Spring, 2022

ME 597 – Solid Mechanics II

Lecture 3
Kinematics of deformations

PURDUE UNIVERSITY | Mechanical Engineering

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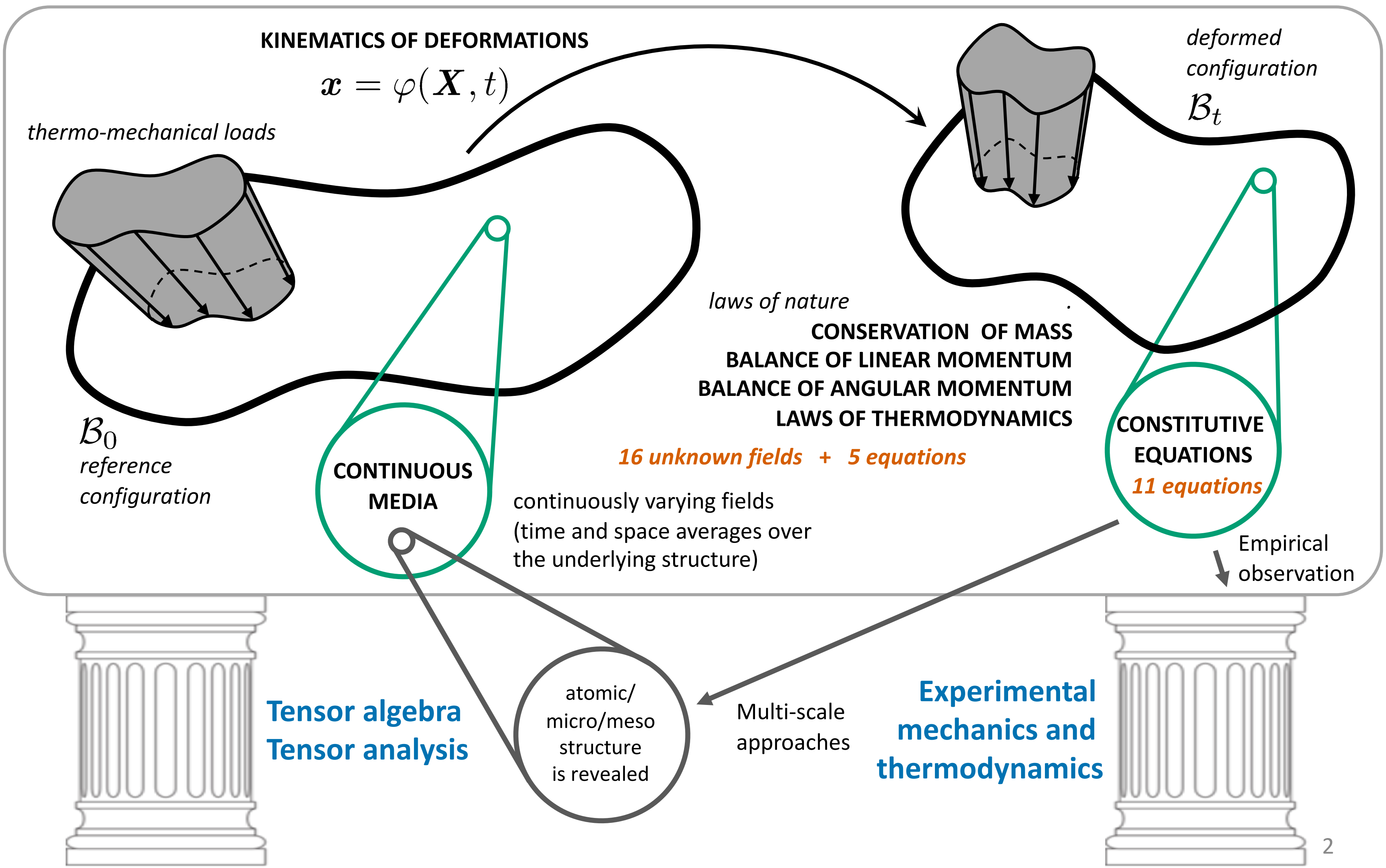
Last modified: 2/1/22 4:07:55 PM

Lecture 3 – Kinematics of deformations

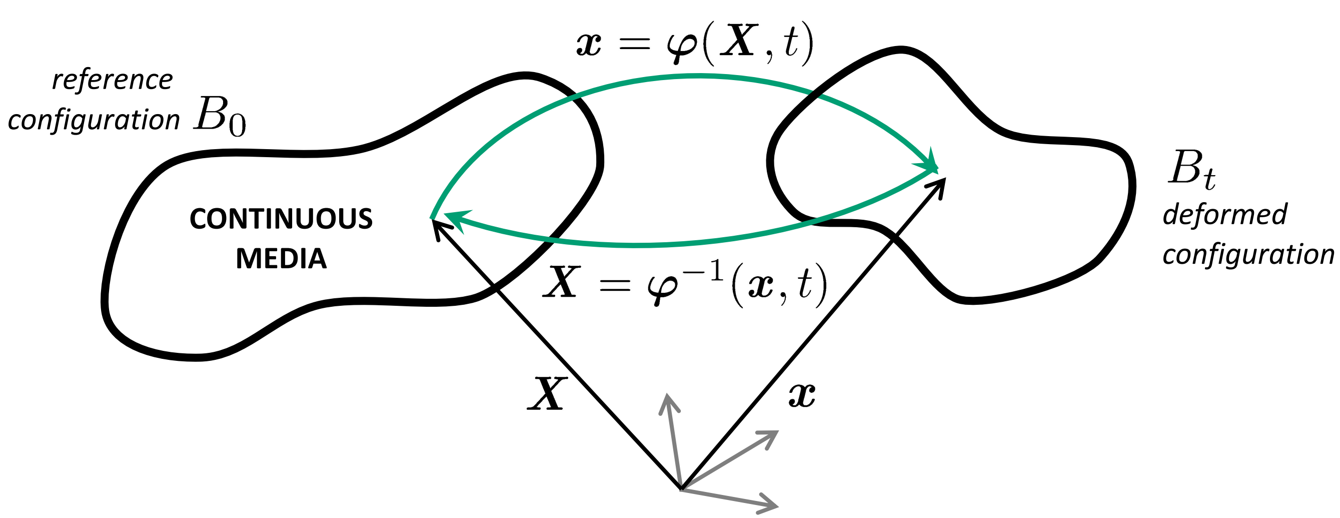


Kinematics of deformations

The deformation mapping

- A body can take on many different shapes or configurations:
  + Reference configuration B_0 (typically chosen by convenience)
  + Deformed configuration B_t

- The deformed configuration is described in terms of the deformation mapping $\boldsymbol{\varphi}(\boldsymbol{X})$ or motion $\boldsymbol{\varphi}(\boldsymbol{X}, t)$.
  (continuum particles cannot be destroyed or created ⇢ one-to-one mapping)



Material and spatial descriptions

Scalar invariant fields

- Spatial or Eulerian description:
  Temperature at a particular position in space regardless of which particle is occupying it at time t .

  $$T(\boldsymbol{x}, t) \qquad \boldsymbol{x} \in B_t$$

  (i.e., temperature in terms of spatial positions)

- Material or Lagrangian description:
  Temperature of a give particle regardless of where the particle is located in space at time t .

  $$\breve{T}(\boldsymbol{X}, t) \qquad \boldsymbol{X} \in B_0$$

  (i.e., temperature in terms of material particles)

Note: The two descriptions are related by:

$$\breve{T}(\boldsymbol{X}, t) \equiv T(\boldsymbol{\varphi}(\boldsymbol{X}, t), t)$$

- $\boldsymbol{X}$ are referred to as material coordinates.
- $\boldsymbol{x}$ are referred to as spatial coordinates.

Material and spatial descriptions

Tensor fields

- A tensor is a multi-linear mapping of vectors …
  … but these vectors can be material vectors or spatial vectors!

$$\boldsymbol{T} : \mathbb{R}^{n_\mathrm{d}} \times \dots \times \mathbb{R}^{n_\mathrm{d}} \to \mathbb{R}$$

- A spatial tensor has spatial vectors as arguments

$$a_{ij} \equiv \boldsymbol{a}[\boldsymbol{e}_i, \boldsymbol{e}_j]$$

- A material tensor has material vectors as arguments

$$A_{IJ} \equiv \boldsymbol{A}[\boldsymbol{e}_I, \boldsymbol{e}_J]$$

- A mixed or two-point tensor has both spatial and material vectors as arguments

$$A_{iJ} \equiv \boldsymbol{A}[\boldsymbol{e}_i, \boldsymbol{e}_J]$$

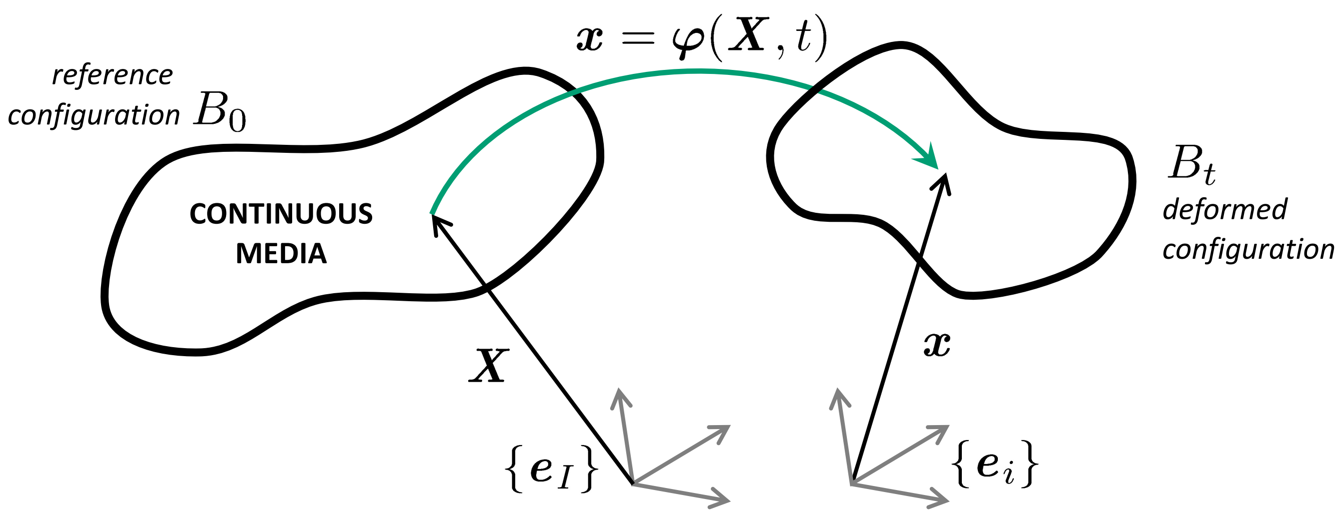
Material and spatial descriptions

Differentiation

- Indicial notation: $\square_{,I} = \dfrac{\partial \square}{\partial X_I}$ $\square_{,i} = \dfrac{\partial \square}{\partial x_i}$

- Direct notation: $\nabla_0 \square$ or $\mathrm{Grad}\square$ $\mathrm{Curl}\square$ $\mathrm{Div}\square$

  $\nabla \square$ or $\mathrm{grad}\square$ $\mathrm{curl}\square$ $\mathrm{div}\square$



Description of local deformation

Deformation gradient

$$x_i + dx_i = \varphi_i(\boldsymbol{X} + d\boldsymbol{X}) \approx \varphi_i(\boldsymbol{X}) + \left.\frac{\partial \varphi_i}{\partial X_J}\right|_{\boldsymbol{X}} dX_J = x_i + F_{iJ}dX_J$$

$$dx_i = F_{iJ}dX_J \iff d\boldsymbol{x} = \boldsymbol{F}d\boldsymbol{X}$$

$$F_{iJ} = \frac{\partial \varphi_i}{\partial X_J} = \frac{\partial x_i}{\partial X_J} \iff \boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \nabla_0 \boldsymbol{x}$$

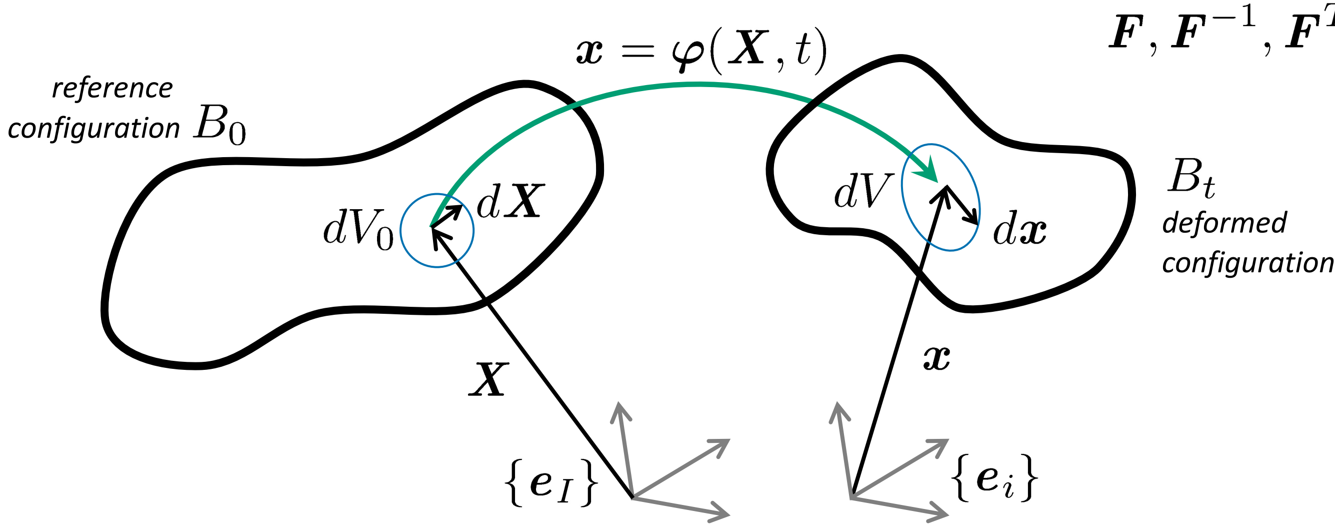
deformation gradient

(*not symmetric, second-order two-point tensor*)

Basis?

$\boldsymbol{F}, \boldsymbol{F}^{-1}, \boldsymbol{F}^T$

DIY



Description of local deformation

Deformation gradient

- Stretch:

  for any $d\boldsymbol{X} = \boldsymbol{M}dS$
  then
  $d\boldsymbol{x} = \boldsymbol{F}d\boldsymbol{X} = \boldsymbol{F}\ \boldsymbol{M}dS$

$$\alpha = \frac{ds}{dS} = \frac{\|d\boldsymbol{x}\|}{\|d\boldsymbol{X}\|} = \|\boldsymbol{F}\boldsymbol{M}\|$$

stretch
(of an infinitesimal element originally oriented along ***M****)*

... alternatively ...

$$ds^2 = \|\boldsymbol{F}\boldsymbol{M}\|^2 dS^2 = \boldsymbol{M}\cdot(\boldsymbol{F}^T\boldsymbol{F})\boldsymbol{M}dS^2 = \boldsymbol{M}\cdot\boldsymbol{C}\boldsymbol{M}dS^2$$

$$C_{IJ} = F_{kI}F_{kJ} \iff \boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$$

right Cauchy-Green deformation tensor
(symmetric, positive-definite, second-order material tensor)

- Volume change:

$J \equiv \det\boldsymbol{F}$: Jacobian of the deformation mapping

$$dV_0 = (d\boldsymbol{X}_1 \times d\boldsymbol{X}_2)\cdot d\boldsymbol{X}_3$$

$$dV = (\boldsymbol{F}d\boldsymbol{X}_1 \times \boldsymbol{F}d\boldsymbol{X}_2)\cdot\boldsymbol{F}d\boldsymbol{X}_3 = \det\boldsymbol{F}dV_0 = J\ dV_0$$

volume preserving deformation — WIT

Description of local deformation

Deformation gradient

- Area change:

Element of oriented area $d\boldsymbol{A}_0 = d\boldsymbol{X}_1 \times d\boldsymbol{X}_2 = \boldsymbol{N} dA_0$

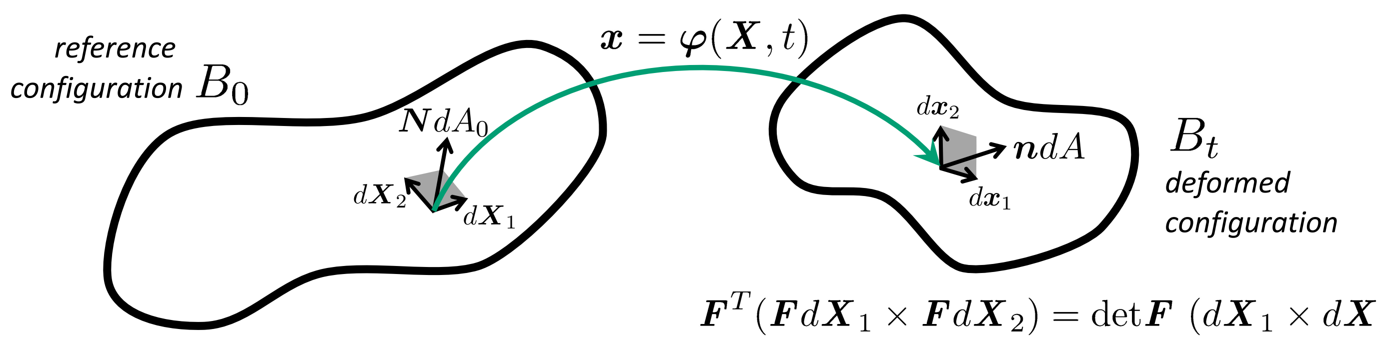
$$dA_0 = \|d\boldsymbol{X}_1 \times d\boldsymbol{X}_2\|$$

$$\boldsymbol{N} = \frac{d\boldsymbol{X}_1 \times d\boldsymbol{X}_2}{\|d\boldsymbol{X}_1 \times d\boldsymbol{X}_2\|}$$

Nanson's formula

$$n_i dA = J F^{-1}_{Ii} N_I dA_0 \Longleftrightarrow \boldsymbol{n} dA = J \boldsymbol{F}^{-T} \boldsymbol{N} dA_0$$

DIY



$$\boldsymbol{F}^T (\boldsymbol{F} d\boldsymbol{X}_1 \times \boldsymbol{F} d\boldsymbol{X}_2) = \det \boldsymbol{F} \; (d\boldsymbol{X}_1 \times d\boldsymbol{X}_2)$$

Description of local deformation

Examples

- Uniform stretching

$$x_1 = \alpha_1 X_1 \qquad x_2 = \alpha_2 X_2 \qquad x_3 = \alpha_3 X_3$$

DIY

- Simple shear

$$x_1 = X_1 + \gamma X_2 \qquad x_2 = X_2 \qquad x_3 = X_3$$

DIY

Description of local deformation

Polar decomposition theorem

- The deformation gradient provides a measure of the deformation of a material particle.
  This deformation includes:
  . Changes in length or stretching
  . Changes in angles or shearing
  . Rotations
- Right and left (unique) polar decomposition of the deformation gradient

$$F_{iJ} = R_{iI}U_{IJ} = V_{ij}R_{jJ} \Longleftrightarrow \boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$$

(this decomposition is unique if $\det \boldsymbol{F} > 0$)

$\boldsymbol{R}$ **proper orthogonal rotation**
(*two point tensor*)

Geometric meaning? DIY

$\boldsymbol{U}$ **right stretch tensor**
(*symmetric, positive-definite, material second order tensor*)

$\boldsymbol{V}$ **left stretch tensor**
(*symmetric, positive-definite, spatial second order tensor*)

Description of local deformation

Polar decomposition theorem

DIY

Let's define:

$$\boldsymbol{U} = \sqrt{\boldsymbol{C}} = \sqrt{\boldsymbol{F}^T\boldsymbol{F}}$$

$$\boldsymbol{V} = \sqrt{\boldsymbol{B}} = \sqrt{\boldsymbol{F}\boldsymbol{F}^T}$$

$\boldsymbol{B}$ **left Cauchy-Green deformation tensor**

(*symmetric, positive definite, second-order spatial tensor*)

$$\det\boldsymbol{U} = \det\sqrt{\boldsymbol{C}} = \sqrt{\det\boldsymbol{C}} = \det\boldsymbol{F}$$

$$\det\boldsymbol{V} = \sqrt{\det\boldsymbol{B}} = \det\sqrt{\boldsymbol{B}} = \det\boldsymbol{F}$$

Description of local deformation

Polar decomposition theorem

DIY

Is $\boldsymbol{R}$ proper orthogonal?

a) $\boldsymbol{R}^T\boldsymbol{R} = \delta_{IJ}\boldsymbol{e}_I \otimes \boldsymbol{e}_J = \boldsymbol{I}$ $\qquad$ $\boldsymbol{R}\boldsymbol{R}^T = \delta_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \boldsymbol{I}$

b) $\det\boldsymbol{R} = +1$

$\boldsymbol{V} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T$ (*congruence relation*)

Description of local deformation

Polar decomposition theorem

DIY

Q: How to calculate $\boldsymbol{U}$ or $\boldsymbol{V}$?

A: Using the spectral decomposition…

Q: How to calculate $\boldsymbol{R}$?

A: Using $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$

Description of local deformation

Deformation measures

- Recall: the right Cauchy-Green deformation tensor

$$C_{IJ} = F_{kI} F_{kJ} \Longleftrightarrow \boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F}$$

$$d\boldsymbol{X} = \boldsymbol{M} dS$$

$$dS^2 = \|d\boldsymbol{X}\| = \boldsymbol{M} \cdot \boldsymbol{M} dS^2 \qquad ds^2 = \boldsymbol{M} \cdot \boldsymbol{C} \boldsymbol{M} dS^2$$

- Change in squared length of an infinitesimal vector

$$ds^2 - dS^2 = \boldsymbol{M} \cdot (\boldsymbol{C} - \boldsymbol{I}) \boldsymbol{M} dS^2$$

$$E_{IJ} = \frac{1}{2}(C_{IJ} - \delta_{IJ}) \Longleftrightarrow \boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2}(\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I})$$

Lagrangian strain tensor
(*symmetric, positive-definite, second-order material tensor*)

DIY

Next lecture: the Lagrangian strain tensor is insensitive to rotations.

Description of local deformation

Examples

- Uniform stretching

$$[\boldsymbol{E}] =$$

DIY

- Simple shear

$$[\boldsymbol{E}] =$$

DIY

Any questions?