Spring, 2022

ME 597 – Solid Mechanics II

Lecture 4
Kinematics of deformations

PURDUE UNIVERSITY | Mechanical Engineering

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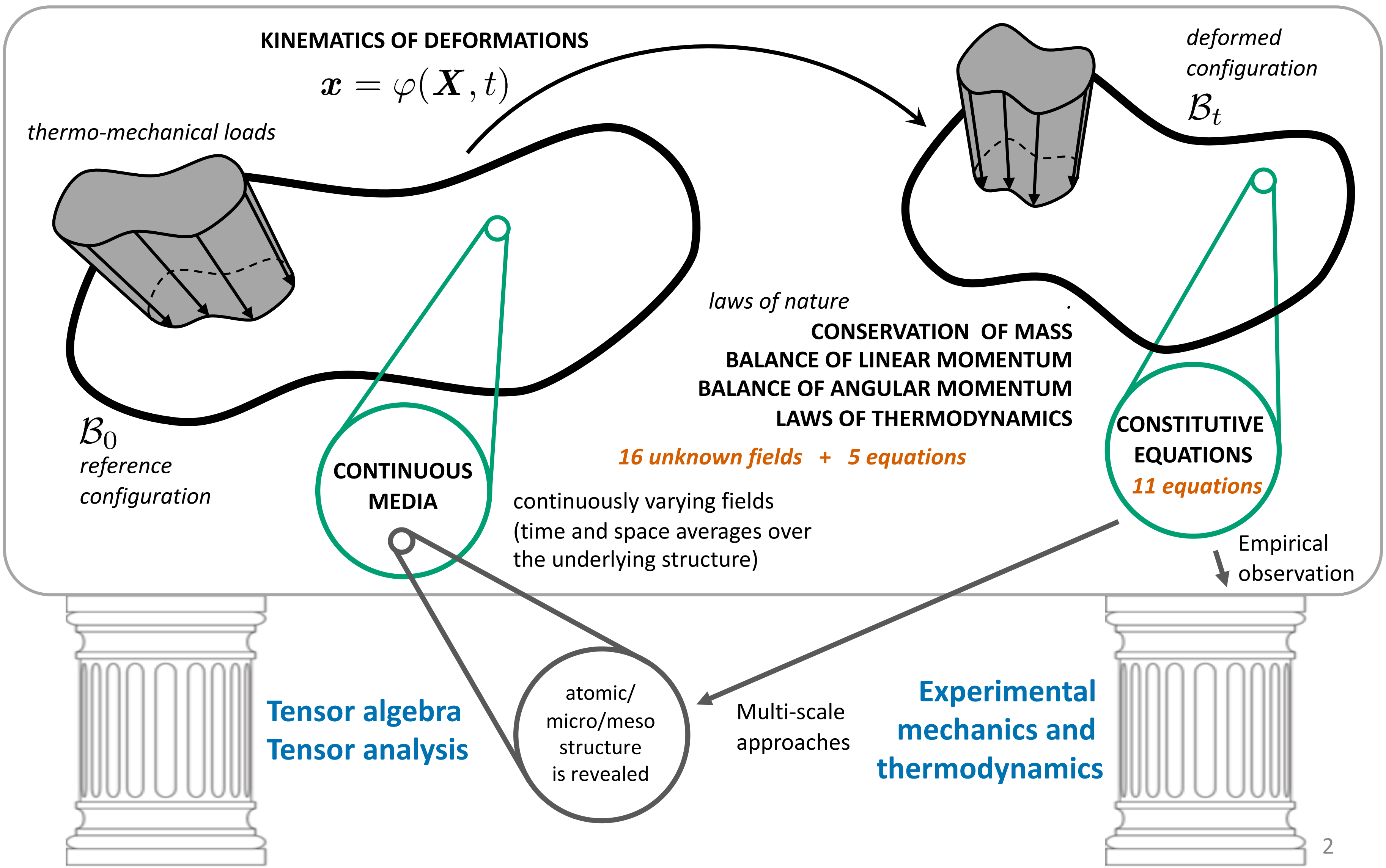
Last modified: 1/20/22 10:34:23 AM

Lecture 4 – Kinematics of deformations



Kinematics of deformation

Review

DIY

- The deformed configuration is described in terms of the <u>deformation mapping</u> $\varphi(\boldsymbol{X}, t)$
- Description of local deformation:
  + Deformation gradient (non symmetric, two-point tensor)

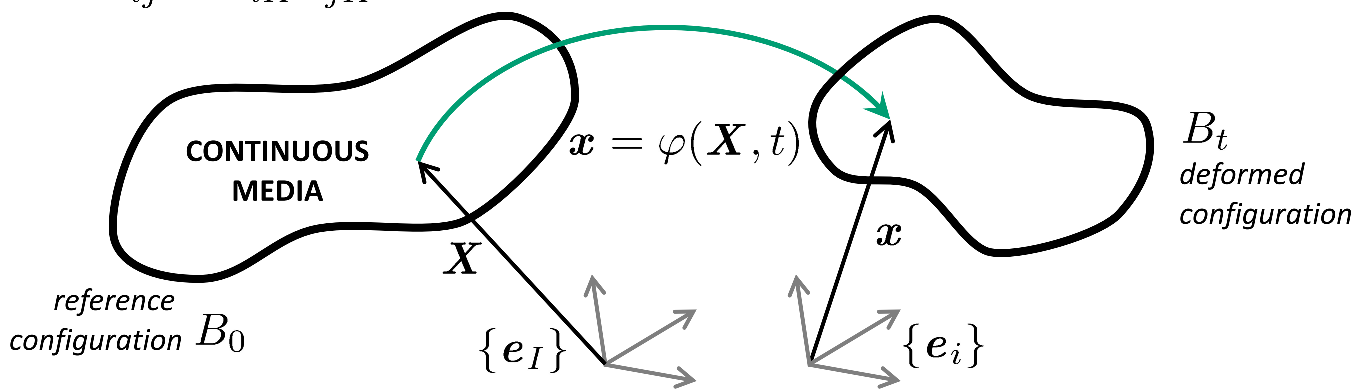
$$F_{iJ} = \frac{\partial x_i}{\partial X_J} \qquad\qquad d\boldsymbol{x} = \boldsymbol{F}\big|_{(\boldsymbol{X},t)}\, d\boldsymbol{X}$$

  + Right Cauchy-Green deformation tensor (symmetric, positive-definite, material tensor)

$$C_{IJ} = F_{kI}F_{kJ} \Longleftrightarrow \boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} \qquad\qquad ds^2 = \boldsymbol{M}\cdot\boldsymbol{C}\boldsymbol{M}\, dS^2$$

  + Left Cauchy-Green deformation tensor (symmetric, positive-definite, spatial tensor)

$$B_{ij} = F_{iK}F_{jK} \Longleftrightarrow \boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$$



Kinematics of deformation

Review

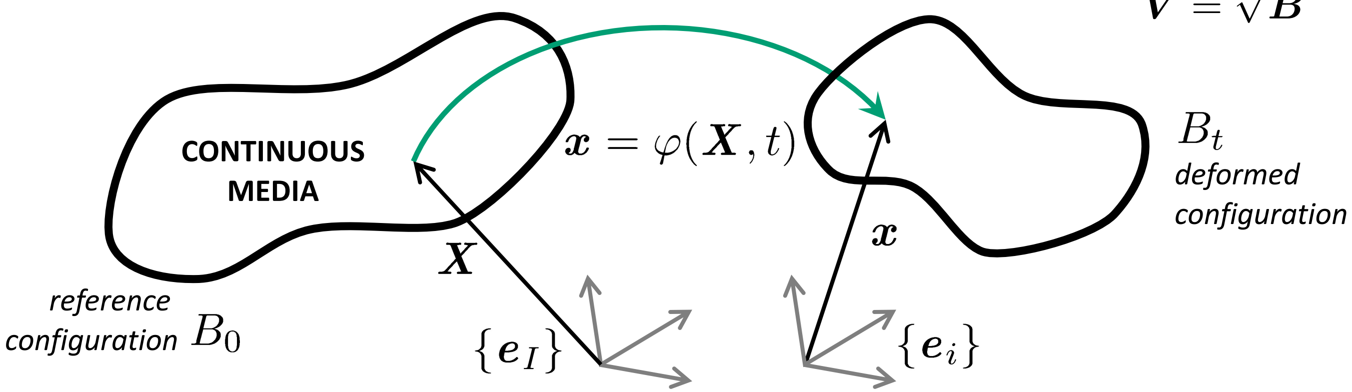
DIY

- The deformed configuration is described in terms of the <u>deformation mapping</u> $\varphi(\boldsymbol{X}, t)$
- Description of local deformation:
  + Right and left polar decomposition of the deformation gradient

$$F_{iJ} = R_{iI}U_{IJ} = V_{ij}R_{jJ} \iff \boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$$

. Proper orthogonal rotation: ***R*** (two point tensor)
. Right stretch tensor: ***U*** (symmetric, positive-definite, material tensor)
. Left stretch tensor: ***V*** (symmetric, positive-definite, spatial tensor)

$$\boldsymbol{U} = \sqrt{\boldsymbol{C}}$$
$$\boldsymbol{V} = \sqrt{\boldsymbol{B}}$$



Kinematics of deformation

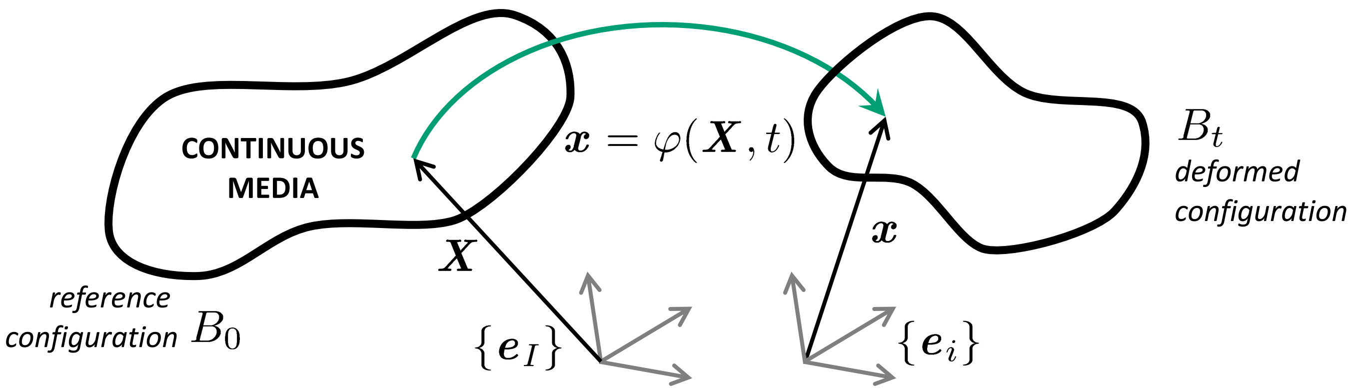
Review

DIY

- The deformed configuration is described in terms of the deformation mapping $\boldsymbol{\varphi}(\boldsymbol{X},t)$
- Description of local deformation:
  + Lagrangian strain tensor (symmetric, positive-definite, material tensor)

$$E_{IJ} = \frac{1}{2}(C_{IJ} - \delta_{IJ}) \Longleftrightarrow \boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$$

$$ds^2 - dS^2 = \boldsymbol{M} \cdot (\boldsymbol{C} - \boldsymbol{I})\boldsymbol{M}\, dS^2$$



Kinematics of deformation

Deformation measures

- Change in squared length of an infinitesimal vector

$$ds^2 - dS^2 = \boldsymbol{M} \cdot (\boldsymbol{C} - \boldsymbol{I})\boldsymbol{M}\, dS^2$$

$$E_{IJ} = \frac{1}{2}(C_{IJ} - \delta_{IJ}) \Longleftrightarrow \boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \frac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})$$

Lagrangian strain tensor
(*symmetric, positive-definite, second-order material tensor*)

DIY

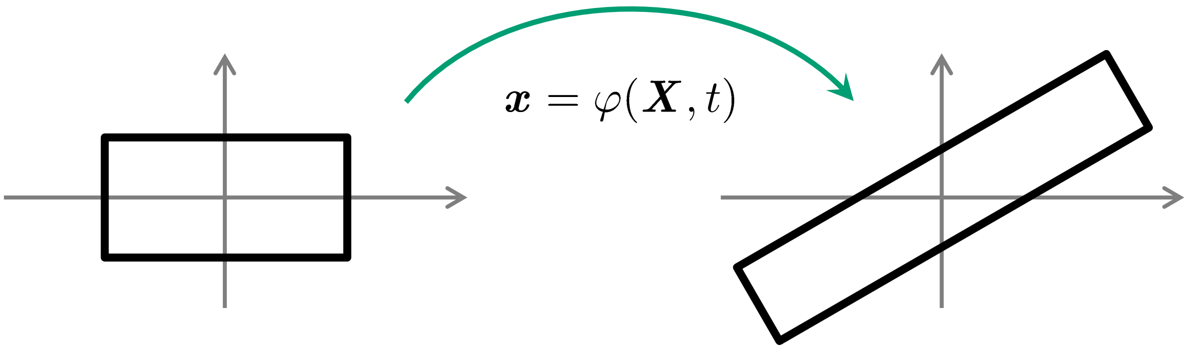
Note: the Lagrangian strain tensor is insensitive to rotations

Deformation gradient

Deformation gradient

- Example: elongation and rotation

DIY

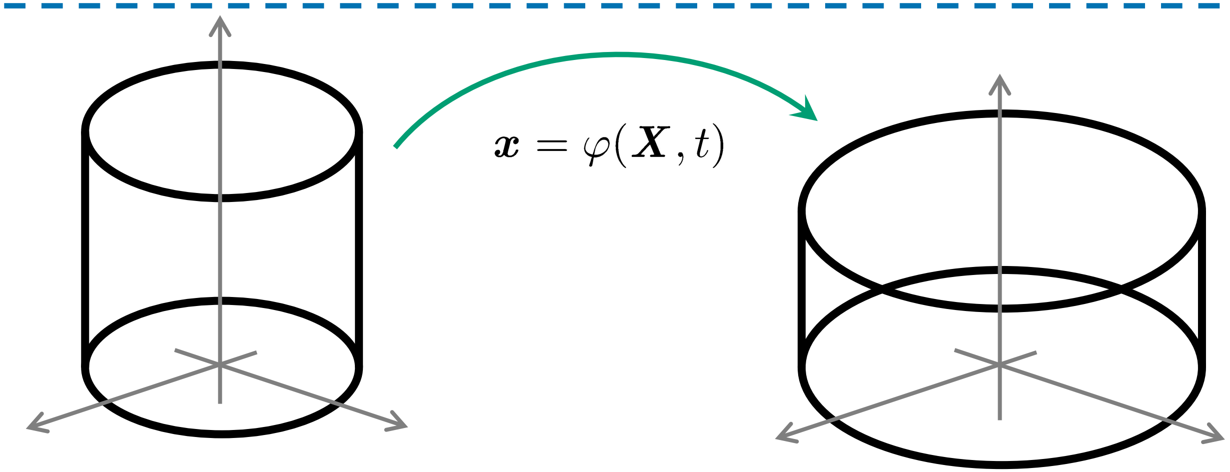


Deformation gradient

Deformation gradient

- Example: use cylindrical coordinates to write the deformation mapping in Cartesian coordinates, and use the deformation gradient to enforce incompressibility

DIY



Deformation gradient

Deformation gradient

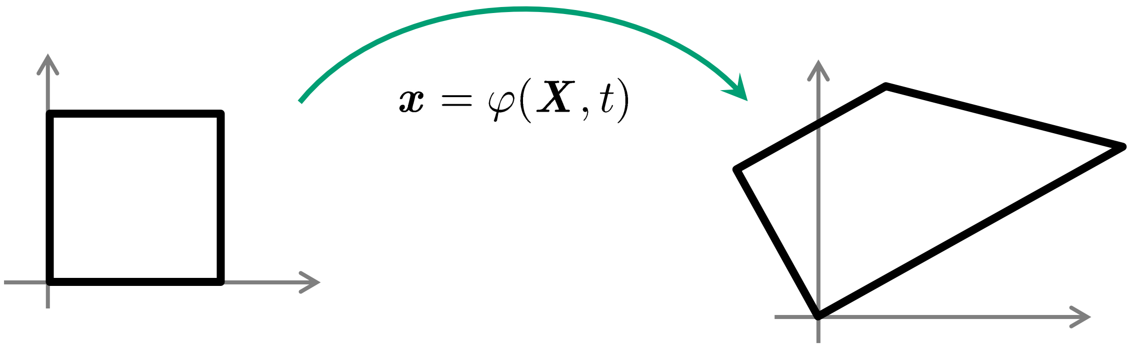
- Example: use cylindrical coordinates to write the deformation mapping in Cartesian coordinates, and use the deformation gradient to enforce incompressibility

DIY

Deformation gradient

- Example: deformation and rotation

DIY

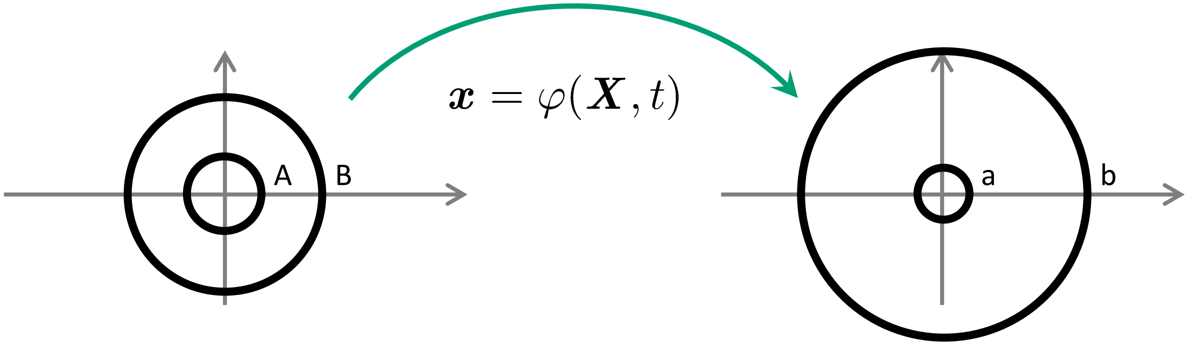


Deformation gradient

Deformation gradient

- Example: deformation – changes in density?

DIY

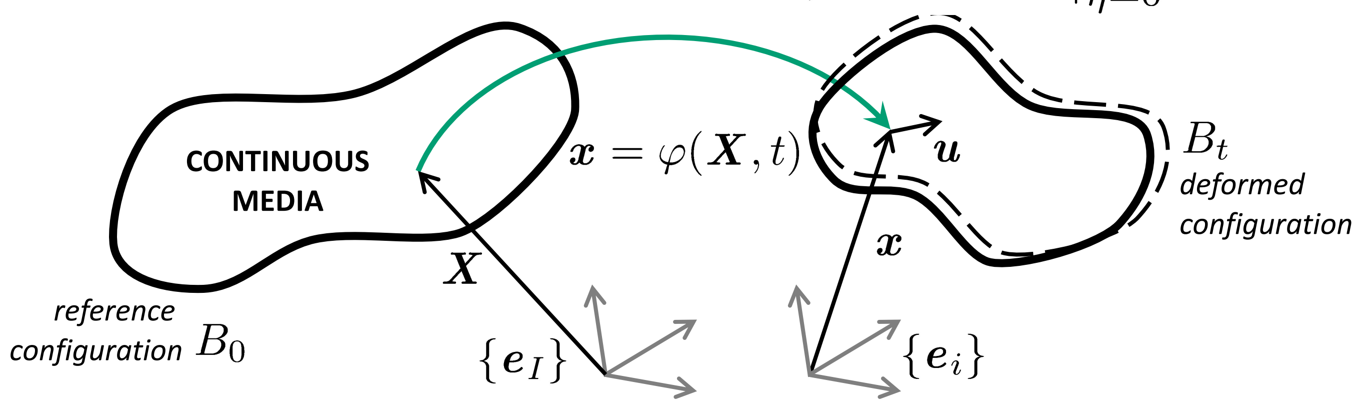


Linearized kinematics

Linearized kinematics

- Linearized or incremental expressions for the kinematic quantities are required when:
  (i) the deformation process is described as a series of small steps
  $$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}) \rightarrow \boldsymbol{\varphi}(\boldsymbol{X}) + \boldsymbol{u}(\boldsymbol{X})$$
  (ii) displacements are indeed small
  $$\boldsymbol{X} \rightarrow \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X})$$
- We define the variation of $\square$ in the direction of the vector field $\boldsymbol{u}(\boldsymbol{X})$

$$\nabla_{\varphi}\square \cdot \boldsymbol{u} = \langle \nabla_{\varphi}\square ; \boldsymbol{u} \rangle = \left. \frac{d}{d\eta} \square[\boldsymbol{\varphi} + \eta \boldsymbol{u}] \right|_{\eta=0}$$



Linearized kinematics

Linear parts of kinematic fields

- Examples: Right Cauchy-Green deformation tensor

DIY

(i) the deformation process is described as a series of small steps

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}) \rightarrow \boldsymbol{\varphi}(\boldsymbol{X}) + \boldsymbol{u}(\boldsymbol{X})$$

(ii) displacements are indeed small

$$\boldsymbol{X} \rightarrow \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X}) \qquad \boldsymbol{F} = \boldsymbol{I},\ \nabla_0 \boldsymbol{u} = \nabla \boldsymbol{u}$$

Linearized kinematics

Linear parts of kinematic fields

- Examples (for $\boldsymbol{x} = \varphi(\boldsymbol{X}) \rightarrow \varphi(\boldsymbol{X}) + \boldsymbol{u}(\boldsymbol{X})$):

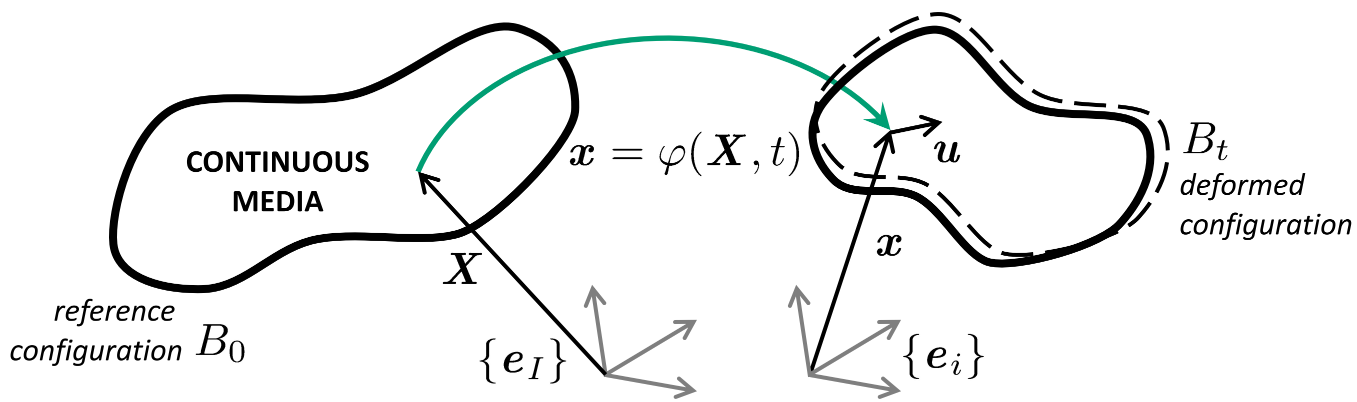
$$\langle \nabla_\varphi \boldsymbol{F}; \boldsymbol{u} \rangle = \frac{d}{d\eta} \left[\frac{\partial}{\partial X_J} (\varphi_i + \eta u_i) \right] \Big|_{\eta=0} \boldsymbol{e}_i \otimes \boldsymbol{e}_J = \nabla_0 \boldsymbol{u}$$

$$\langle \nabla_\varphi \boldsymbol{E}; \boldsymbol{u} \rangle = \tfrac{1}{2} [\boldsymbol{F}^T \nabla_0 \boldsymbol{u} + (\boldsymbol{F}^T \nabla_0 \boldsymbol{u})^T]$$

$$\langle \nabla_\varphi J; \boldsymbol{u} \rangle = \frac{\partial \det \boldsymbol{F}}{\partial \boldsymbol{F}} : \nabla_0 \boldsymbol{u} = J \mathrm{tr}((\nabla_0 \boldsymbol{u}) \boldsymbol{F}^{-1})$$

DIY

linearized quantities used in computational mechanics



Linearized kinematics

Linear parts of kinematic fields

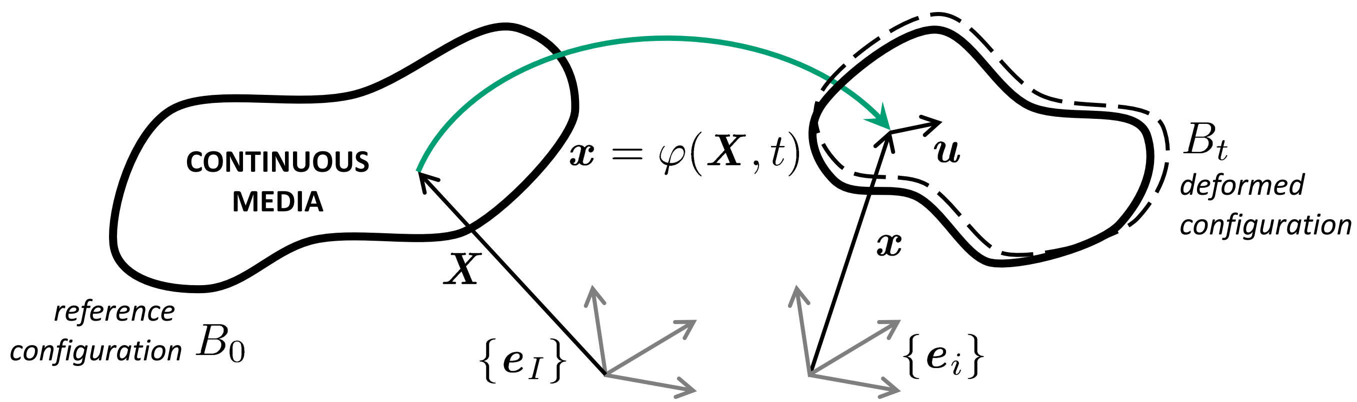
- Examples (for $\boldsymbol{X} \to \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X})$, i.e., $\boldsymbol{F} = \boldsymbol{I},\ \nabla_0 \boldsymbol{u} = \nabla \boldsymbol{u}$):
  [the linearization is evaluated in the undeformed configuration]

$$\langle \nabla_\varphi \boldsymbol{E}; \boldsymbol{u} \rangle = \tfrac{1}{2}[\boldsymbol{F}^T \nabla_0 \boldsymbol{u} + (\boldsymbol{F}^T \nabla_0 \boldsymbol{u})^T] = \tfrac{1}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T] = \boldsymbol{\epsilon}$$

small-strain tensor
(*employed in elasticity theory*)

$$\langle \nabla_\varphi J; \boldsymbol{u} \rangle = J \mathrm{tr}((\nabla_0 \boldsymbol{u}) \boldsymbol{F}^{-1}) = \mathrm{tr} \nabla \boldsymbol{u} = \mathrm{tr} \boldsymbol{\epsilon}$$

dilatation
(*employed in elasticity theory*)



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Any questions?