state of stress



$$\int_{A_0} \dots \, dA_0$$

Integrate over the cross-section

requirements for equilibrium?
... recall simple shear



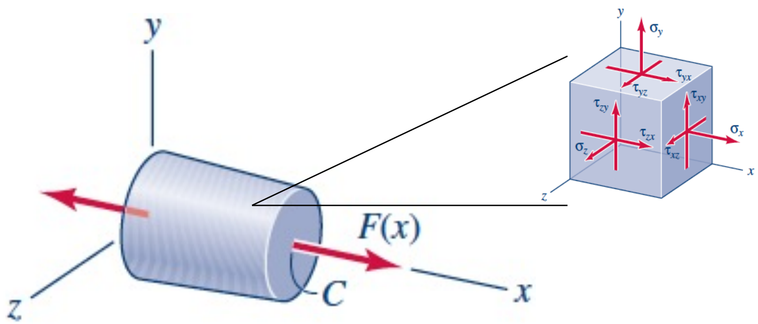
$$\tau_{xy} = \tau_{yx} \qquad \tau_{yz} = \tau_{zy} \qquad \tau_{zx} = \tau_{xz}$$

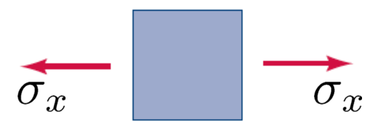
Note: these are Cartesian coordinates

Transformation of stress

Plane stress

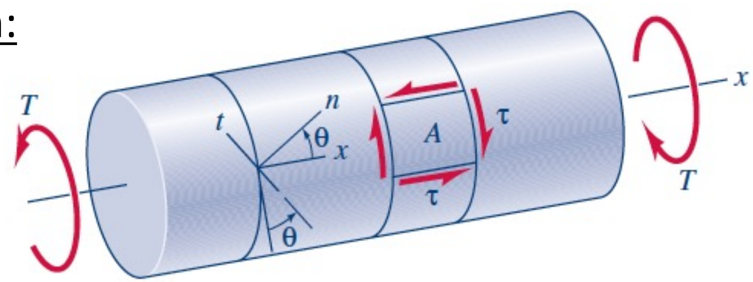
- Axial deformation:
  (Lectures 6-9)

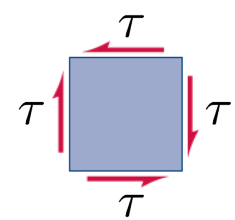




$$\sigma_x = F(x)/A(x)$$

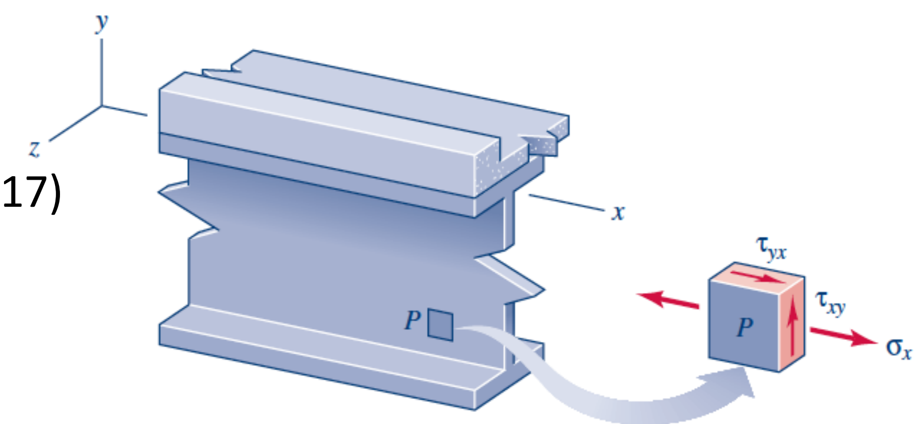
- Torsional deformation:
  (Lectures 10-12)

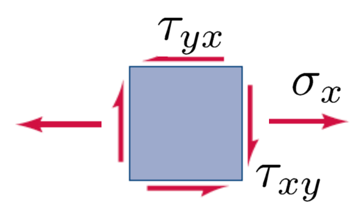




$$\tau = T\rho/I_p$$

- Bending:
  (Lectures 13-17)





$$\sigma_x = -My/I$$

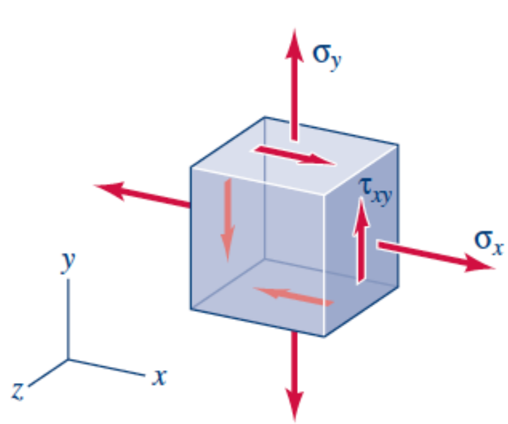
$$\tau_{xy} = VQ/It$$

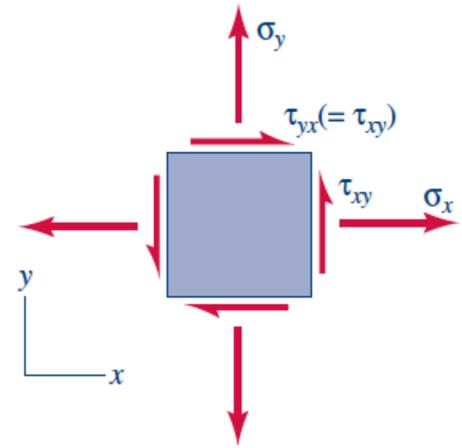
Transformation of stress

Plane stress

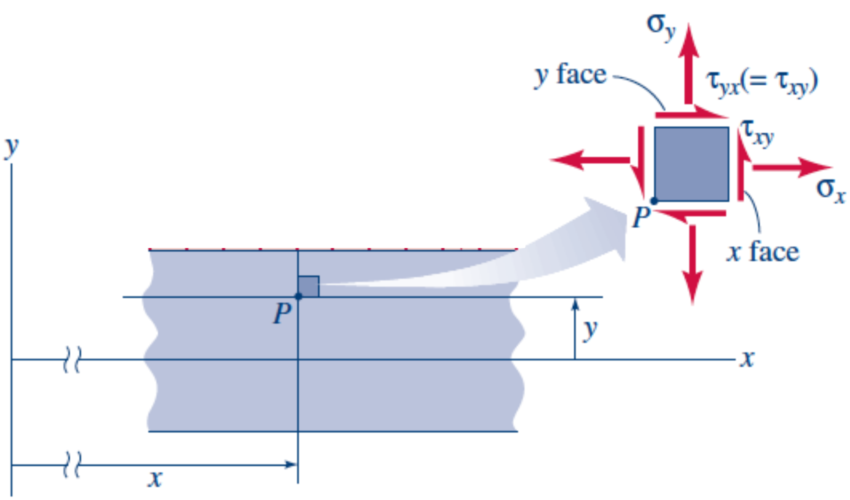
- <u>In general:</u>

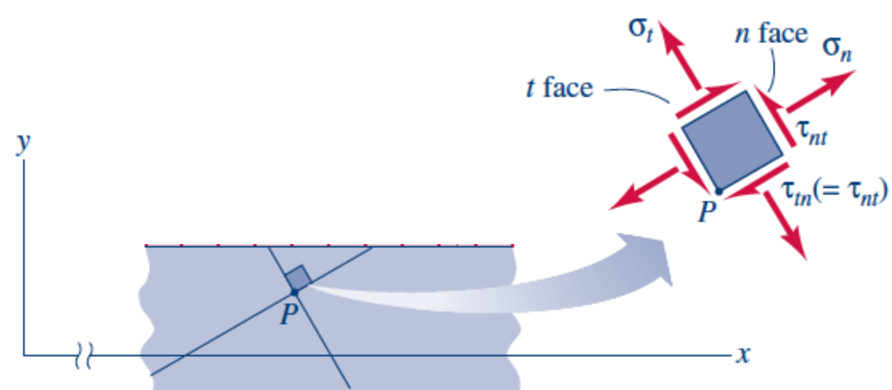
$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$





- Many times it is convenient to rotate the differential element. However, **there is one and only one state of stress at a given point!!!**





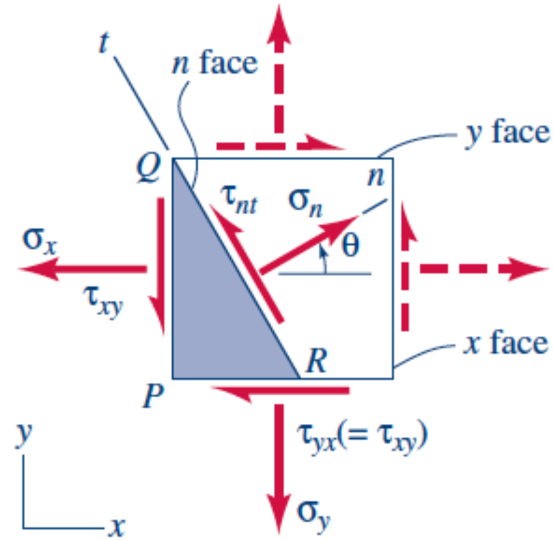
Transformation of stress

Stress transformation for plane stress

Equilibrium of forces:

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}(2\sin\theta\cos\theta)$$

$$\tau_{nt} = -(\sigma_x - \sigma_y)\cos\theta\sin\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

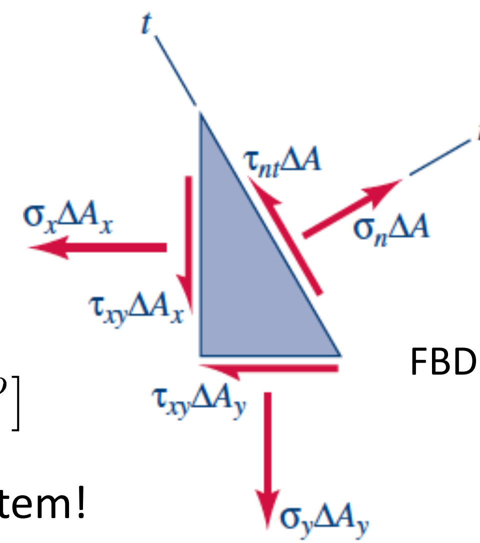


… using some trigonometry:

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

with $\theta \in [-90^o, 90^o]$



FBD

<u>Note</u>: the *n*-*t*-*z* axes must form a right-handed coordinate system!

Transformation of stress

Stress transformation for plane stress

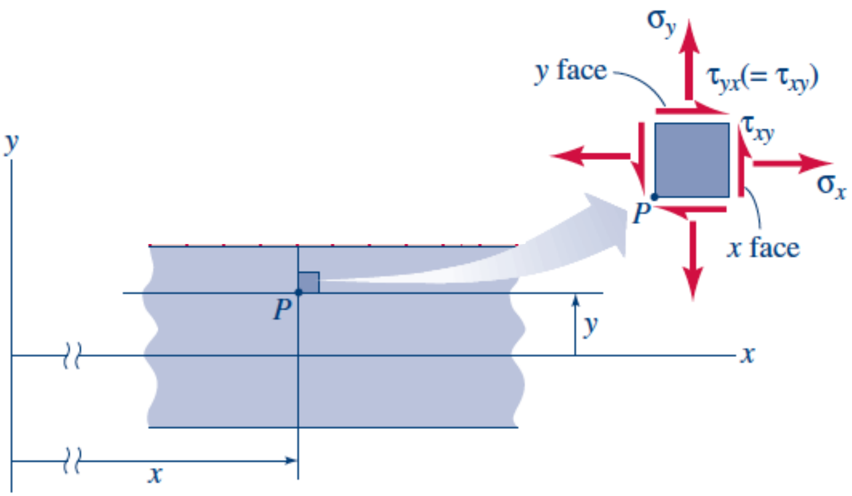
$$\sigma_{x'} = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

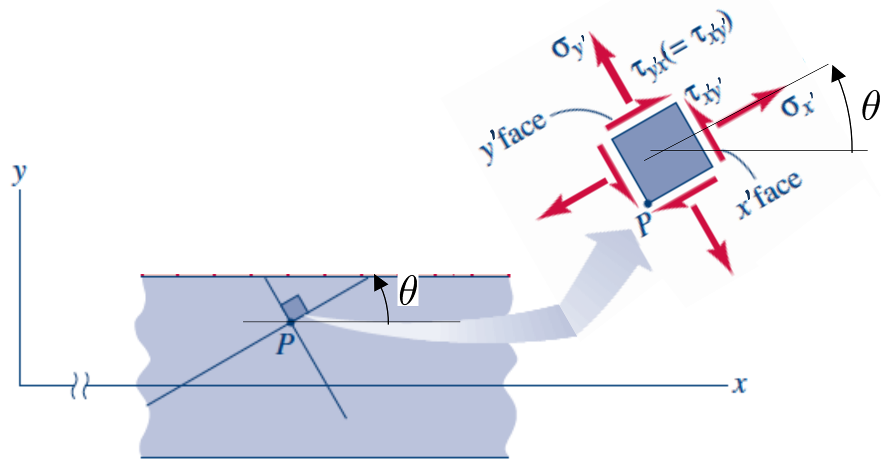
$$\sigma_{y'} = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

(measured counterclockwise from x to x')

with $\theta \in [-90^o, 90^o]$

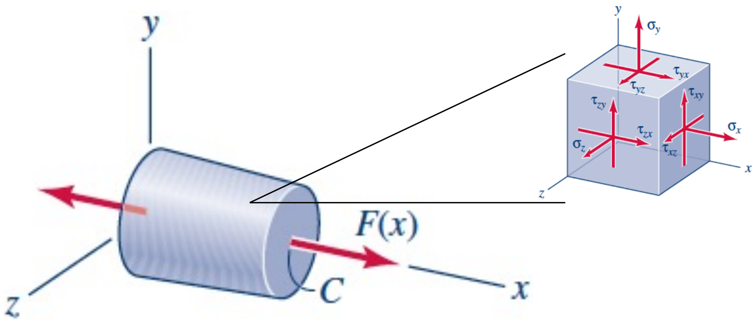


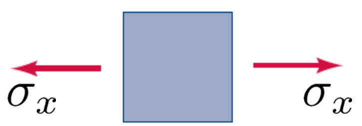


Transformation of stress

Problem 66:

For the state of stress of a rod under axial load, determine the stresses on faces that are rotated 30° counterclockwise from the direction of the load.





$$\sigma_x = F(x)/A(x)$$

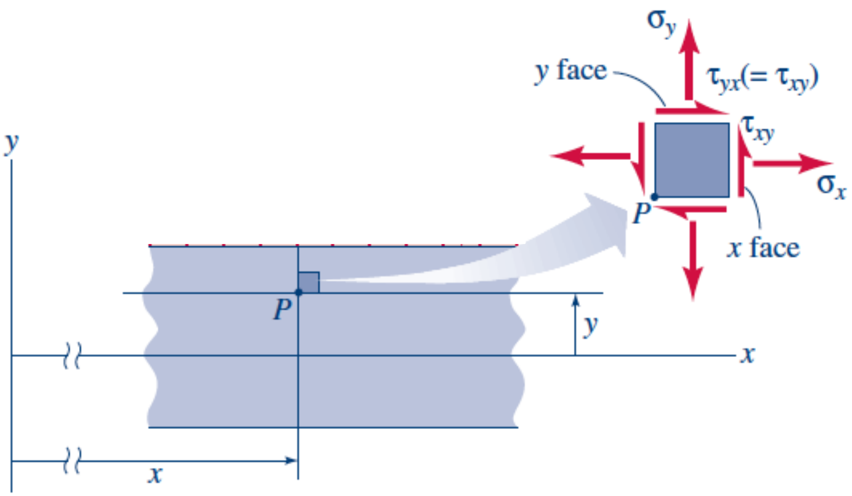
Stress transformation for plane stress

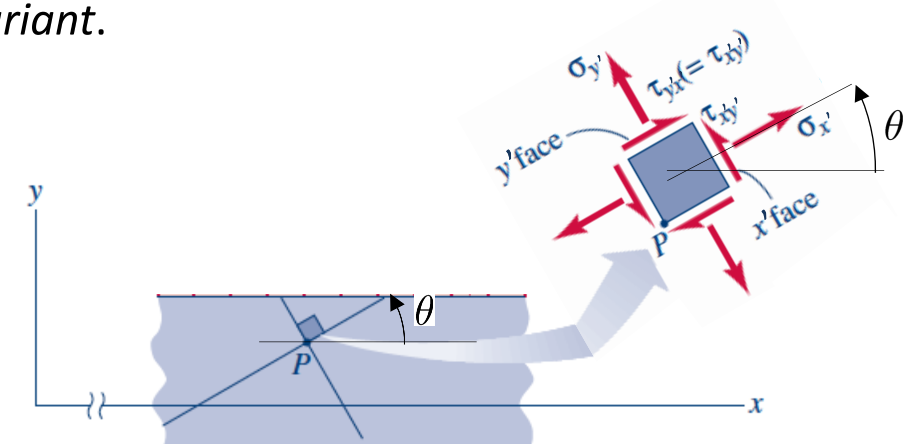
$$\sigma_{x'} = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta$$

Notice that (for orthogonal faces) $\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$ for any angle!

The sum of normal stresses is a *stress invariant*.





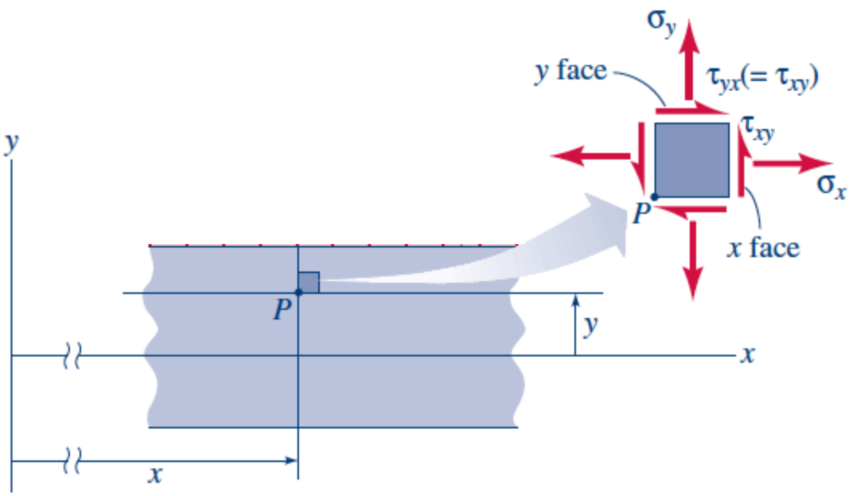
Transformation of stress

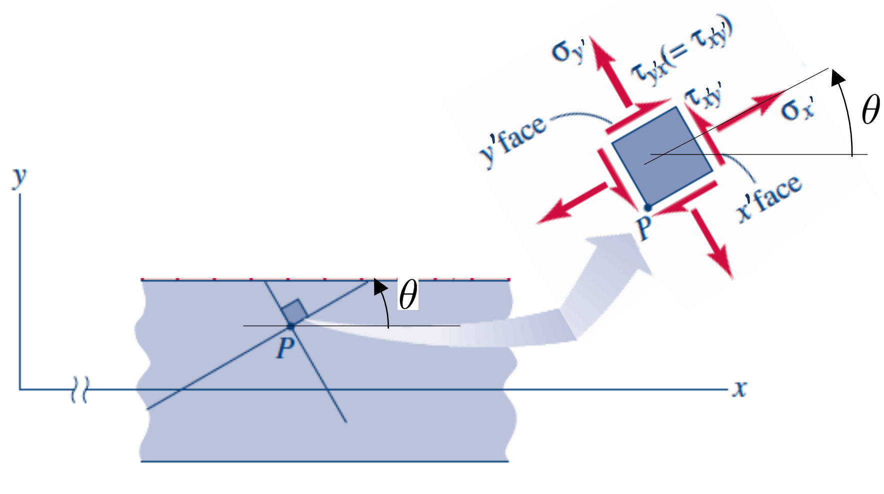
Problem 67:

Given the components of a state of plane stress, find the maximum normal and shear stresses, and the planes on which they act.

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

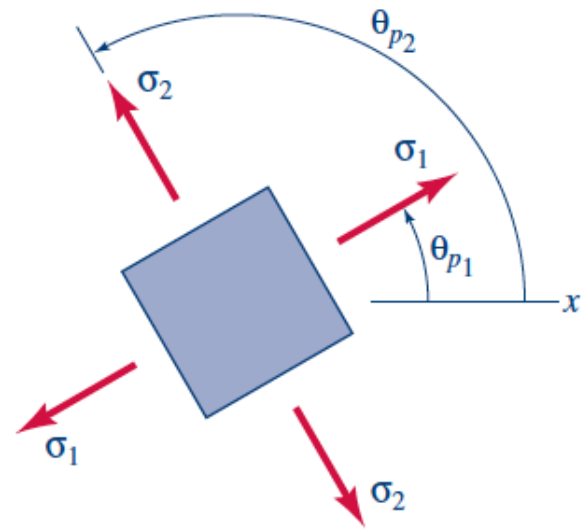




Principal stresses - Maximum shear stress

Principal stresses

Find the orientation θ such that the normal stress σ_n is maximum:



$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

Principal stresses: Principal directions:

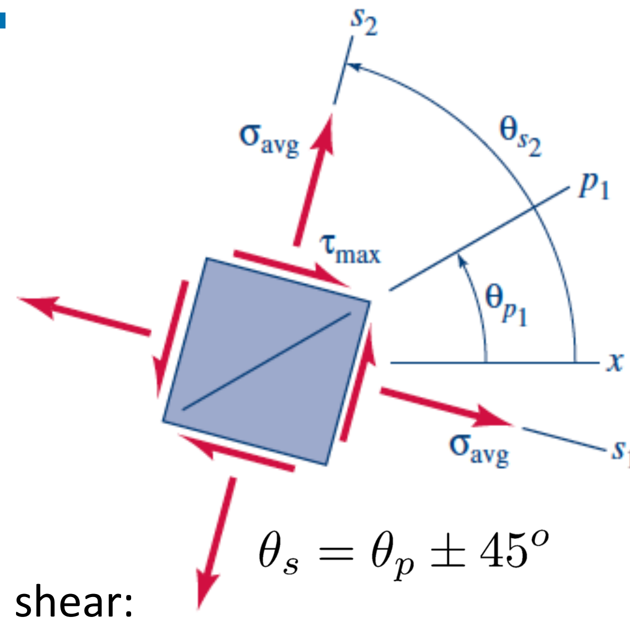
maximum in plane normal stress $\sigma_1 = \sigma_{\text{avg}} + R$ $\quad \sin 2\theta_{p1} = \frac{\tau_{xy}}{R} \quad \cos 2\theta_{p1} = \frac{\sigma_x - \sigma_y}{2R}$

minimum in plane normal stress $\sigma_2 = \sigma_{\text{avg}} - R$ $\quad \sin 2\theta_{p2} = -\frac{\tau_{xy}}{R} \quad \cos 2\theta_{p2} = -\frac{\sigma_x - \sigma_y}{2R}$

with $\sigma_{\text{avg}} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

shear stress on principal planes $\tau_{nt}(\theta_{p1}) = \tau_{nt}(\theta_{p2}) = 0$ no shear stress on the principal planes!!

Principal stresses - Maximum shear stress



Maximum in-plane shear stresses

Find the orientation θ such that the normal stress τ_{nt} is maximum:

$$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\theta_s = \theta_p \pm 45^o$$

Maximum in-plane shear stress

$$\tau_{\max} = R$$

Planes of maximum shear:

$$\sin 2\theta_{s1} = -\frac{\sigma_x - \sigma_y}{2R} \qquad \cos 2\theta_{s1} = \frac{\tau_{xy}}{R}$$

$$\sin 2\theta_{s2} = \frac{\sigma_x - \sigma_y}{2R} \qquad \cos 2\theta_{s2} = -\frac{\tau_{xy}}{R}$$

with $\sigma_{\text{avg}} = \dfrac{\sigma_x + \sigma_y}{2}$ $\quad R = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

normal stress on planes of max. shear $\quad \sigma_{s1} = \sigma_{\text{avg}} \quad \sigma_{s2} = \sigma_{\text{avg}}$

the planes of maximum shear stress are not free of normal stress!!

Transformation of stress

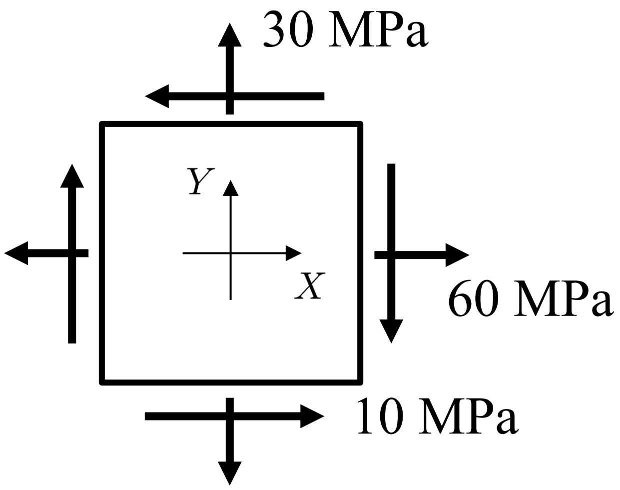
Problem 68:

The state of plane stress in indicated in the figure. Determine:

(a) the normal and shear stresses on a face rotated 40 counterclockwise from the x face;

(b) the magnitude of the principal stresses and the orientation of the principal planes;

(c) the magnitude of the maximum in-plane shear stress and orientation of the planes of maximum shear stress.



Transformation of stress

Any questions?