# Lecture 5
# Conservation and balance laws


PURDUE UNIVERSITY | Mechanical Engineering

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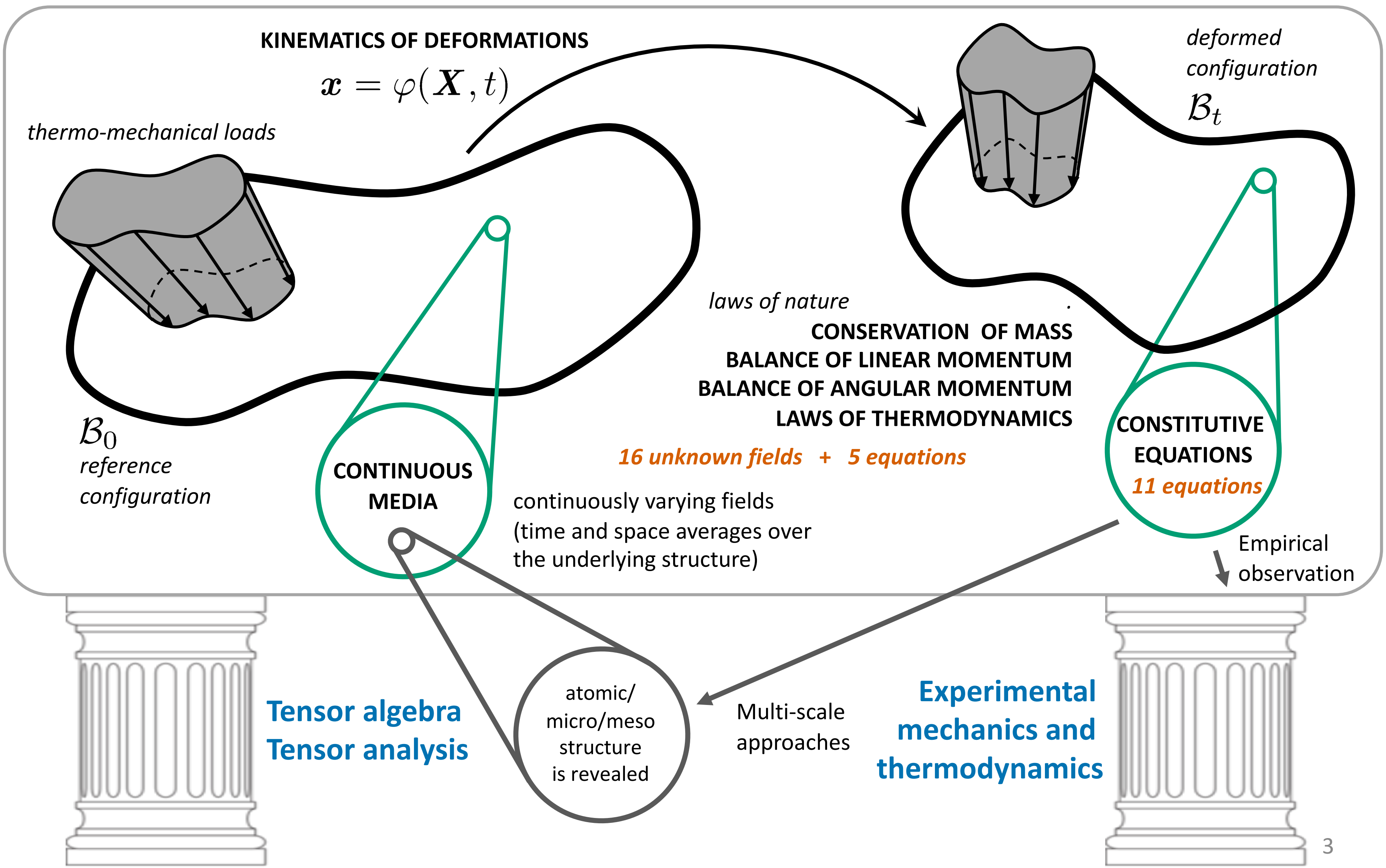
# Lecture 5 – Conservation and balance laws



# Kinematic rates

## Material time derivative

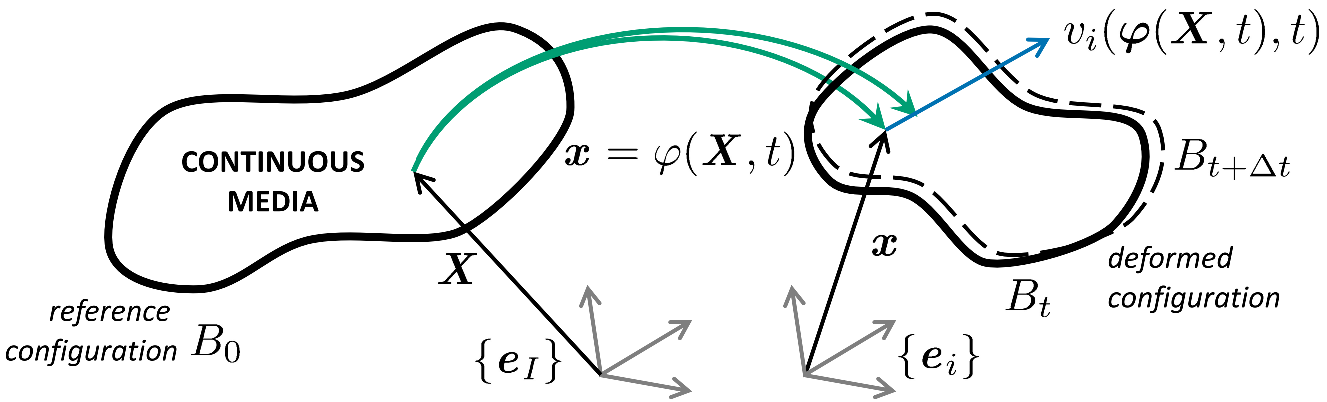
- Material time derivative of the field $\square$ : $\left.\dfrac{\partial \square(\boldsymbol{X}, t)}{\partial t}\right|_{\boldsymbol{X}} = \dot{\square} = \dfrac{D\square}{Dt}$
  (*follows a particular material particle*)

- Local rate of change of the field $\square$ : $\left.\dfrac{\partial \square(\boldsymbol{x}, t)}{\partial t}\right|_{\boldsymbol{x}}$
  (*at a fixed spatial position*)

- Example: the motion $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t)$

**velocity**

$$\dot{\boldsymbol{x}} = \frac{\partial \boldsymbol{\varphi}(\boldsymbol{X}, t)}{\partial t} = \breve{\boldsymbol{v}}(\boldsymbol{X}, t) = \breve{\boldsymbol{v}}(\boldsymbol{\varphi}^{-1}(\boldsymbol{x}, t), t) \equiv \boldsymbol{v}(\boldsymbol{x}, t)$$



# Kinematic rates

## Rate of change of local deformation measures

- Rate of change of the deformation gradient

$$\overline{d\boldsymbol{x}}^{\boldsymbol{\cdot}} = \overline{\boldsymbol{F}d\boldsymbol{X}}^{\boldsymbol{\cdot}} = \nabla \boldsymbol{v} \boldsymbol{F} d\boldsymbol{X} = \nabla \boldsymbol{v} d\boldsymbol{x}$$

$$\dot{F}_{iJ} = \overline{\frac{\partial x_i}{\partial X_J}}^{\boldsymbol{\cdot}} = \frac{\partial \breve{v}_i}{\partial X_J} = \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_J}$$

DIY

Note: in a dynamical spatial setting, the velocity gradient plays a role similar to the deformation gradient.

$$\overline{d\boldsymbol{x} \cdot d\boldsymbol{x}}^{\boldsymbol{\cdot}} = (\nabla \boldsymbol{v} d\boldsymbol{x}) \cdot d\boldsymbol{x} + d\boldsymbol{x} \cdot (\nabla \boldsymbol{v} d\boldsymbol{x}) = d\boldsymbol{x} \cdot \left[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\right] d\boldsymbol{x}$$

$$v_{i,j} \iff \boldsymbol{\nabla v}$$

**velocity gradient**
*(spatial gradient of the velocity field)*

$$d_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right)$$

**rate of deformation tensor**
(*symmetric part of the velocity gradient*)

$$w_{ij} = \frac{1}{2}\left(v_{i,j} - v_{j,i}\right)$$

**spin tensor**
(*antisymmetric part of the velocity gradient*)

# Conservation and balance laws

## Review: Conservation of mass

DIY

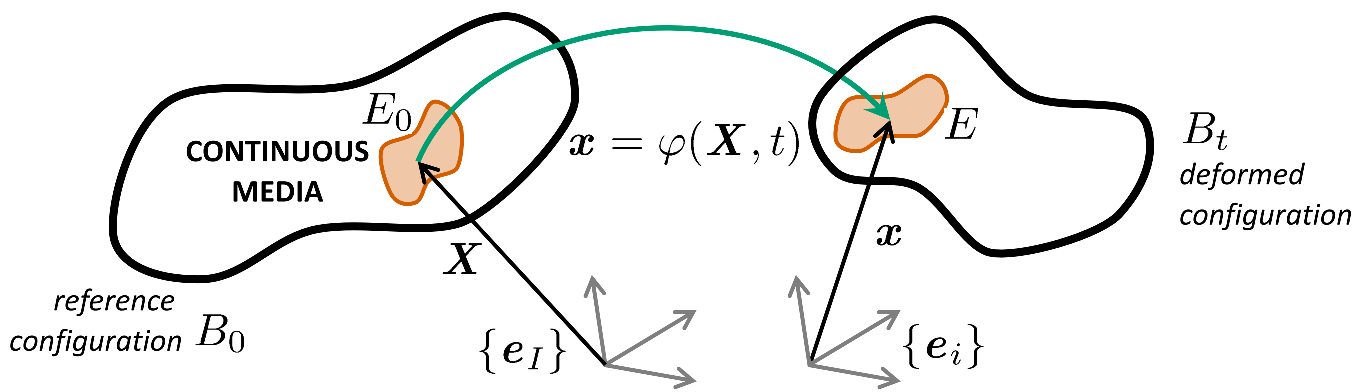
- Divergence theorem: $\int_{\partial\Omega} \boldsymbol{\omega} \cdot \boldsymbol{n} \, dA = \int_{\Omega} \mathrm{div}\boldsymbol{\omega} \, dV \qquad \int_{\partial\Omega} \boldsymbol{T}\boldsymbol{n} \, dA = \int_{\Omega} \mathrm{div}\, \boldsymbol{T} \, dV$

- Reynolds transport theorem for extensive properties:

$$\frac{D}{Dt}\int_{E\subset B} \rho\psi dV = \int_{E\subset B} [\dot{\overline{\rho\psi}} + (\rho\psi)\ \mathrm{div}\boldsymbol{v}]dV = \int_{E\subset B} \rho\dot{\psi} dV$$

- Conservation of mass (i.e., $\psi = 1$ ):

$$J\rho = \rho_0 \quad \forall \boldsymbol{X} \in B_0$$ **material form of the conservation of mass**



# Conservation and balance laws

## Conservation of linear momentum

$$\frac{D}{Dt}\boldsymbol{L} = \boldsymbol{F}^{\text{ext}}$$

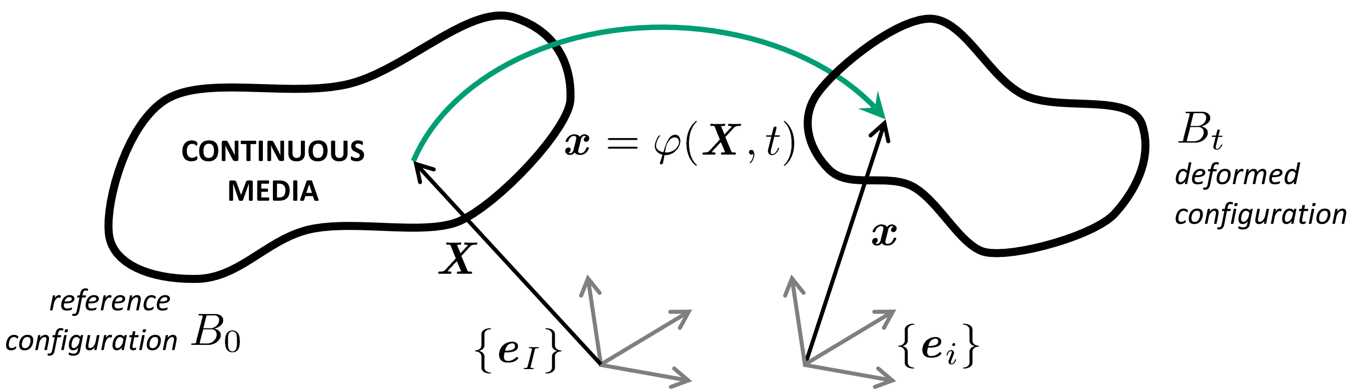
- Newton's second law (for a system of particles)

$$\frac{D}{Dt}(m^\alpha \dot{\boldsymbol{r}}^\alpha) = \boldsymbol{f}^\alpha \qquad \frac{D}{Dt}\sum_{\alpha=1}^{N} m^\alpha \dot{\boldsymbol{r}}^\alpha = \sum_{\alpha=1}^{N} \boldsymbol{f}^\alpha$$

- Continuum system (or continuum body)

$$\boldsymbol{L}(B) = \int_B d\boldsymbol{L} dV = \int_B \dot{\boldsymbol{x}} dm = \int_B \dot{\boldsymbol{x}} \rho dV$$

... but what about $\boldsymbol{F}^{\text{ext}}(B)$?



# Conservation and balance laws

## Conservation of linear momentum

- External forces in a continuum body:

+ Body forces per unit mass (e.g., gravity and electromagnetic fields) $\int_B \boldsymbol{b}\rho dV$

+ Surface forces per unit *spatial* area – Traction or stress vector

$$\bar{\boldsymbol{t}} \equiv \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{f}^{\text{surf}}}{\Delta A} = \frac{d\boldsymbol{f}^{\text{surf}}}{dA}$$

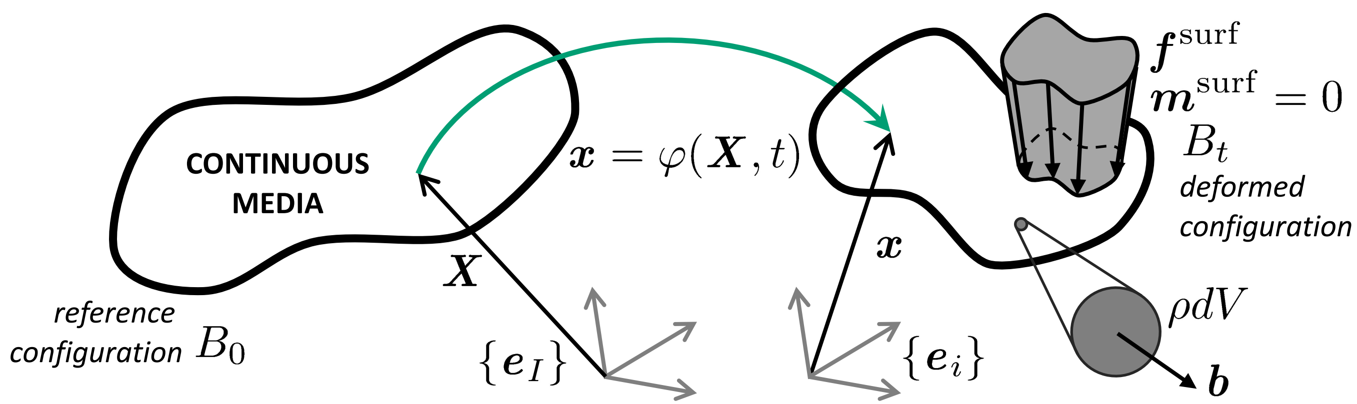
$$\Delta \boldsymbol{m}^{\text{surf}} = 0$$

Note: these are assumptions!

$$\int_{\partial B} d\boldsymbol{f}^{\text{surf}} = \int_{\partial B} \bar{\boldsymbol{t}} dA$$

**spatial form of the global balance of linear momentum**

$$\frac{D}{Dt}\int_B \rho \dot{\boldsymbol{x}} dV = \boldsymbol{F}^{\text{ext}}(B) \iff \int_B \ddot{\boldsymbol{x}}\rho dV = \int_B \boldsymbol{b}\rho dV + \int_{\partial B} \bar{\boldsymbol{t}} dA$$



# Conservation and balance laws

## Conservation of linear momentum

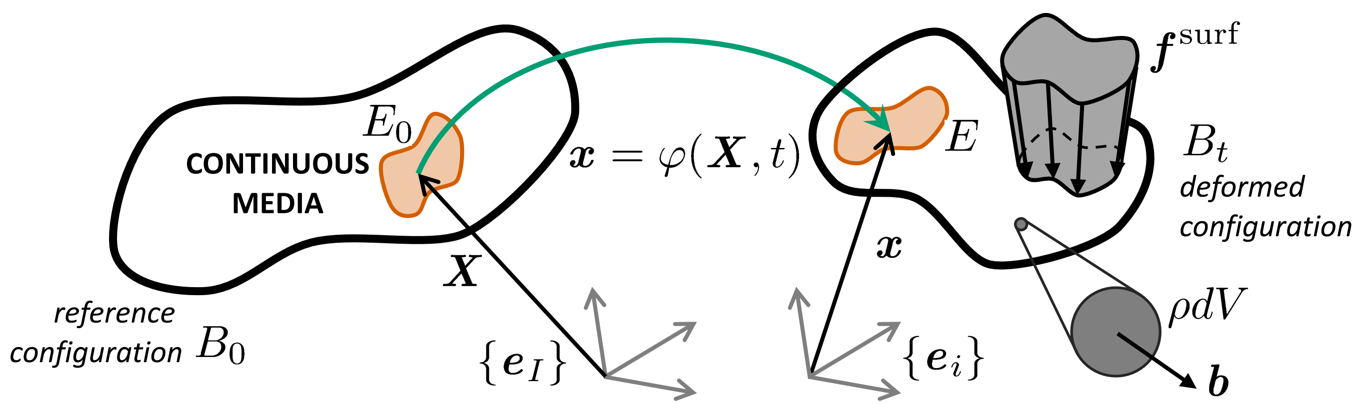
- **Cauchy's stress principle**:
  *"Material interactions across an internal surface of a body can be described as a distribution of tractions in the same way that the effect of external forces on physical surfaces of the body are described"*

$$\int_E (\ddot{\boldsymbol{x}} - \boldsymbol{b})\rho dV = \int_{\partial E} \boldsymbol{t} dA = \int_{\partial P_t} \boldsymbol{t} dA + \int_{\partial P_b} \boldsymbol{t} dA + \int_{\partial P_c} \boldsymbol{t} dA$$

DIY

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{x}, -\boldsymbol{n}) \quad \forall \boldsymbol{x} \in B$$ **Cauchy's lemma**

$$\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}) = \bar{\boldsymbol{t}}(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \partial B$$



# Conservation and balance laws

## Conservation of linear momentum

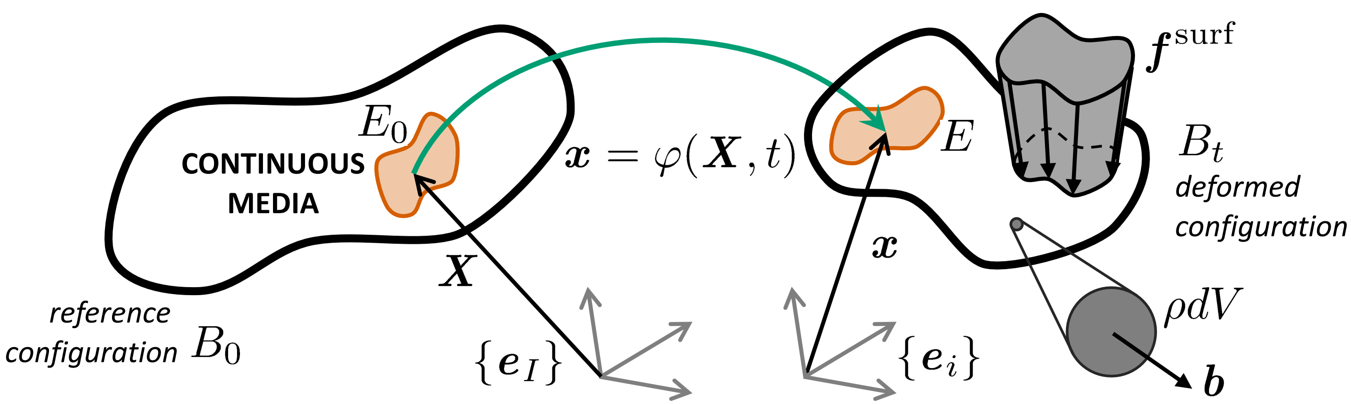
- Stress state at $\boldsymbol{x} \in B$: infinite number of $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{n})$
  (it is an odd function of $\boldsymbol{n}$, is it also a linear function of $\boldsymbol{n}$?)
- Let's follow Cauchy …

$$\int_T (\ddot{\boldsymbol{x}} - \boldsymbol{b})\rho dV = \int_{\partial T} \boldsymbol{t} dA = \int_{\partial T_1} \boldsymbol{t} dA + \int_{\partial T_2} \boldsymbol{t} dA + \int_{\partial T_3} \boldsymbol{t} dA + \int_{\partial T_n} \boldsymbol{t} dA$$

DIY

$$\boldsymbol{t}(\boldsymbol{n}) = \boldsymbol{t}(\boldsymbol{e}_1)n_1 + \boldsymbol{t}(\boldsymbol{e}_2)n_2 + \boldsymbol{t}(\boldsymbol{e}_3)n_3 = \boldsymbol{t}(\boldsymbol{e}_j)n_j$$

$$\boldsymbol{e}_i \cdot \boldsymbol{t}(\boldsymbol{n}) = t_i(\boldsymbol{n}) = \boldsymbol{e}_i \cdot \boldsymbol{t}(\boldsymbol{e}_j)n_j \equiv \sigma_{ij} n_j$$



# Conservation and balance laws

## Conservation of linear momentum

- Cauchy stress tensor

$$t_i(\boldsymbol{n}) = \sigma_{ij} n_j \Longleftrightarrow \boldsymbol{t}(\boldsymbol{n}) = \boldsymbol{\sigma}\boldsymbol{n}$$ **Cauchy stress tensor**

DIY

$\sigma_{ij}$ is the component of the traction (i.e., the stress) acting in the direction of $\boldsymbol{e}_i$ on the face normal to $\boldsymbol{e}_j$ .

Examples:

$\sigma_{11}$ is a normal (tensile/compressive) stress

$\sigma_{12}$ is a shear stress

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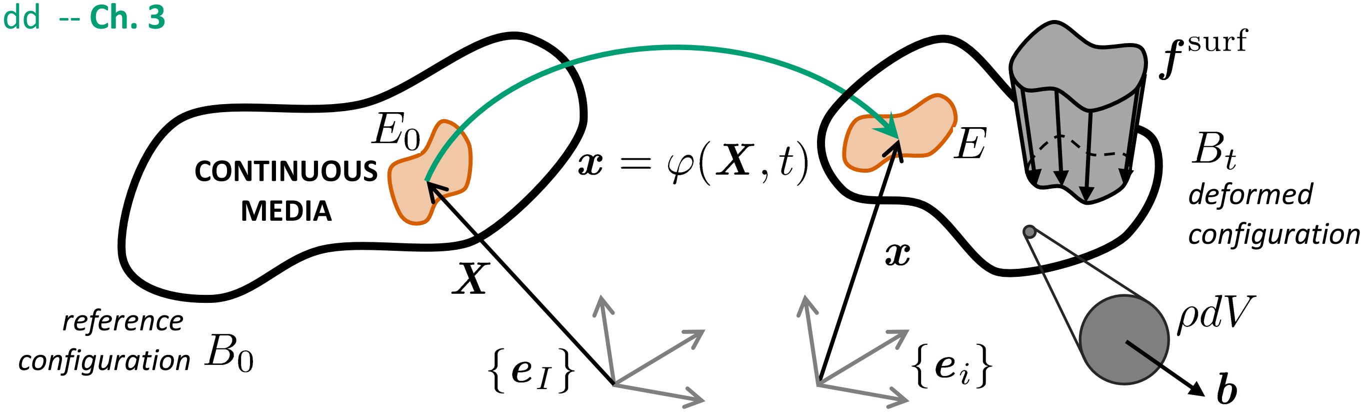


# Conservation and balance laws

## Conservation of linear momentum

- Additive decomposition of the Cauchy stress tensor

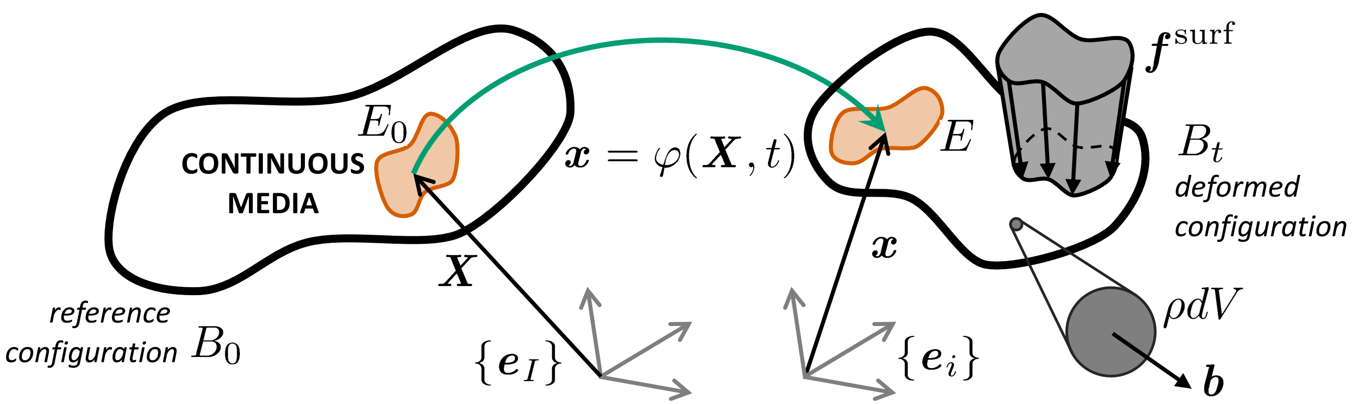
$$\sigma_{ij} = s_{ij} - p\delta_{ij} \iff \boldsymbol{\sigma} = \boldsymbol{s} - p\boldsymbol{I}$$

$p = -\frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}$ is the hydrostatic stress or pressure

$\boldsymbol{s}$ is the deviatoric part of the stress tensor (only includes information on shear stress)

DIY

$\mathrm{tr}\boldsymbol{s} =$



# Conservation and balance laws

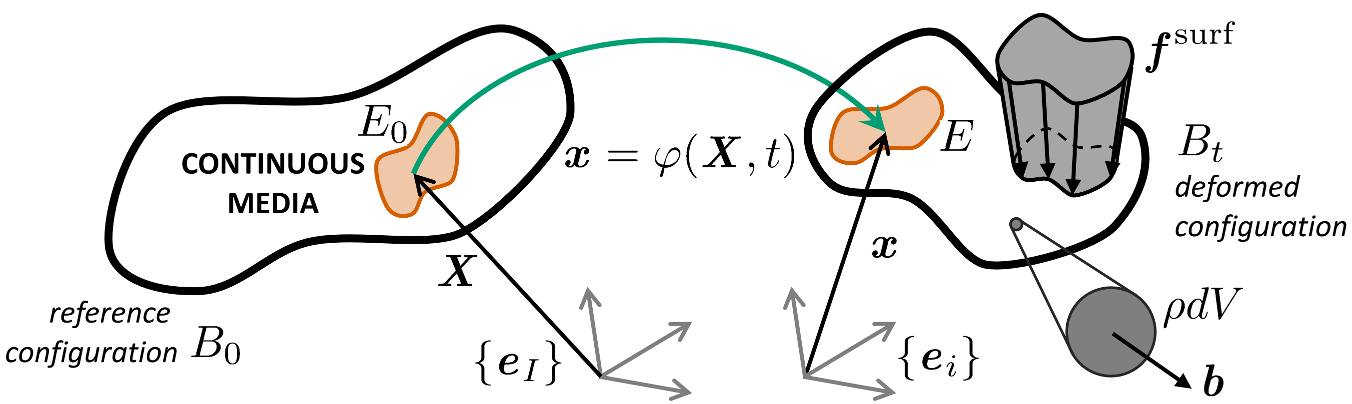
## Conservation of linear momentum

- Local form of the balance of linear momentum

$$\int_E (\ddot{\boldsymbol{x}} - \boldsymbol{b})\rho dV = \int_{\partial E} \boldsymbol{t} dA = \int_{\partial E} \boldsymbol{\sigma n} dA = \int_E \mathrm{div}\boldsymbol{\sigma} dV \quad \forall E \subset B$$

$$\sigma_{ij,j} + \rho b_i = \rho a_i \Longleftrightarrow \mathrm{div}\boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho \boldsymbol{a} \quad \forall \boldsymbol{x} \in B$$

**local spatial form of the balance of linear momentum**



# Conservation and balance laws

## Conservation of linear momentum

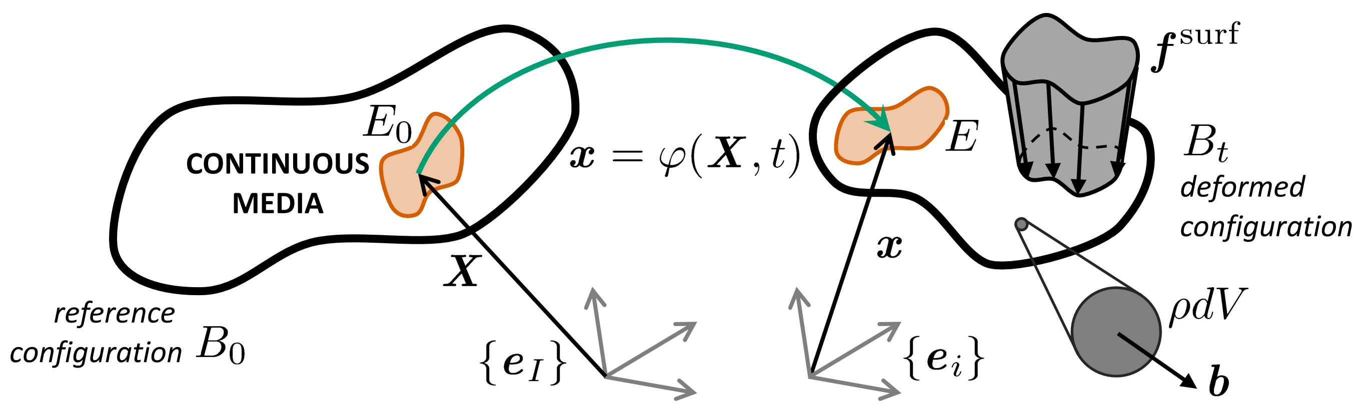
- Local form of the balance of linear momentum

$$\sigma_{ij,j} + \rho b_i = \rho a_i \iff \mathrm{div}\boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho \boldsymbol{a} \quad \forall \boldsymbol{x} \in B$$

DIY

$$\mathrm{div}\boldsymbol{\sigma} + \rho \boldsymbol{b} = 0 \quad \forall \boldsymbol{x} \in B$$

**stress equilibrium equation**



# Conservation and balance laws

## Conservation of angular momentum

$$\frac{D}{Dt}\boldsymbol{H}_o = \boldsymbol{M}_o^{\text{ext}}$$

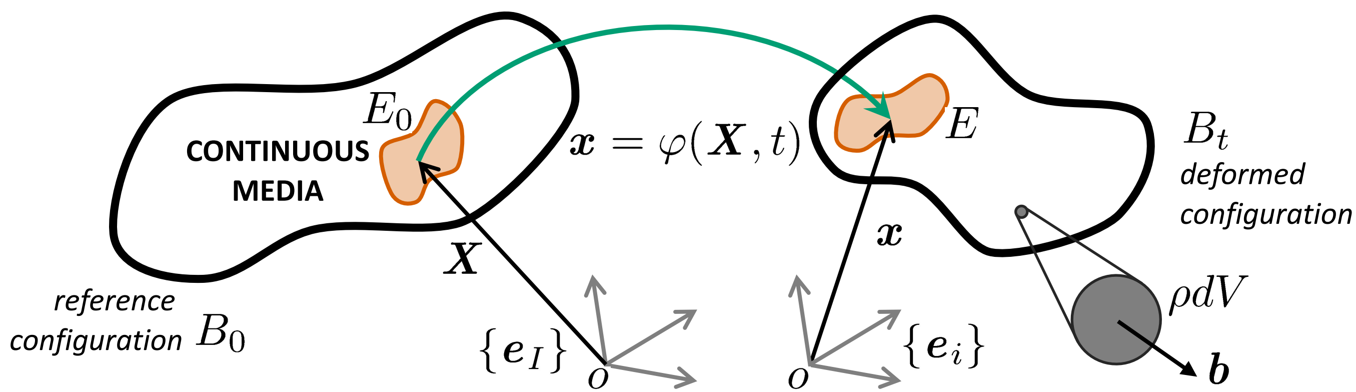
$o$ : the origin

- Moment of momentum principle (for a system of particles)

$$\frac{D}{Dt}\sum_{\alpha=1}^{N}\boldsymbol{r}^{\alpha}\times(m^{\alpha}\dot{\boldsymbol{r}}^{\alpha}) = \sum_{\alpha=1}^{N}\boldsymbol{r}^{\alpha}\times\boldsymbol{f}^{\alpha}$$

- Continuum system (or continuum body)

$$\boldsymbol{H}_o(E) = \int_E \boldsymbol{x}\times(\rho\dot{\boldsymbol{x}})dV \qquad \boldsymbol{M}_o^{\text{ext}}(E) = \int_E \boldsymbol{x}\times(\rho\boldsymbol{b})dV + \int_{\partial E}\boldsymbol{x}\times\boldsymbol{t}\ dA$$



# Conservation and balance laws

## Conservation of angular momentum

$$\frac{D}{Dt}\int_E \boldsymbol{x} \times (\rho \dot{\boldsymbol{x}}) dV = \int_E \boldsymbol{x} \times (\rho \boldsymbol{b}) dV + \int_{\partial E} \boldsymbol{x} \times \boldsymbol{t}\, dA$$

DIY

What can we say about the **symmetry** of the Cauchy stress tensor?

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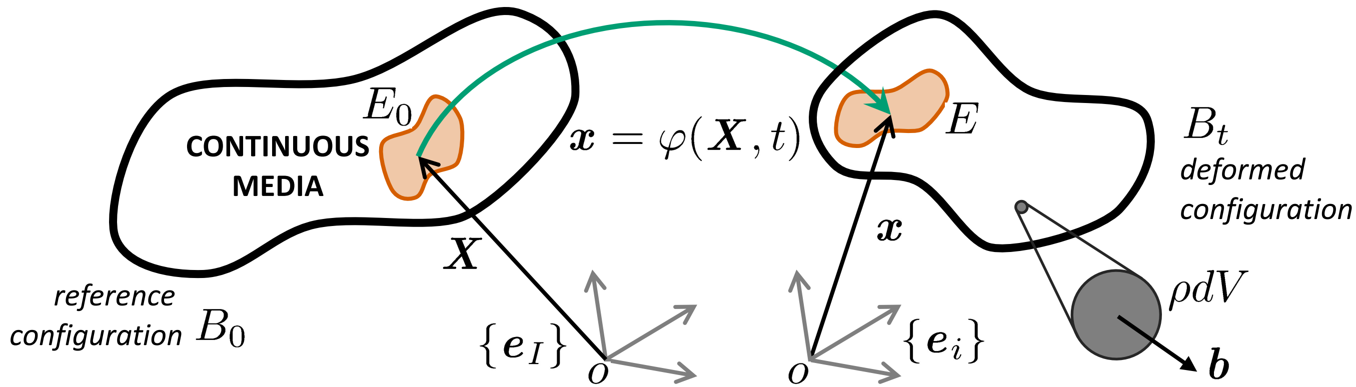

# Conservation and balance laws

## Example:

DIY

Given:

$$[\boldsymbol{\sigma}] =$$

$$[\boldsymbol{n}] =$$

Determine:

Normal stress $\sigma_{\boldsymbol{n}} = \boldsymbol{n} \cdot \boldsymbol{t} = \boldsymbol{n} \cdot (\boldsymbol{\sigma} \boldsymbol{n})$

Tangential stress $\boldsymbol{\tau}_{\boldsymbol{n}} = \boldsymbol{t} - [\boldsymbol{n} \cdot (\boldsymbol{\sigma} \boldsymbol{n})] \boldsymbol{n}$

Principal stresses

Principal directions

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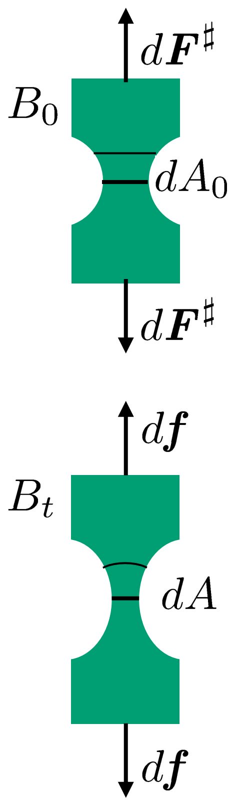
# Conservation and balance laws

## True stress – Engineering stress




True traction:

$$t_i = \frac{df_i}{dA} = \sigma_{ij} n_j \iff \boldsymbol{t} = \boldsymbol{\sigma n}$$

**Cauchy stress tensor**
(*symmetric, spatial tensor,*
*a.k.a., true stress*)

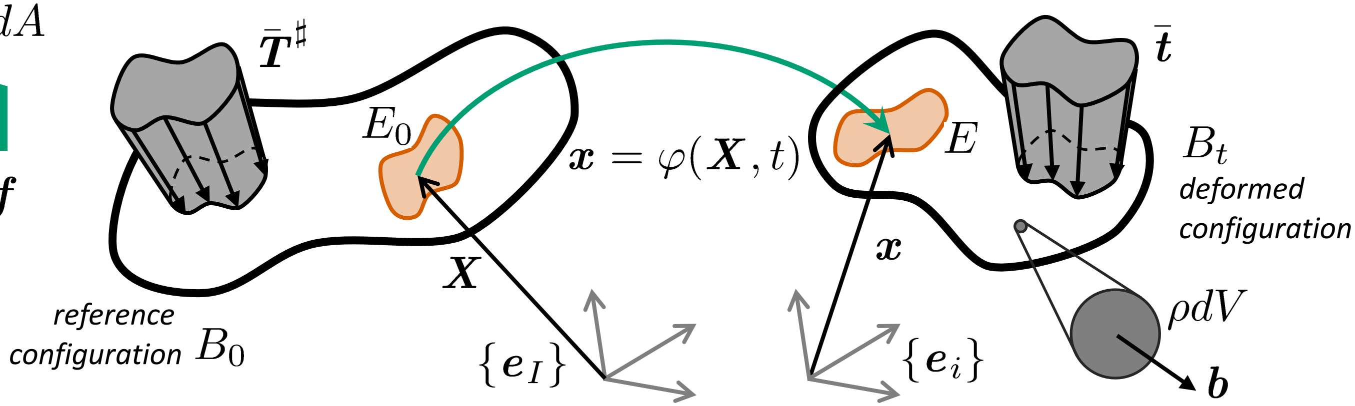
Nominal traction:

$$T_i = \frac{df_i}{dA_0} = P_{iJ} N_J \iff \boldsymbol{T} = \boldsymbol{PN}$$

**first Piola-Kirchhoff stress tensor**
(*non-symmetric, two-point tensor,*
*a.k.a. engineering stress*)

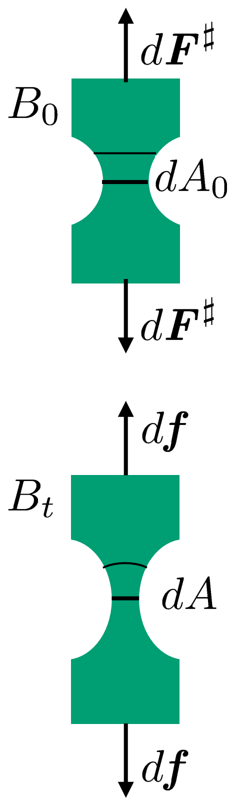
$$P_{iJ} = J \sigma_{ij} F^{-1}_{Jj} \iff \boldsymbol{P} = J \boldsymbol{\sigma} \boldsymbol{F}^{-T}$$

DIY



# Conservation and balance laws

## True stress – Engineering stress – …




True traction:

$$t_i = \frac{df_i}{dA} = \sigma_{ij} n_j \iff \boldsymbol{t} = \boldsymbol{\sigma} \boldsymbol{n}$$

**Cauchy stress tensor**
(*symmetric, spatial tensor, a.k.a., true stress*)

Nominal traction:

$$T_i = \frac{df_i}{dA_0} = P_{iJ} N_J \iff \boldsymbol{T} = \boldsymbol{P} \boldsymbol{N}$$

**first Piola-Kirchhoff stress tensor**
(*non-symmetric, two-point tensor, a.k.a. engineering stress*)

$$P_{iJ} = J \sigma_{ij} F^{-1}_{Jj} \iff \boldsymbol{P} = J \boldsymbol{\sigma} \boldsymbol{F}^{-T}$$

Pull-back of the nominal traction:

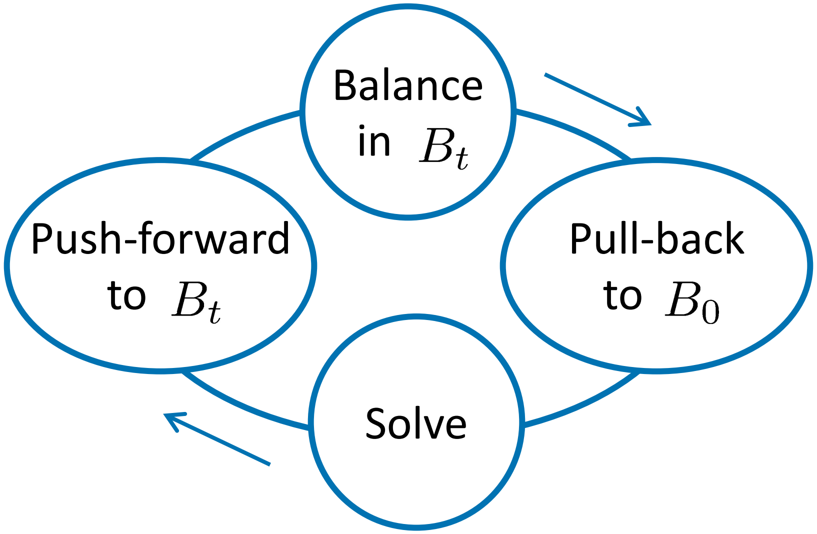
$$\boldsymbol{T}^\sharp = \frac{d\boldsymbol{F}^\sharp}{dA_0} = \boldsymbol{F}^{-1} \boldsymbol{P} \boldsymbol{N} = \boldsymbol{S} \boldsymbol{N}$$

**second Piola-Kirchhoff stress tensor**
(*symmetric, material tensor*)

$$S_{IJ} = J F^{-1}_{Ii} \sigma_{ij} F^{-1}_{Jj} \iff \boldsymbol{S} = J \boldsymbol{F}^{-1} \boldsymbol{\sigma} \boldsymbol{F}^{-T}$$

# Lecture 5 – Conservation and balance laws

Any questions?




Recall: Lagrangian descriptions are very convenient in solving the non-linear PDEs numerically.