Lecture 13
Constitutive relations

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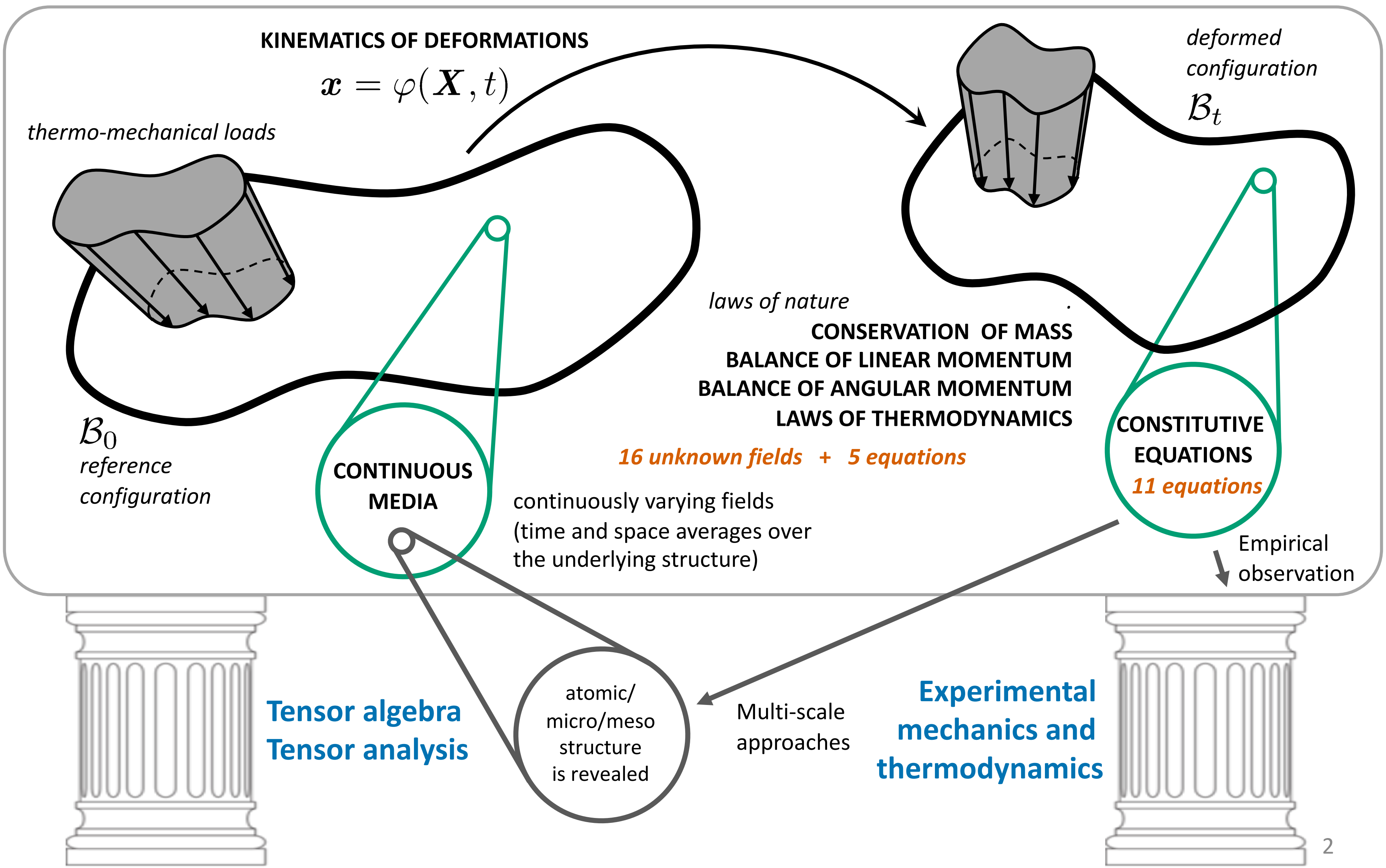
ME 3061M

Lecture 13 – Constitutive relations



Continuum mechanics – Laws of nature

Summary (spatial local forms)

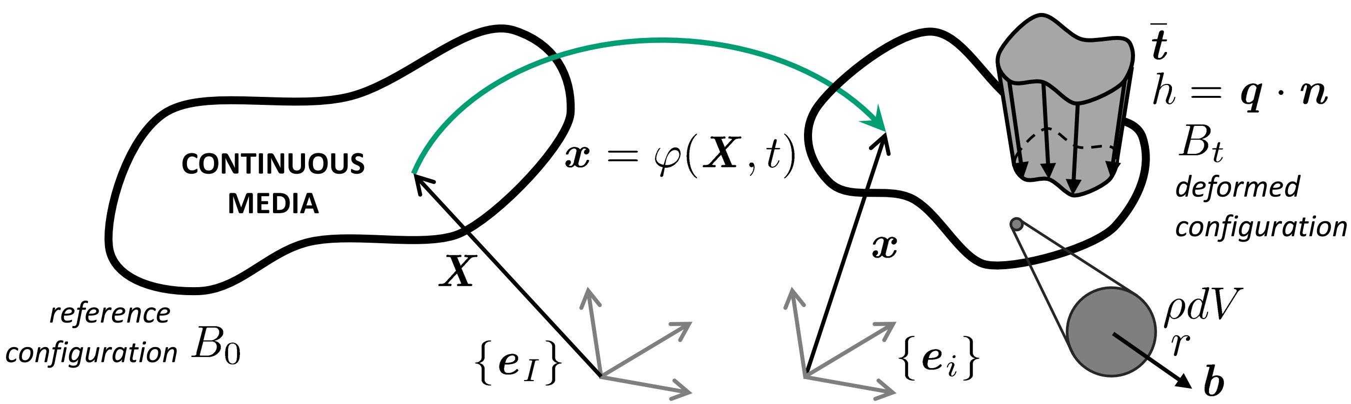
$$\dot{\rho} + \rho \,\mathrm{div}\boldsymbol{v} = 0 \qquad [[\rho(v_s - \boldsymbol{v}\cdot\boldsymbol{n}_s)]] = 0$$ **conservation of mass**

$$\mathrm{div}\boldsymbol{\sigma} + \rho\boldsymbol{b} = \rho\boldsymbol{a} \qquad [[\rho\boldsymbol{v}(v_s - \boldsymbol{v}\cdot\boldsymbol{n}_s)]] + [[\boldsymbol{\sigma}]]\boldsymbol{n}_s = 0$$ **balance of linear momentum**

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$ **balance of angular momentum**

$$\rho\dot{u} = \boldsymbol{\sigma} : \boldsymbol{d} + \rho r - \mathrm{div}\boldsymbol{q}$$
$$[[\rho(u + \frac{1}{2}\|\boldsymbol{v}\|^2)(v_s - \boldsymbol{v}\cdot\boldsymbol{n}_s)]] + [[\boldsymbol{\sigma}\boldsymbol{v} + \boldsymbol{q}]]\cdot\boldsymbol{n}_s = 0$$ **conservation of energy**

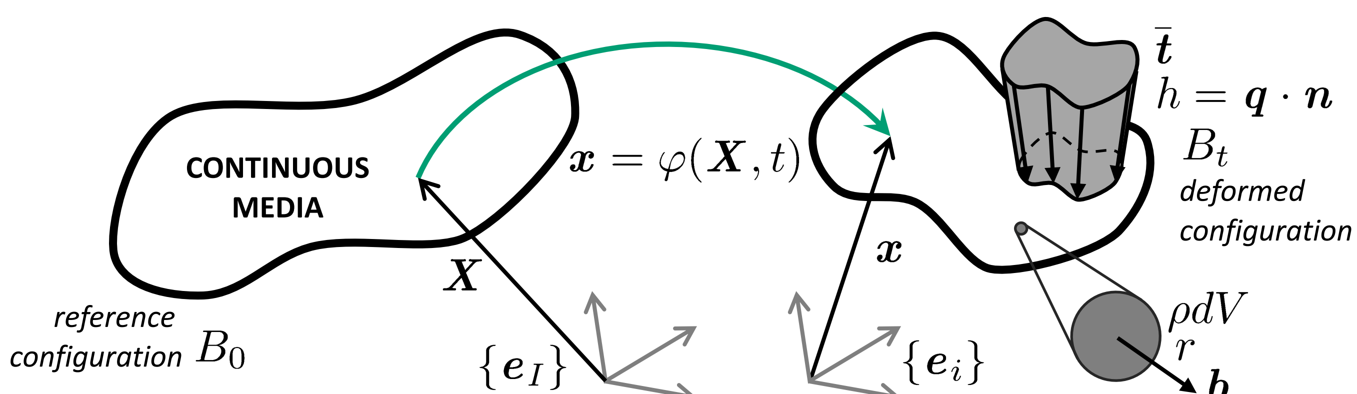
$$\dot{s} \geq \frac{r}{T} - \frac{1}{\rho}\mathrm{div}\frac{\boldsymbol{q}}{T} \qquad [[\rho s(v_s - \boldsymbol{v}\cdot\boldsymbol{n}_s)]] + [[\boldsymbol{q}/T]]\cdot\boldsymbol{n}_s \geq 0$$ **Clausius-Duhem inequality**



Continuum mechanics – Laws of nature

Summary (spatial local forms)

$\dot{\rho} + \rho \, \mathrm{div}\boldsymbol{v} = 0$	**conservation of mass**	(1 equation)
$\mathrm{div}\boldsymbol{\sigma} + \rho\boldsymbol{b} = \rho\boldsymbol{a}$	**balance of linear momentum**	(3 equations)
$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$	**balance of angular momentum**	(constraint)
$\rho\dot{u} = \boldsymbol{\sigma} : \boldsymbol{d} + \rho r - \mathrm{div}\boldsymbol{q}$	**conservation of energy**	(1 equation)
$\dot{s} \geq \frac{r}{T} - \frac{1}{\rho}\mathrm{div}\frac{\boldsymbol{q}}{T}$	**Clausius-Duhem inequality**	(constraint)
	$\rho,\ \boldsymbol{x},\ \boldsymbol{\sigma},\ \boldsymbol{q},\ u,\ s,\ T$	(16 unknowns)



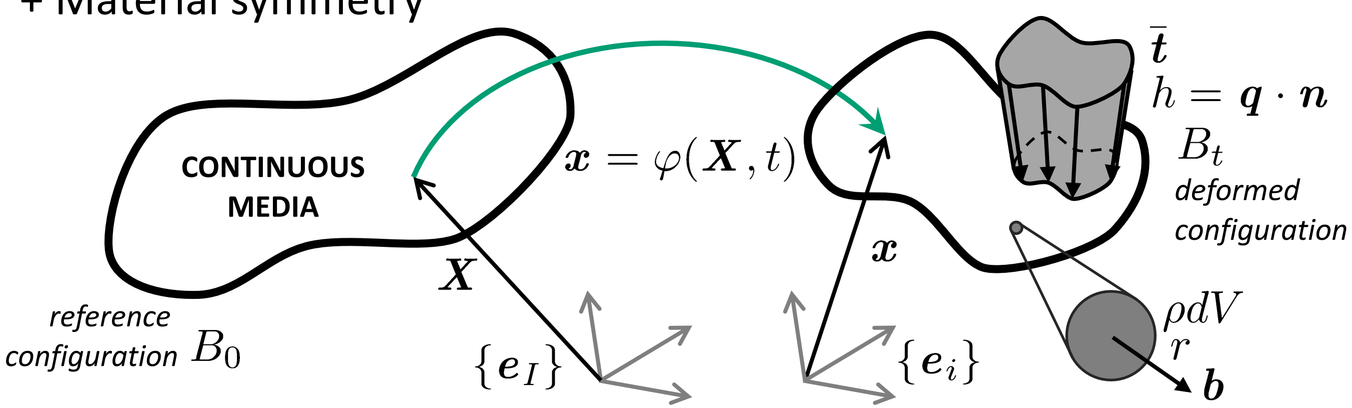
Lecture 13 – Constitutive relations

Constitutive relations

- Relations that describe the response of the material to mechanical and thermal loading. $\boldsymbol{\sigma},\ \boldsymbol{q},\ u,\ T$ (11 constitutive equations)
- Can these constitutive relations be selected arbitrarily? NO!

  They must follow fundamental principles:

  + Principle of determinism
  + Principle of local action
  + Second law of thermodynamics restrictions (Clausius-Duhem inequality)
  + Principle of material frame indifference (objectivity)
  + Material symmetry



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Constraints on constitutive relations

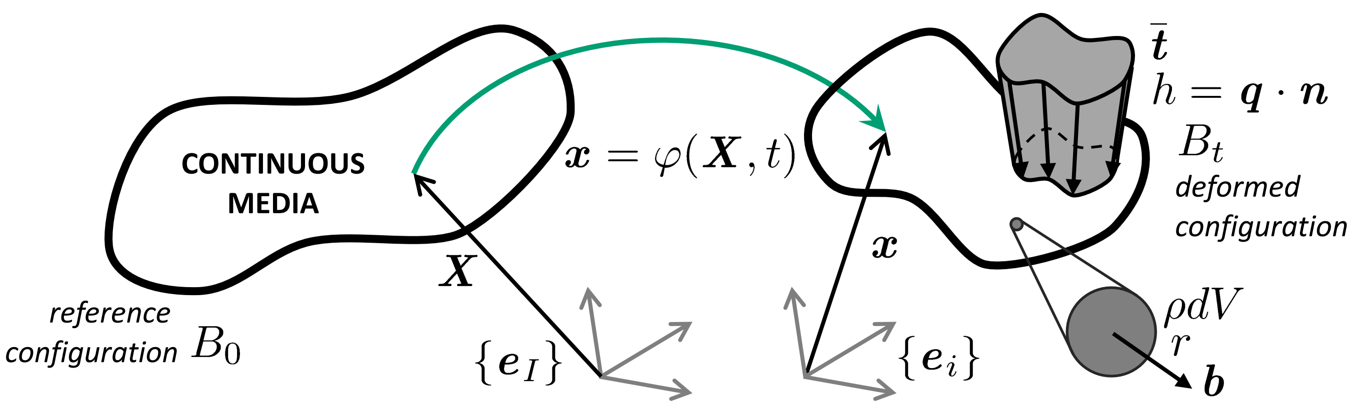
- Principle of determinism
  (causal determinism, the concept of cause and effect, …)

  "The current value of any physical variable can be determined from the knowledge of the present and the past values of other variables"

$$\boldsymbol{\sigma}(\boldsymbol{X},t) = \boldsymbol{f}(\varphi^t, T^t, ..., \boldsymbol{X}, t)$$

$\square^t$ materials with memory

$\boldsymbol{f}(..., t)$ materials with aging



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Constraints on constitutive relations

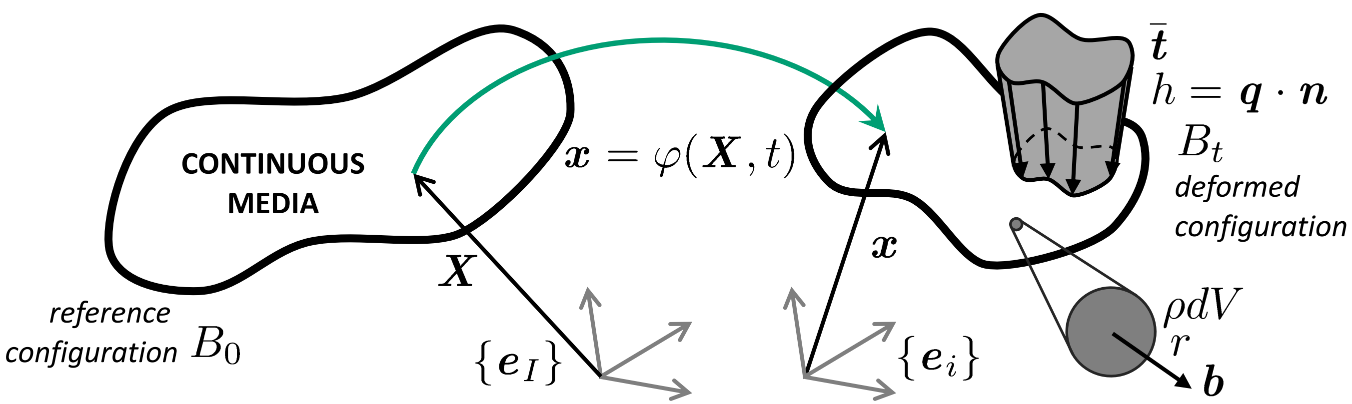
- Principle of local action

"The material response at a point depends only on the conditions within an arbitrarily small region about that point"

$$\boldsymbol{\sigma}(\boldsymbol{X},t) = \boldsymbol{f}(\varphi^t, \boldsymbol{F}^t, ..., T^t, \nabla_0 T^t, ..., \boldsymbol{X}, t)$$

In general $\boldsymbol{F}(\boldsymbol{X},t)$, therefore $\boldsymbol{f}(..., \boldsymbol{F}, \dot{\boldsymbol{F}}, ...)$.
However, we restrict attention to $u = \bar{u}(\boldsymbol{F}, s)$.
[the caloric equation of state]

Simple elastic material: $u = \bar{u}(\boldsymbol{F}, s)$



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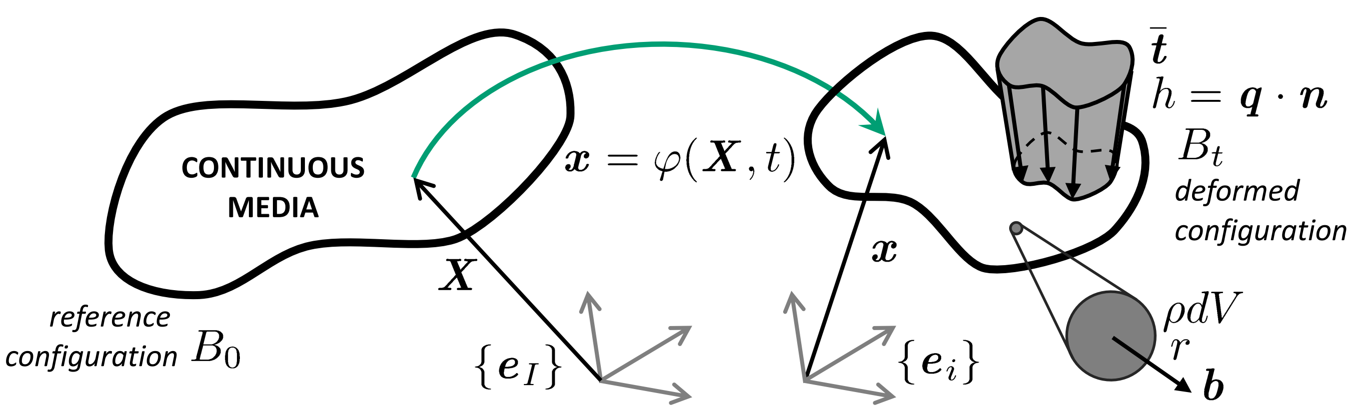
Constraints on constitutive relations

- Second law restrictions

“A constitutive equation cannot violate the second law of thermodynamics”

$$\dot{s} \geq \frac{r}{T} - \frac{1}{\rho}\mathrm{div}\frac{\boldsymbol{q}}{T}$$

The application of the Clausius-Duhem inequality to constitutive equations is known as the Coleman-Noll procedure (1963)

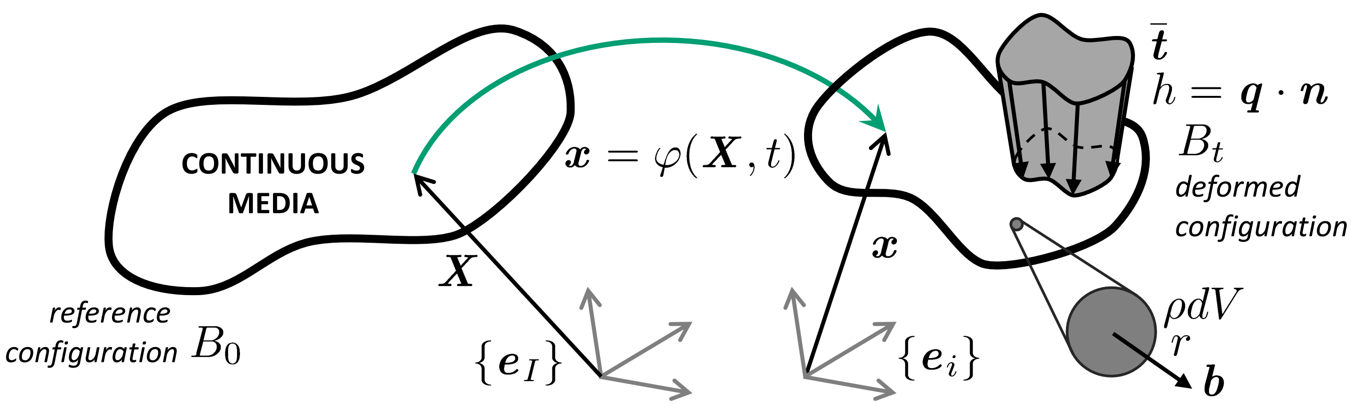


Constraints on constitutive relations

- Principle of material frame indifference (objectivity)

"All physical variables for which constitute relations are required must be objective tensors"

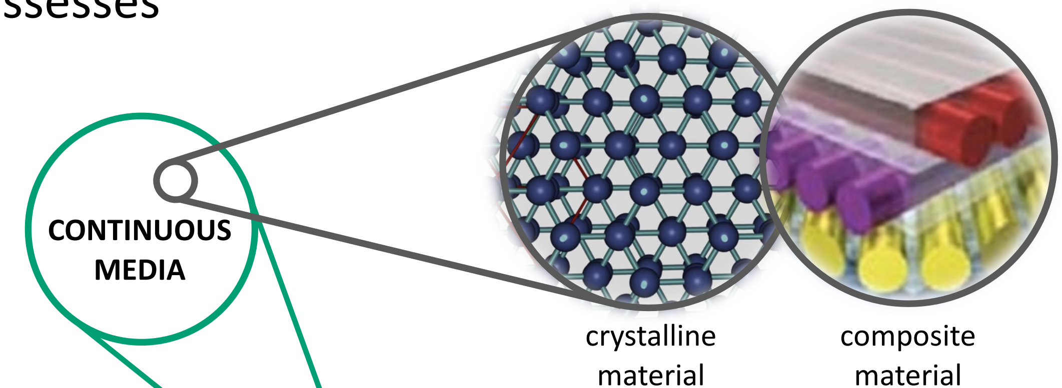
Definition: an objective tensor is a tensor which is physically the same in all frames of reference.
Recall: a frame of reference is an Euclidean point space, which represents points, and a clock, which represent time (relativistic phenomena is not considered).



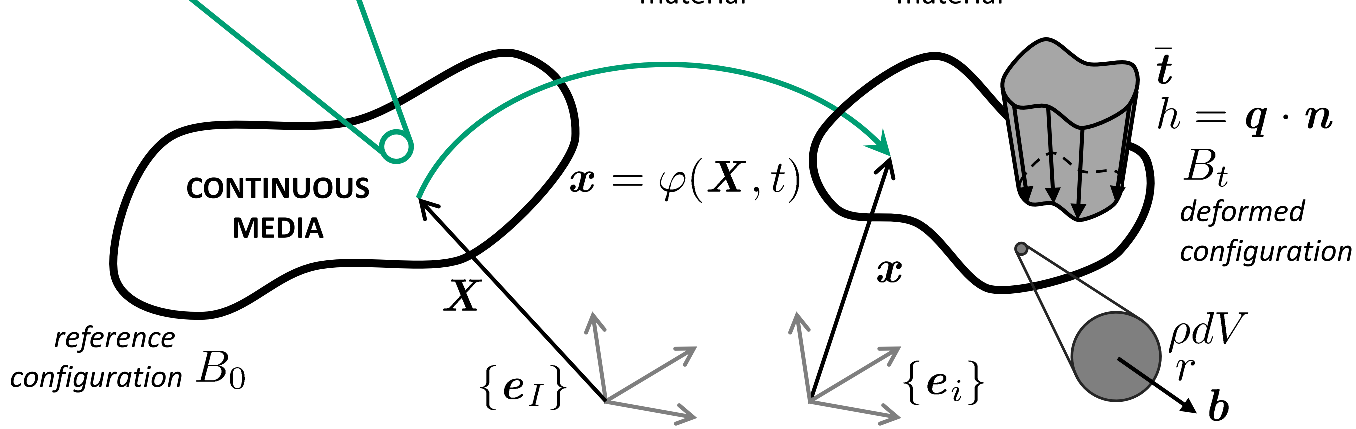
Constraints on constitutive relations

- Material symmetry

  "A constitutive relation must represent any symmetries that the material possesses"



There are eight possible different symmetries at the macroscopic level



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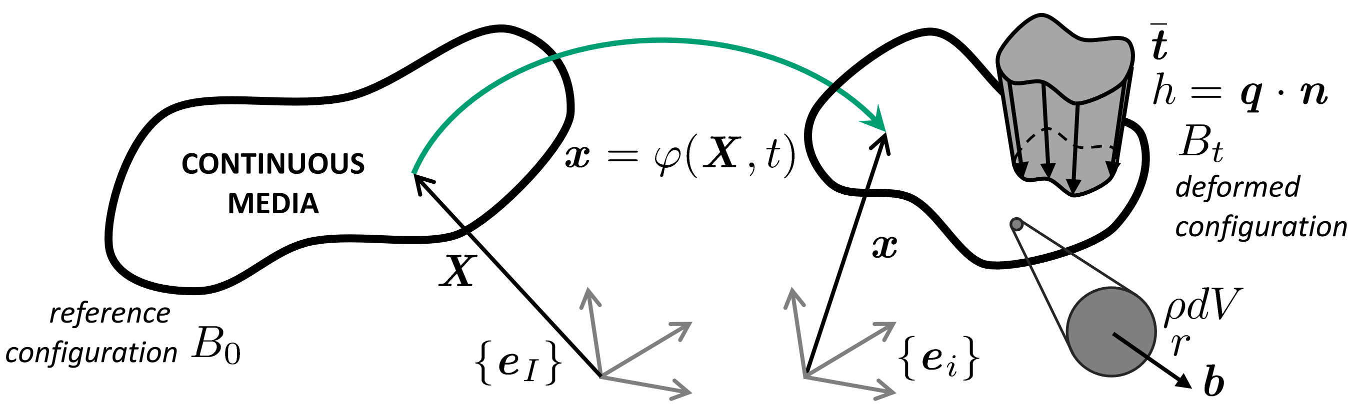
Coleman-Noll procedure

$$\rho\left[T-\frac{\partial\bar{u}}{\partial s}\right]\dot{s}+\left[\boldsymbol{\sigma}\boldsymbol{F}^{-T}-\rho\frac{\partial\bar{u}}{\partial\boldsymbol{F}}\right]:\dot{\boldsymbol{F}}-\frac{1}{T}\boldsymbol{q}\cdot\nabla T\geq 0$$

DIY

$$\dot{s}^{\text{int}}\equiv\dot{s}-\frac{r}{T}+\frac{1}{\rho}\text{div}\frac{\boldsymbol{q}}{T}\geq 0\Longrightarrow\rho T\dot{s}-\rho r+T\text{div}\frac{\boldsymbol{q}}{T}\geq 0$$



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Coleman-Noll procedure

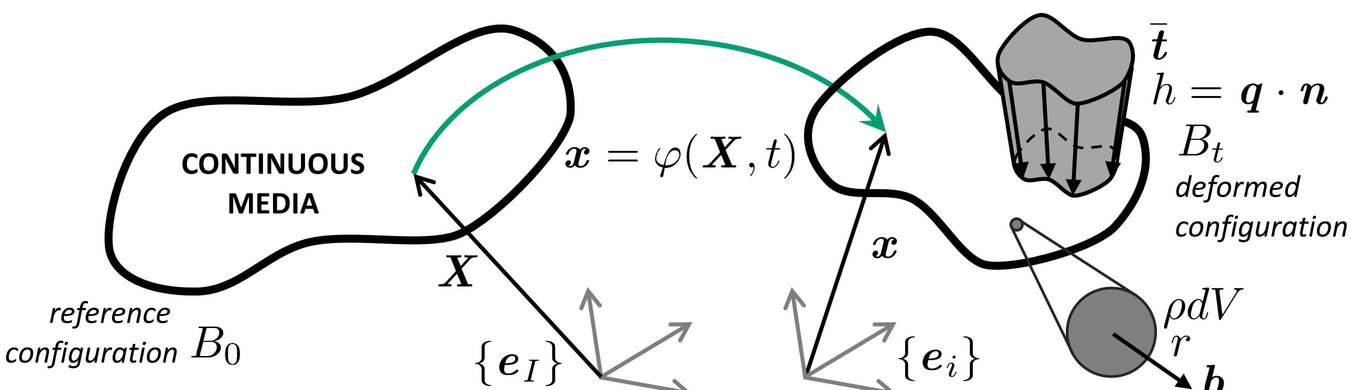
- Temperature constitutive relation $T = \bar{T}(s, \boldsymbol{F}) \equiv \dfrac{\partial \bar{u}}{\partial s}$

DIY

$$\rho\left[T - \frac{\partial \bar{u}}{\partial s}\right]\dot{s} + \left[\boldsymbol{\sigma}\boldsymbol{F}^{-T} - \rho\frac{\partial \bar{u}}{\partial \boldsymbol{F}}\right] : \dot{\boldsymbol{F}} - \frac{1}{T}\boldsymbol{q}\cdot\nabla T \geq 0$$

$$u = \bar{u}(\boldsymbol{F}, s)$$

The inequality has to be verified by any arbitrary process. (i.e., state variables can be chosen arbitrarily)
In particular, a process which has a deformation constant in time and a uniform temperature. Thus, $\rho\left[T - \dfrac{\partial \bar{u}}{\partial s}\right]\dot{s} \geq 0 \quad \forall \dot{s}$



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Coleman-Noll procedure

- Heat flux constitutive relation $\boldsymbol{q} = \bar{\boldsymbol{q}}(s, \boldsymbol{F}, \nabla T)$

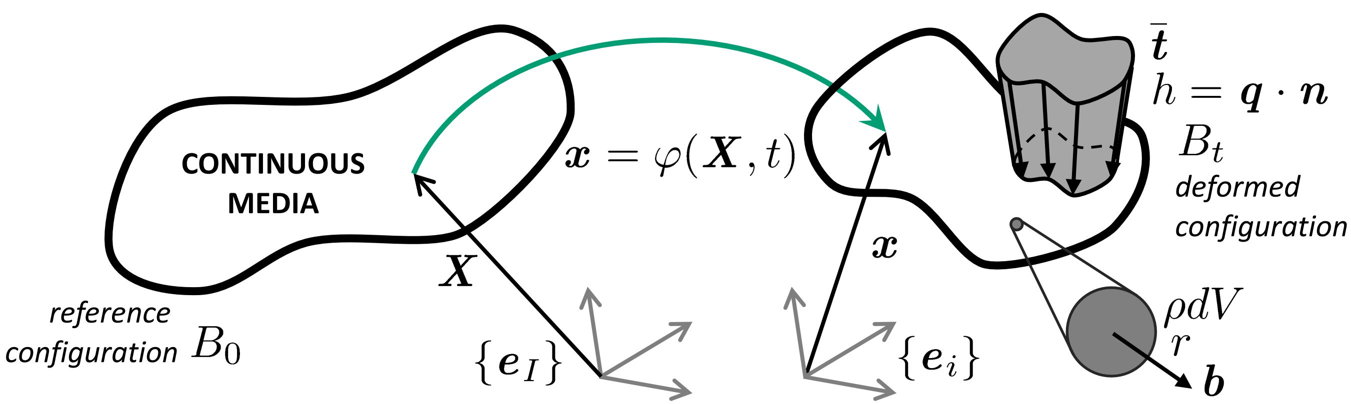
DIY

$$\left[\boldsymbol{\sigma}\boldsymbol{F}^{-T} - \rho\frac{\partial \bar{u}}{\partial \boldsymbol{F}}\right] : \dot{\boldsymbol{F}} - \frac{1}{T}\boldsymbol{q}\cdot\nabla T \geq 0$$

$$T = \frac{\partial \bar{u}}{\partial s}$$

The inequality has to be verified by any arbitrary process.
In particular, one which has a deformation constant in time.
Then,

$$-\frac{1}{T}\boldsymbol{q}\cdot\nabla T \geq 0$$



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Coleman-Noll procedure

- Cauchy stress constitutive relation

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(e)} + \boldsymbol{\sigma}^{(v)} \qquad\qquad \boldsymbol{\sigma}^{(v)}(s, \boldsymbol{F}, \boldsymbol{d})$$

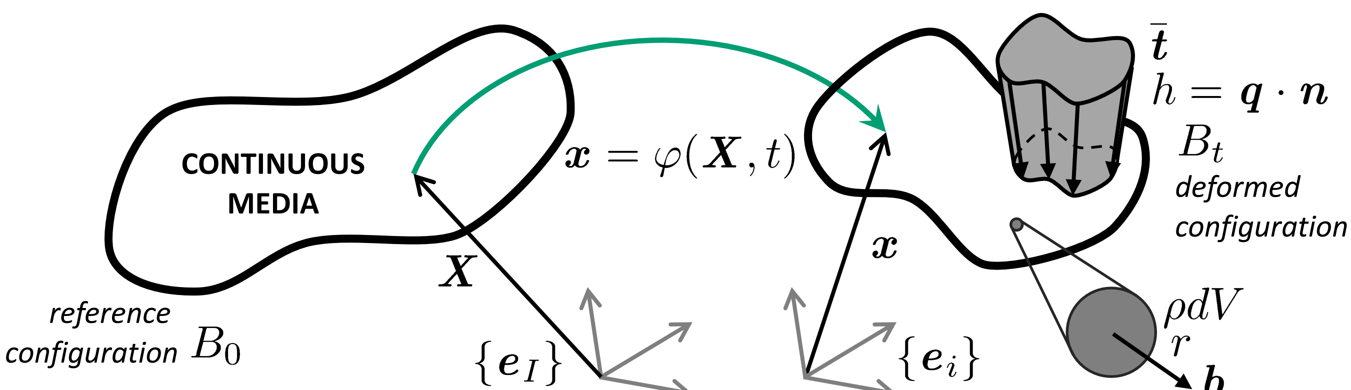
DIY

$$\left[\boldsymbol{\sigma}\boldsymbol{F}^{-T} - \rho\frac{\partial \bar{u}}{\partial \boldsymbol{F}}\right] : \dot{\boldsymbol{F}} - \frac{1}{T}\boldsymbol{q}\cdot\nabla T \geq 0$$

$$T = \frac{\partial \bar{u}}{\partial s}$$

$$\frac{1}{T}\boldsymbol{q}\cdot\nabla T \leq 0$$

Since the stress tensor is not a state variable, we first partition it into an elastic reversible part (which is a state variable) and an irreversible part (which is not associated with an equilibrium state and, therefore, it is not a state variable). Here, the partition is an additive decomposition (but it doesn't have to be the case in general).
The irreversible process has to produce entropy, that is $\boldsymbol{\sigma}^{(v)} : \boldsymbol{d} \geq 0$



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Coleman-Noll procedure

- Cauchy stress constitutive relation

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(e)} + \boldsymbol{\sigma}^{(v)}$$

$$\bar{\boldsymbol{\sigma}}^{(e)}(s, \boldsymbol{F}) \equiv \rho \frac{\partial \bar{u}}{\partial \boldsymbol{F}} \boldsymbol{F}^T$$

$$\bar{\boldsymbol{\sigma}}^{(v)}(s, \boldsymbol{F}, \boldsymbol{d})$$

DIY

$$\left[\boldsymbol{\sigma} \boldsymbol{F}^{-T} - \rho \frac{\partial \bar{u}}{\partial \boldsymbol{F}}\right] : \dot{\boldsymbol{F}} - \frac{1}{T} \boldsymbol{q} \cdot \nabla T \geq 0$$

$$T = \frac{\partial \bar{u}}{\partial s}$$

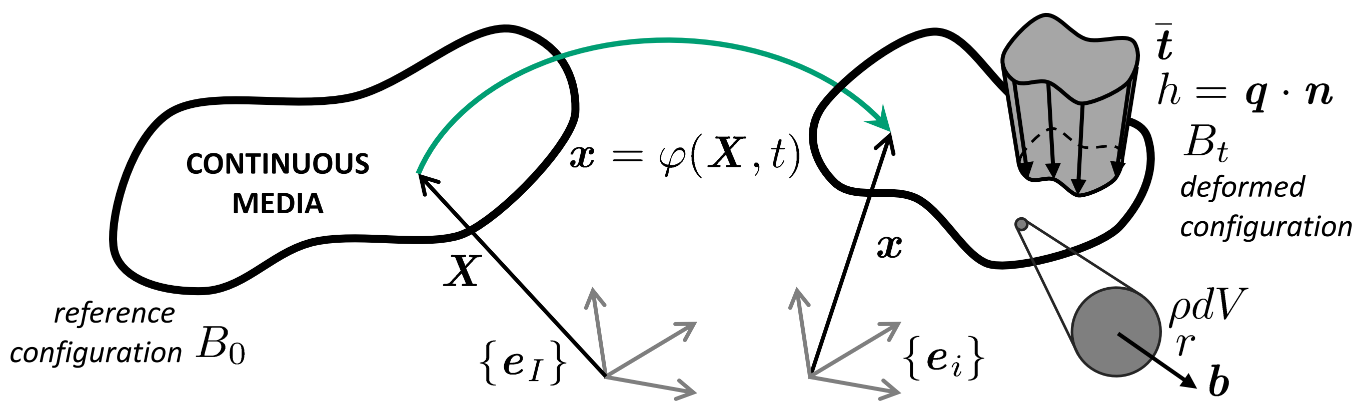
$$\frac{1}{T} \boldsymbol{q} \cdot \nabla T \leq 0$$

$$\boldsymbol{\sigma}^{(v)} : \boldsymbol{d} \geq 0$$

Since the stress tensor is not a state variable, we first partition it into an elastic reversible part (which is a state variable) and an irreversible part (which is not associated with an equilibrium state and, therefore, it is not a state variable).

Finally,

$$\left[\boldsymbol{\sigma}^{(e)} \boldsymbol{F}^{-T} - \rho \frac{\partial \bar{u}}{\partial \boldsymbol{F}}\right] : \dot{\boldsymbol{F}} \geq 0 \quad \forall \dot{\boldsymbol{F}}$$



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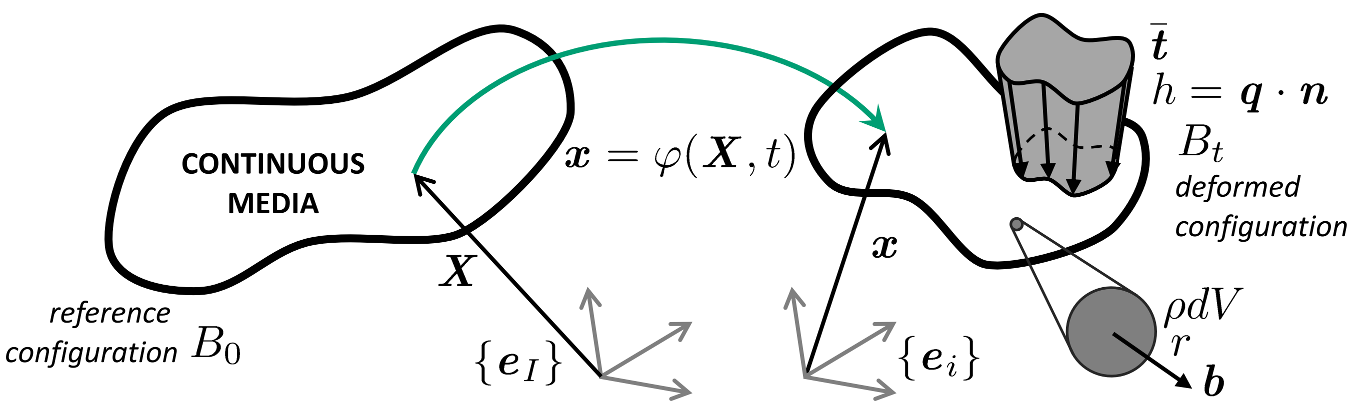
Coleman-Noll procedure

- Energy change in reversible and irreversible processes

$$\dot{s} = \frac{r}{T} - \frac{1}{\rho T}\mathrm{div}\boldsymbol{q} + \frac{1}{\rho T}\boldsymbol{\sigma}^{(v)} : \boldsymbol{d}$$

DIY

$$\rho T\dot{s} - \rho r + T\mathrm{div}\frac{\boldsymbol{q}}{T} = \boldsymbol{\sigma}^{(v)} : \boldsymbol{d} - \frac{1}{T}\boldsymbol{q}\cdot\nabla T$$



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Coleman-Noll procedure (60s)

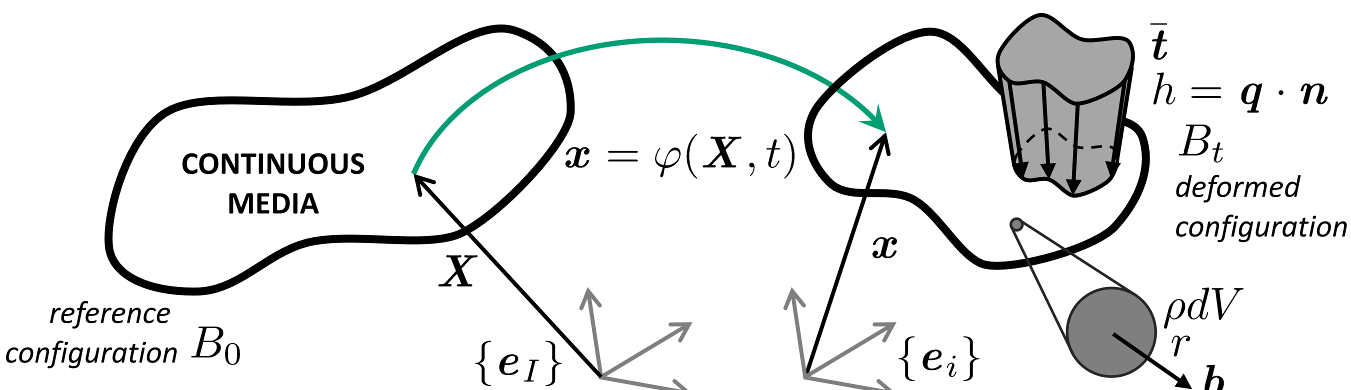
- Energy change in reversible and irreversible processes

<u>For any process</u> $$\rho T \dot{s}^{\mathrm{int}} = \boldsymbol{\sigma}^{(v)} : \boldsymbol{d} - \frac{1}{T}\boldsymbol{q} \cdot \nabla T \geq 0$$

$$\dot{s} = \frac{r}{T} - \frac{1}{\rho T}\mathrm{div}\boldsymbol{q} + \frac{1}{\rho T}\boldsymbol{\sigma}^{(v)} : \boldsymbol{d} \qquad u = \bar{u}(\boldsymbol{F}, s)$$

$$T = \bar{T}(s, \boldsymbol{F}) \equiv \frac{\partial \bar{u}}{\partial s} \qquad \boldsymbol{q} = \bar{\boldsymbol{q}}(s, \boldsymbol{F}, \nabla T) \qquad -\frac{1}{T}\boldsymbol{q} \cdot \nabla T \geq 0$$

$$\bar{\boldsymbol{\sigma}}^{(e)}(s, \boldsymbol{F}) \equiv \rho \frac{\partial \bar{u}}{\partial \boldsymbol{F}}\boldsymbol{F}^T \qquad \bar{\boldsymbol{\sigma}}^{(v)}(s, \boldsymbol{F}, \boldsymbol{d}) \qquad \boldsymbol{\sigma}^{(v)} : \boldsymbol{d} \geq 0$$



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Coleman-Noll procedure (60s)

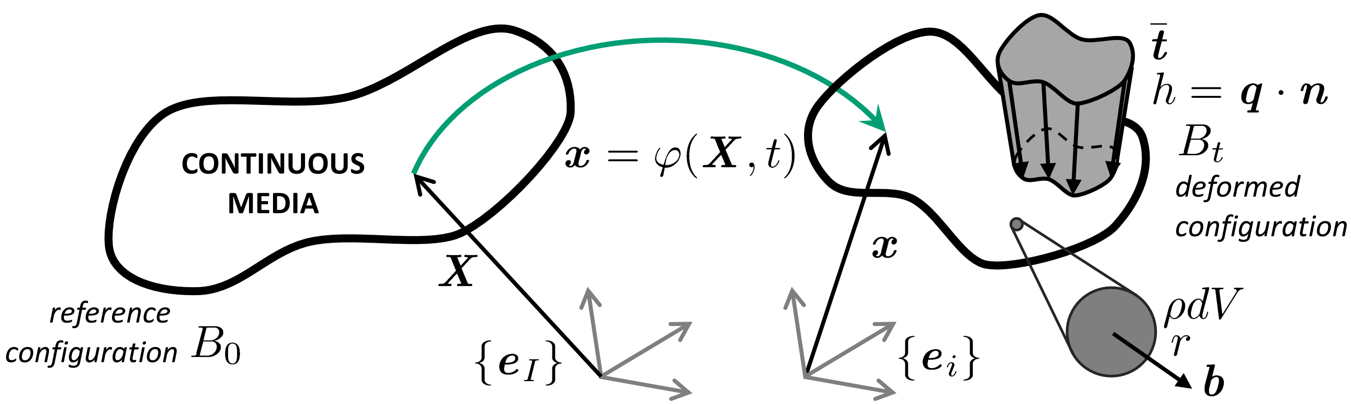
- Energy change in reversible and irreversible processes

<u>For reversible processes</u> $\boldsymbol{\sigma}^{(v)} : \boldsymbol{d} = 0 \qquad \frac{1}{T}\boldsymbol{q} \cdot \nabla T = 0$

$$\dot{s}_{\text{rev}} = \frac{r}{T} - \frac{1}{\rho T}\text{div}\boldsymbol{q} \qquad u = \bar{u}(\boldsymbol{F}, s)$$

$$T = \bar{T}(s, \boldsymbol{F}) \equiv \frac{\partial \bar{u}}{\partial s} \qquad \boldsymbol{q} = \bar{\boldsymbol{q}}(s, \boldsymbol{F}, \nabla T)$$

$$\bar{\boldsymbol{\sigma}}^{(e)}(s, \boldsymbol{F}) \equiv \rho \frac{\partial \bar{u}}{\partial \boldsymbol{F}} \boldsymbol{F}^T \qquad \bar{\boldsymbol{\sigma}}^{(v)}(s, \boldsymbol{F}, \boldsymbol{d})$$



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Any questions?