Spring, 2022

ME 597 – Solid Mechanics II

Lecture 7
Constitutive relations

PURDUE UNIVERSITY | Mechanical Engineering

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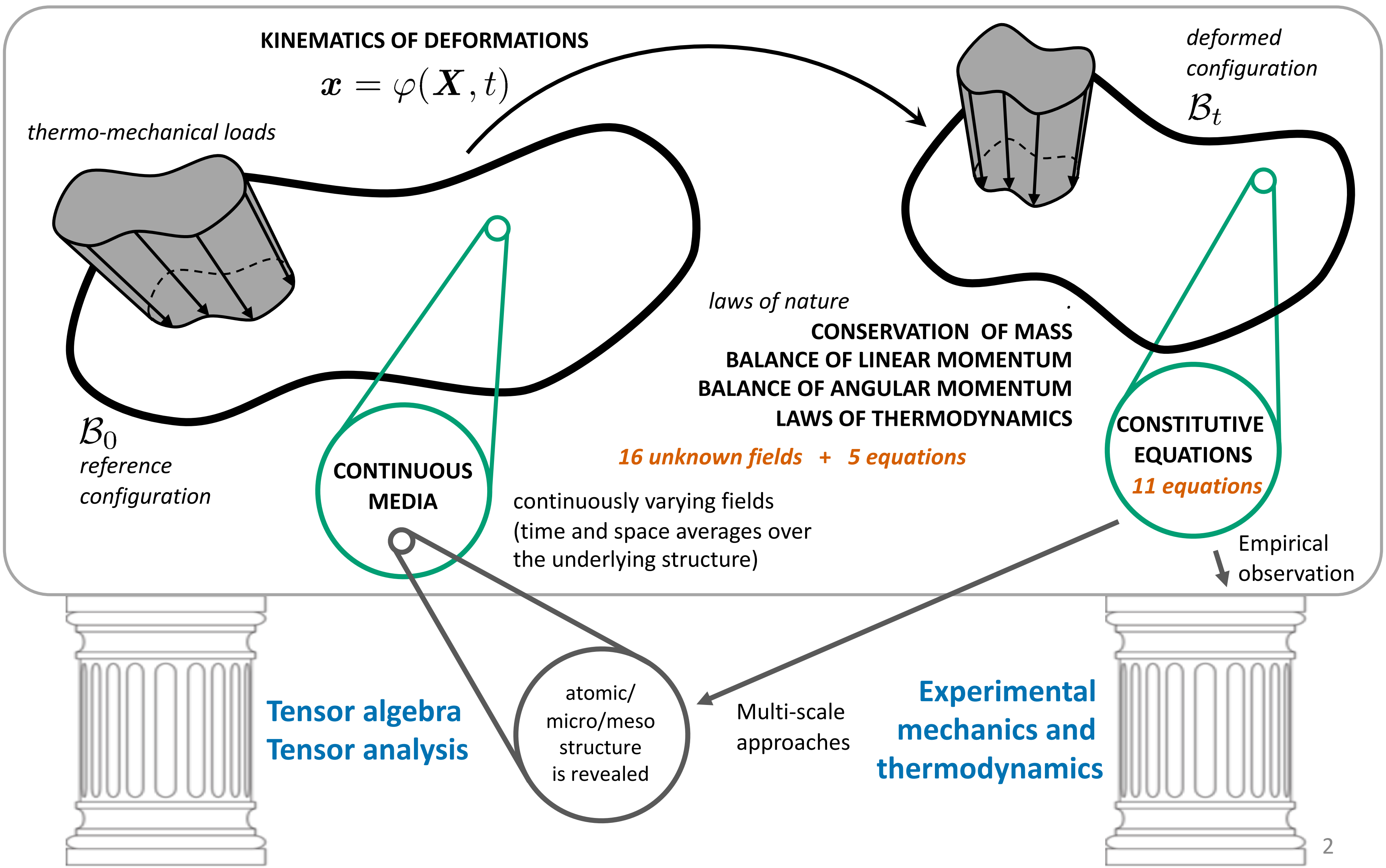
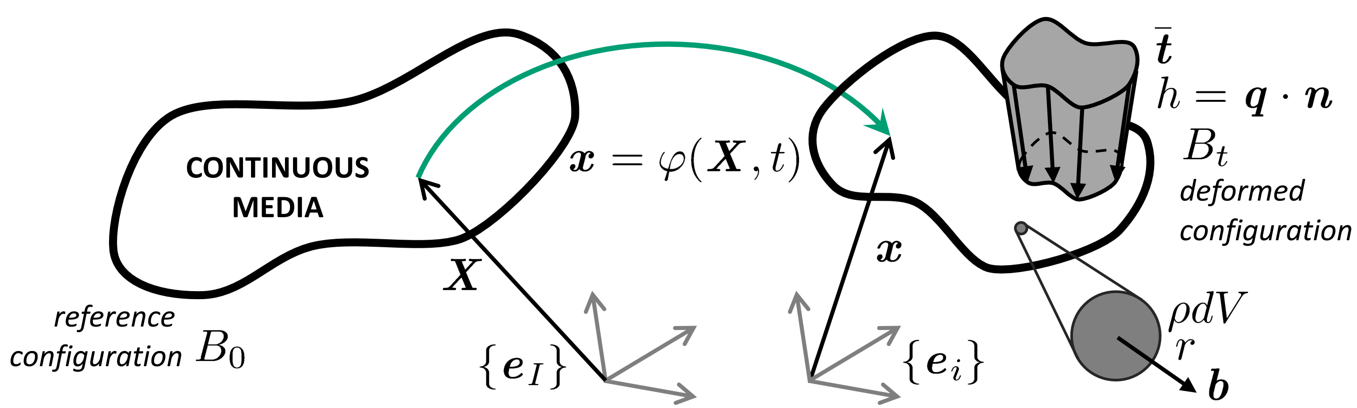
Lecture 7 – Constitutive relations



Continuum mechanics – Laws of nature

Summary

$J\rho = \rho_0$	$\forall \boldsymbol{X} \in B_0$	**conservation of mass**	(1 equation)
$\mathrm{div}\boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho \boldsymbol{a}$	$\forall \boldsymbol{x} \in B$	**balance of linear momentum**	(3 equations)
$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$	$\forall \boldsymbol{x} \in B$	**balance of angular momentum**	(constraint)
$\rho \dot{u} = \boldsymbol{\sigma} : \boldsymbol{d} + \rho r - \mathrm{div}\boldsymbol{q}$	$\forall \boldsymbol{x} \in B$	**conservation of energy**	(1 equation)
$\dot{s} \geq \frac{r}{T} - \frac{1}{\rho}\mathrm{div}\frac{\boldsymbol{q}}{T}$	$\forall \boldsymbol{x} \in B$	**Clausius-Duhem inequality**	(constraint)
		$\rho,\ \boldsymbol{x},\ \boldsymbol{\sigma},\ \boldsymbol{q},\ u,\ s,\ T$	(16 unknowns)



Constitutive relations

Constitutive relations

- Relations that describe the response of the material to mechanical and thermal loading, e.g.,

Simple elastic material

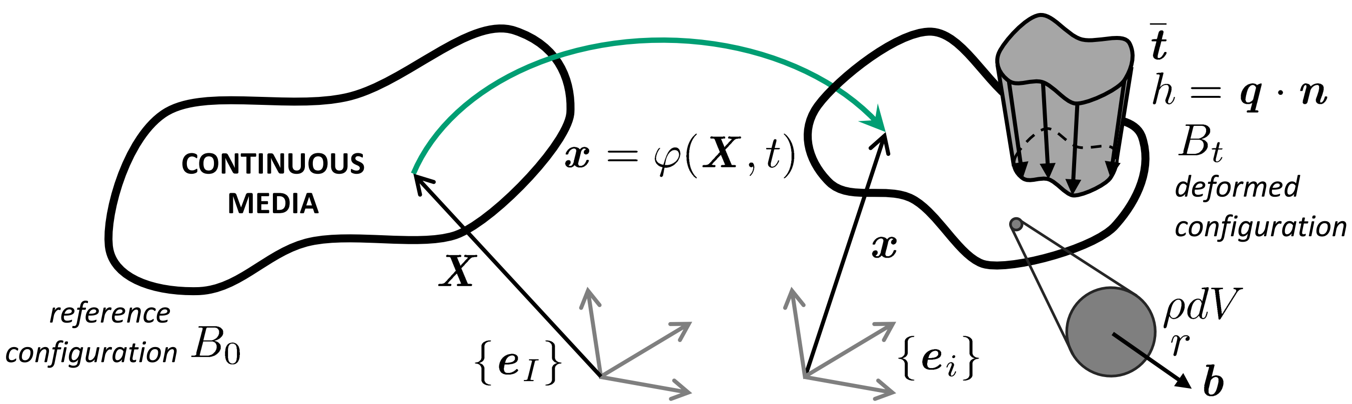
… based on the internal energy $u = \bar{u}(\boldsymbol{F}, s)$

$$\boldsymbol{\sigma},\ \boldsymbol{q},\ u,\ T$$ (11 constitutive equations)

Simple elastic material

… based on the Helmholtz free energy $W = \overline{W}(\boldsymbol{F}, T)$

$$\boldsymbol{\sigma},\ \boldsymbol{q},\ W,\ s$$ (11 constitutive equations)



Constitutive relations

- Relations that describe the response of the material to mechanical and thermal loading, e.g., $\boldsymbol{\sigma},\ \boldsymbol{q},\ W,\ s$ (11 constitutive equations)

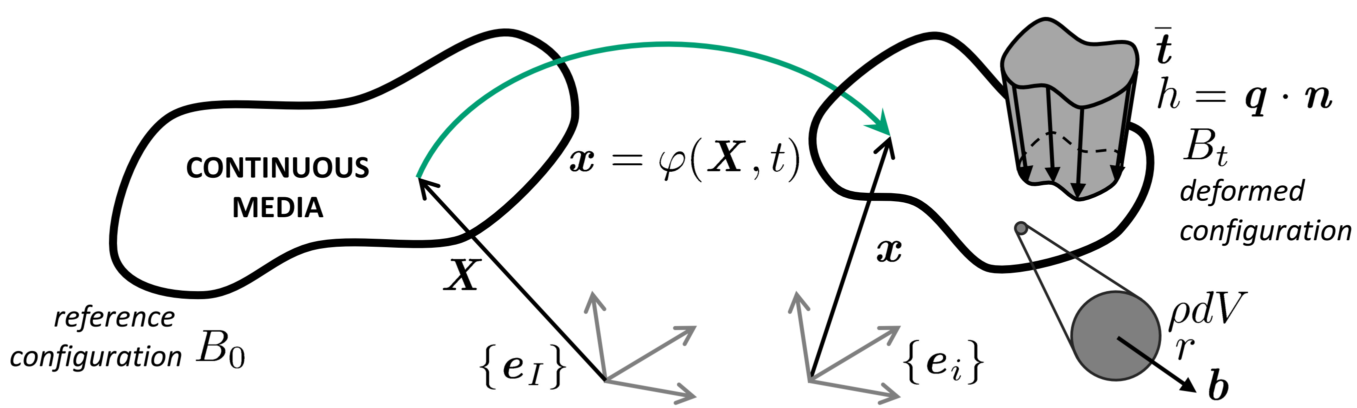
Simple elastic material

… based on the Helmholtz free energy $W = \overline{W}\left(\boldsymbol{F}, T\right)$

entropy $s = \overline{s}\left(\boldsymbol{F}, T, \ldots\right)$

heat flux $\boldsymbol{q} = \overline{\boldsymbol{q}}\left(\boldsymbol{F}, T, \ldots\right)$

Cauchy's stress tensor $\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}}\left(\boldsymbol{F}, T, \ldots\right)$



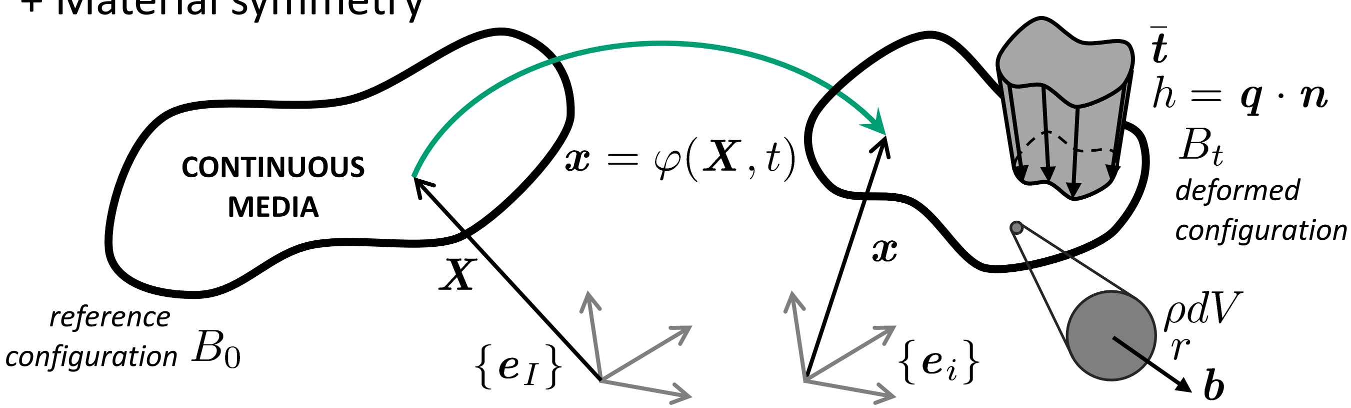
Constitutive relations

Constitutive relations

- Relations that describe the response of the material to mechanical and thermal loading, e.g., $\boldsymbol{\sigma},\ \boldsymbol{q},\ W,\ s$ (11 constitutive equations)
- Can these constitutive relations be selected arbitrarily? **NO!**

  They must follow fundamental principles:

  + Principle of determinism
  + Principle of local action
  + Second law of thermodynamics restrictions (Clausius-Duhem inequality)
  + Principle of material frame indifference (objectivity)
  + Material symmetry



Constitutive relations

Constraints on constitutive relations

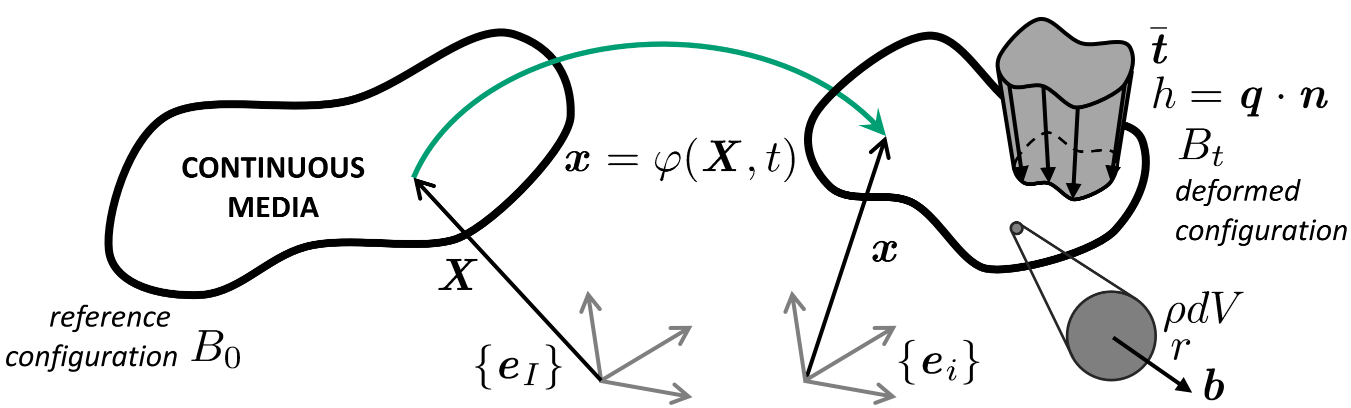
- Principle of determinism
  (causal determinism, the concept of cause and effect, …)

  "The current value of any physical variable can be determined from the knowledge of the present and the past values of other variables"

$$\boldsymbol{\sigma}(\boldsymbol{X}, t) = \boldsymbol{f}(\varphi^t, T^t, ..., \boldsymbol{X}, t)$$

time series $\square^t$ materials with memory

$\boldsymbol{f}(..., t)$ materials with aging



Constitutive relations

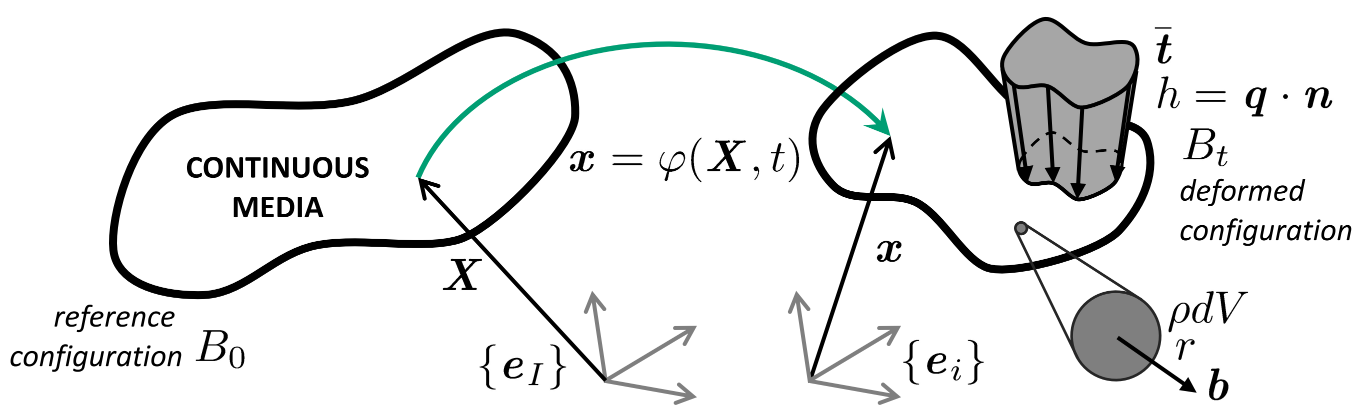
Constraints on constitutive relations

- Principle of local action

"The material response at a point depends only on the conditions within an arbitrarily small region about that point"

$$\boldsymbol{\sigma}(\boldsymbol{X},t) = \boldsymbol{f}(\varphi^t, \boldsymbol{F}^t, ..., T^t, \nabla_0 T^t, ..., \boldsymbol{X}, t)$$

In general $\boldsymbol{F}(\boldsymbol{X},t)$, therefore $\boldsymbol{f}(..., \boldsymbol{F}, \dot{\boldsymbol{F}}, ...)$.



Constitutive relations

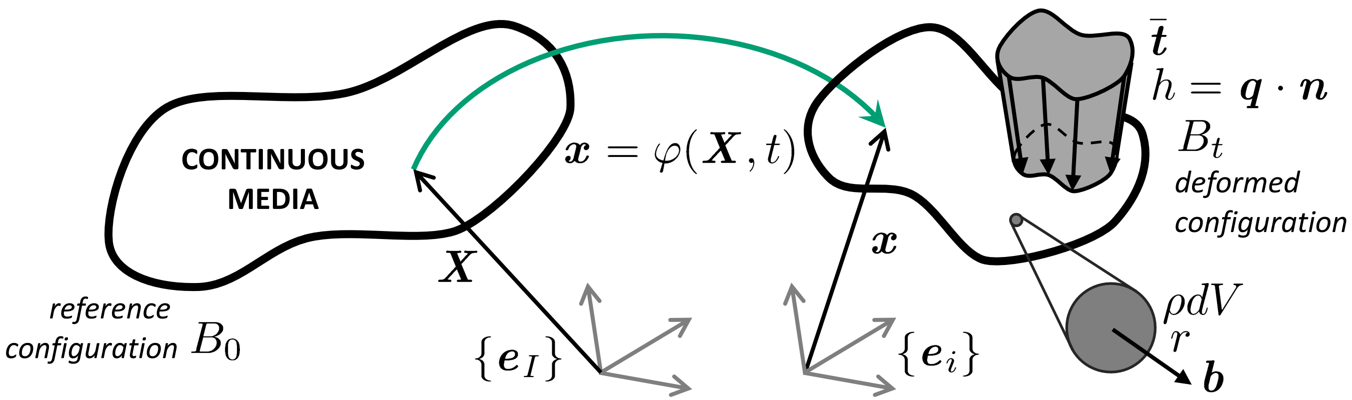
Constraints on constitutive relations

- Second law restrictions

  "A constitutive equation cannot violate the second law of thermodynamics"

$$\dot{s} \geq \frac{r}{T} - \frac{1}{\rho}\mathrm{div}\frac{\boldsymbol{q}}{T}$$

The application of the Clausius-Duhem inequality to constitutive equations is known as the Coleman-Noll procedure (1963)



Constitutive relations

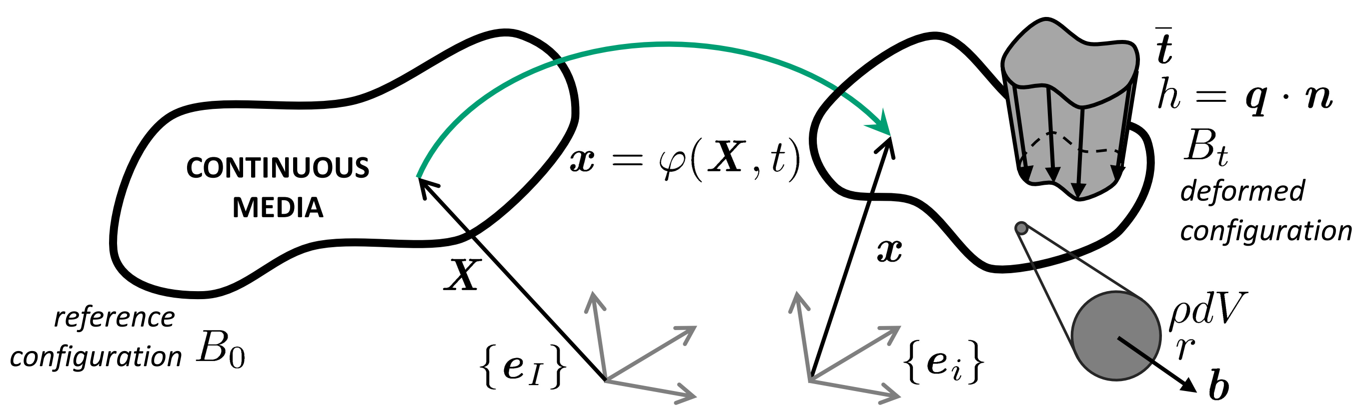
Coleman-Noll procedure

Using conservation of energy, the Clausius-Duhem inequality can be written as

$$\rho T \dot{s}^{\text{int}} = -\rho \left[\frac{1}{\rho_0} \frac{\partial W}{\partial T} + s \right] \dot{T} + \left[\boldsymbol{\sigma} \boldsymbol{F}^{-T} - \frac{\rho}{\rho_0} \frac{\partial W}{\partial \boldsymbol{F}} \right] : \dot{\boldsymbol{F}} - \frac{1}{T} \boldsymbol{q} \nabla T \geq 0$$

Coleman and Noll made the argument that this inequality must be satisfied for every admissible process.

The application of the Clausius-Duhem inequality to constitutive equations is known as the Coleman-Noll procedure (1963)



Constitutive relations

Coleman-Noll procedure

- Entropy constitutive relation $s = \overline{s}\left(\boldsymbol{F}, T\right) \equiv -\frac{1}{\rho_0}\frac{\partial W}{\partial T}$

DIY

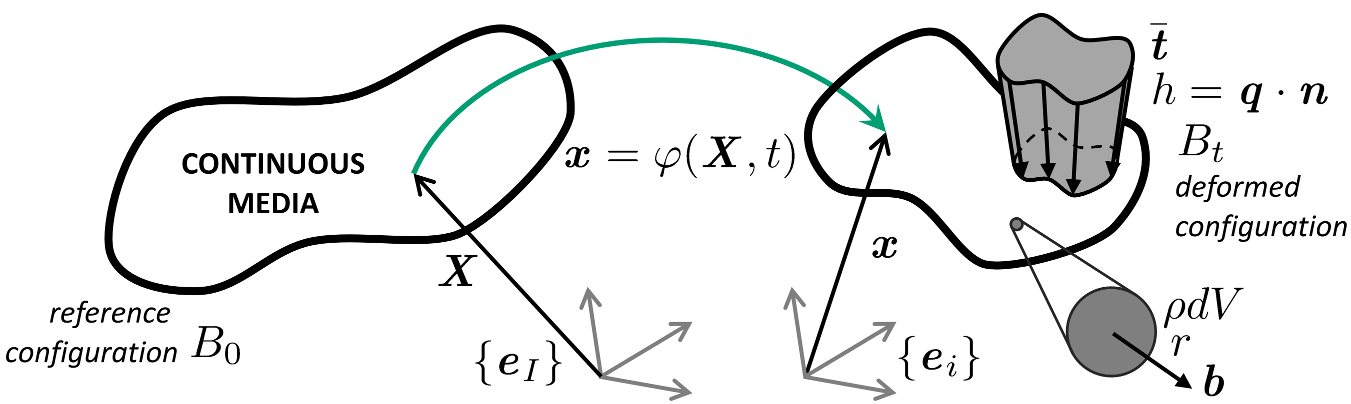
$$\rho T \dot{s}^{\text{int}} = -\rho\left[\frac{1}{\rho_0}\frac{\partial W}{\partial T} + s\right]\dot{T} + \left[\boldsymbol{\sigma}\boldsymbol{F}^{-T} - \frac{\rho}{\rho_0}\frac{\partial W}{\partial \boldsymbol{F}}\right] : \dot{\boldsymbol{F}} - \frac{1}{T}\boldsymbol{q}\nabla T \geq 0$$

$$W = \overline{W}\left(\boldsymbol{F}, T\right)$$

The inequality has to be verified by any arbitrary process. (i.e., state variables can be chosen arbitrarily)

In particular, a process which has a deformation constant in time and a uniform temperature. Thus,

$$-\rho\left[\frac{1}{\rho_0}\frac{\partial W}{\partial T} + s\right]\dot{T} \geq 0 \quad \forall \dot{T}$$



Constitutive relations

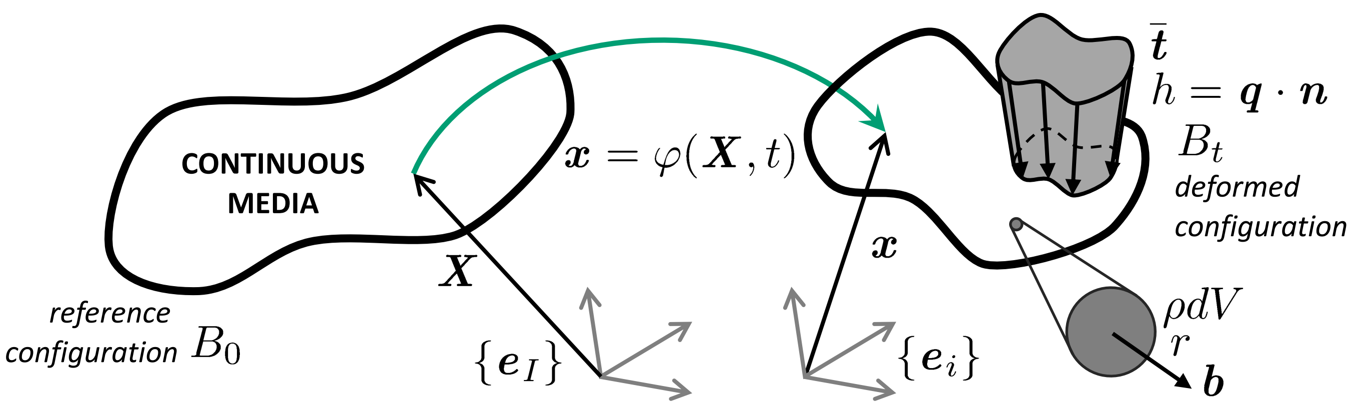
Coleman-Noll procedure

- Heat flux constitutive relation $$\boldsymbol{q} = \overline{\boldsymbol{q}}\left(\boldsymbol{F}, T, \nabla T\right)$$

DIY

$$\left[\boldsymbol{\sigma}\boldsymbol{F}^{-T} - \frac{\rho}{\rho_0}\frac{\partial W}{\partial \boldsymbol{F}}\right] : \dot{\boldsymbol{F}} - \frac{1}{T}\boldsymbol{q}\nabla T \geq 0$$

The inequality has to be verified by any arbitrary process.
In particular, one which has a deformation constant in time.
Then, $$-\frac{1}{T}\boldsymbol{q}\cdot\nabla T \geq 0$$



Constitutive relations

Coleman-Noll procedure

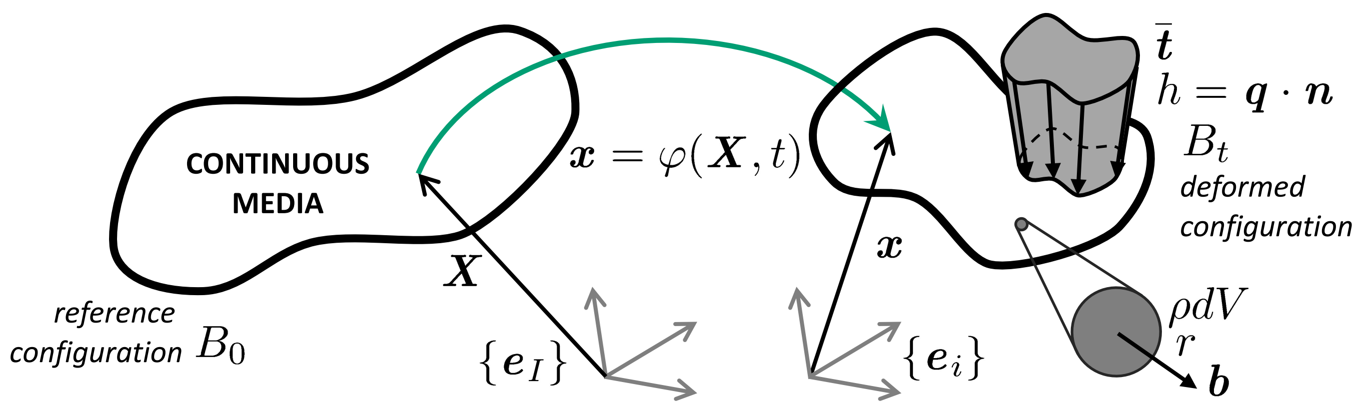
- Cauchy stress constitutive relation

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(e)} + \boldsymbol{\sigma}^{(v)}$$

DIY

$$\left[\boldsymbol{\sigma}\boldsymbol{F}^{-T} - \frac{\rho}{\rho_0}\frac{\partial W}{\partial \boldsymbol{F}}\right] : \dot{\boldsymbol{F}} - \frac{1}{T}\boldsymbol{q}\nabla T \geq 0$$

Since the stress tensor <u>is not a state variable</u>, we first partition it into an elastic reversible part (which is a state variable) and an irreversible part (which is not associated with an equilibrium state and, therefore, it is not a state variable). Here, the partition is an additive decomposition (but it doesn't have to be the case in general).
The irreversible process has to produce entropy, that is $\boldsymbol{\sigma}^{(v)} : \boldsymbol{d} \geq 0$



Constitutive relations

Coleman-Noll procedure

- Cauchy stress constitutive relation

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(e)} + \boldsymbol{\sigma}^{(v)}$$

$$\boldsymbol{\sigma}^{(e)} \equiv \frac{1}{J}\frac{\partial W}{\partial \boldsymbol{F}}\boldsymbol{F}^T$$

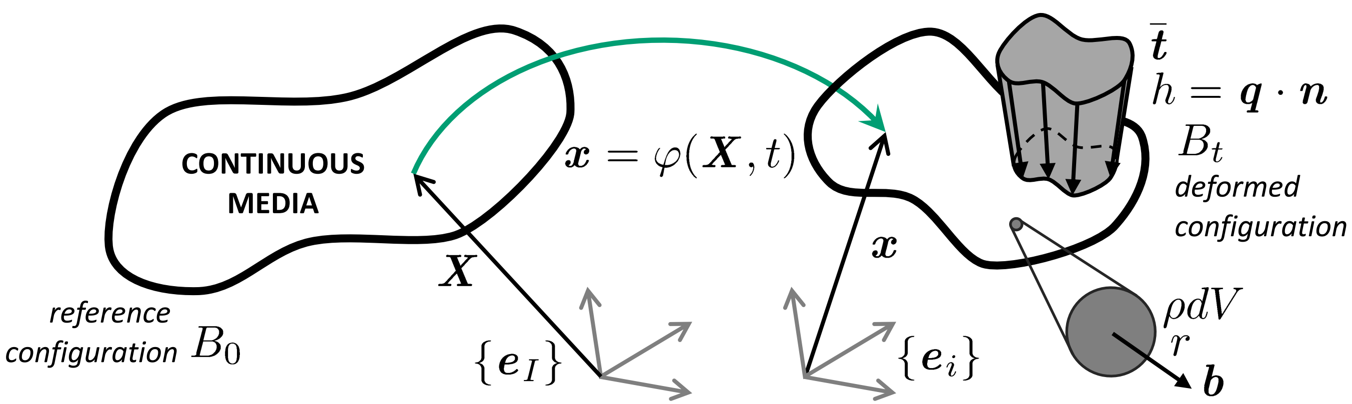
$$\bar{\boldsymbol{\sigma}}^{(v)}\left(\boldsymbol{F}, T, \boldsymbol{d}\right)$$

DIY

$$\left[\boldsymbol{\sigma}\boldsymbol{F}^{-T} - \frac{\rho}{\rho_0}\frac{\partial W}{\partial \boldsymbol{F}}\right] : \dot{\boldsymbol{F}} - \frac{1}{T}\boldsymbol{q}\nabla T \geq 0$$

Since the stress tensor is not a state variable, we first partition it into an elastic reversible part (which is a state variable) and an irreversible part (which is not associated with an equilibrium state and, therefore, it is not a state variable).

Finally, $\boldsymbol{\sigma}^{(v)} : \boldsymbol{d} \geq 0$ $\quad \left[\boldsymbol{\sigma}^{(e)}\boldsymbol{F}^{-T} - \frac{\rho}{\rho_0}\frac{\partial W}{\partial \boldsymbol{F}}\right] : \dot{\boldsymbol{F}} \geq 0 \quad \forall \dot{\boldsymbol{F}}$



Constitutive relations

Coleman-Noll procedure

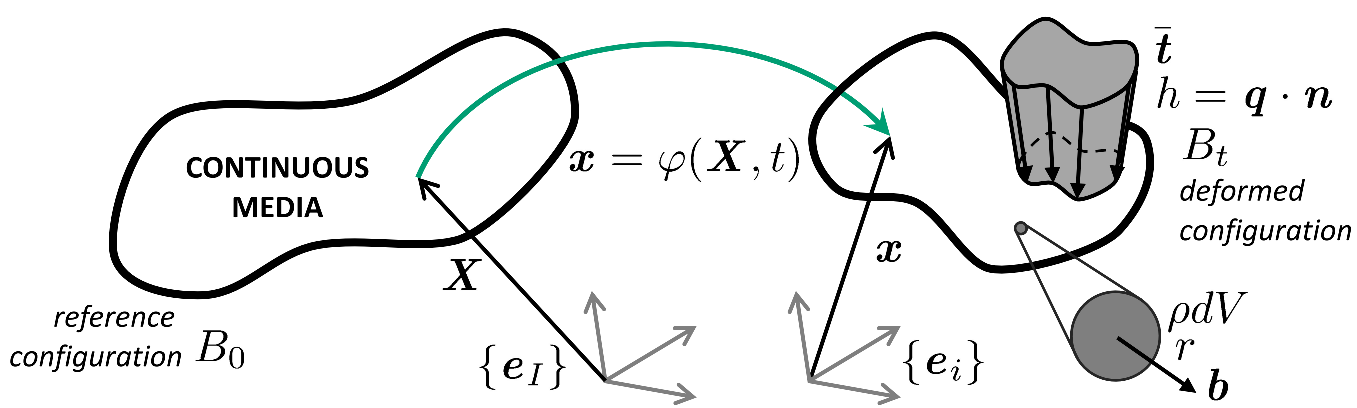
- Energy change in reversible and irreversible processes

<u>For any process</u> $$\dot{s} = \frac{r}{T} - \frac{1}{\rho T}\mathrm{div}\boldsymbol{q} + \frac{1}{\rho T}\boldsymbol{\sigma}^{(v)} : \boldsymbol{d}$$

$$\rho T \dot{s}^{\mathrm{int}} = \boldsymbol{\sigma}^{(v)} : \boldsymbol{d} - \frac{1}{T}\boldsymbol{q} \cdot \nabla T \geq 0 \qquad W = \overline{W}\left(\boldsymbol{F}, T\right)$$

$$s = \overline{s}\left(\boldsymbol{F}, T\right) \equiv -\frac{1}{\rho_0}\frac{\partial W}{\partial T} \qquad \boldsymbol{q} = \overline{\boldsymbol{q}}\left(\boldsymbol{F}, T, \nabla T\right) \qquad -\frac{1}{T}\boldsymbol{q} \cdot \nabla T \geq 0$$

$$\boldsymbol{\sigma}^{(e)} \equiv \frac{1}{J}\frac{\partial W}{\partial \boldsymbol{F}}\boldsymbol{F}^T \qquad \bar{\boldsymbol{\sigma}}^{(v)}\left(\boldsymbol{F}, T, \boldsymbol{d}\right) \qquad \boldsymbol{\sigma}^{(v)} : \boldsymbol{d} \geq 0$$



Constitutive relations

Coleman-Noll procedure – Isothermal processes

- *Isothermal* processes, where the motion and deformation occur at such a low temporal rate that the temperature is uniform and constant.

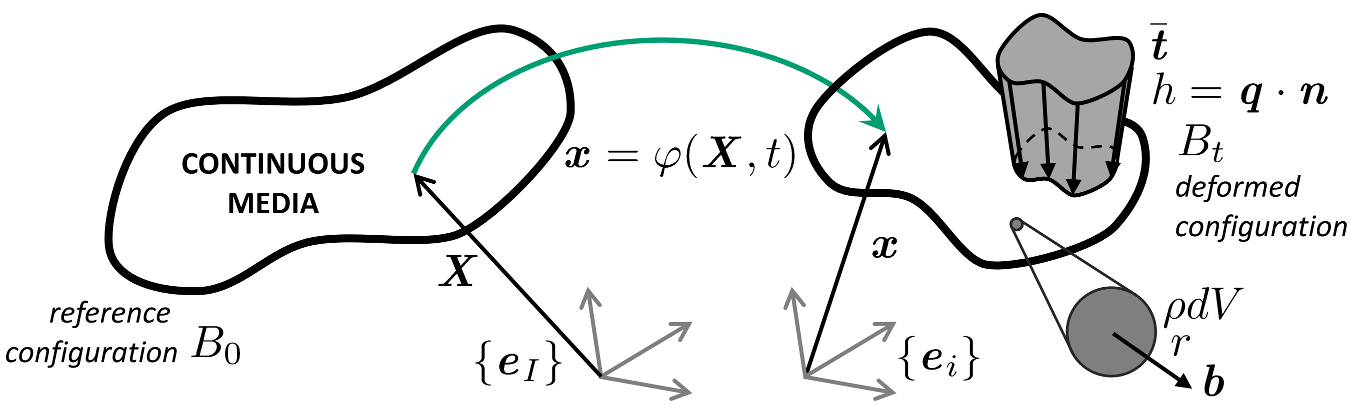
… based on the Helmholtz free energy

$$\dot{T} = 0 \qquad \nabla T = 0 \qquad \boldsymbol{q} = -k\nabla T = 0 \qquad W = \overline{W}\left(\boldsymbol{F}, T\right)$$

$$s = \overline{s}\left(\boldsymbol{F}, T\right) \equiv -\frac{1}{\rho_0}\frac{\partial W}{\partial T} \qquad \dot{s} \neq 0$$

$$\boldsymbol{\sigma}^{(e)} \equiv \frac{1}{J}\frac{\partial W}{\partial \boldsymbol{F}}\boldsymbol{F}^T \qquad \bar{\boldsymbol{\sigma}}^{(v)}\left(\boldsymbol{F}, T, \boldsymbol{d}\right)$$

If $\boldsymbol{\sigma}^{(v)} = \boldsymbol{0}$ then the process is reversible, and the material is called **hyperelastic**



Constitutive relations

Coleman-Noll procedure – Isentropic processes

- *Isentropic* processes, where the motion and deformation occur at a high temporal rate that no flow of heat occurs.

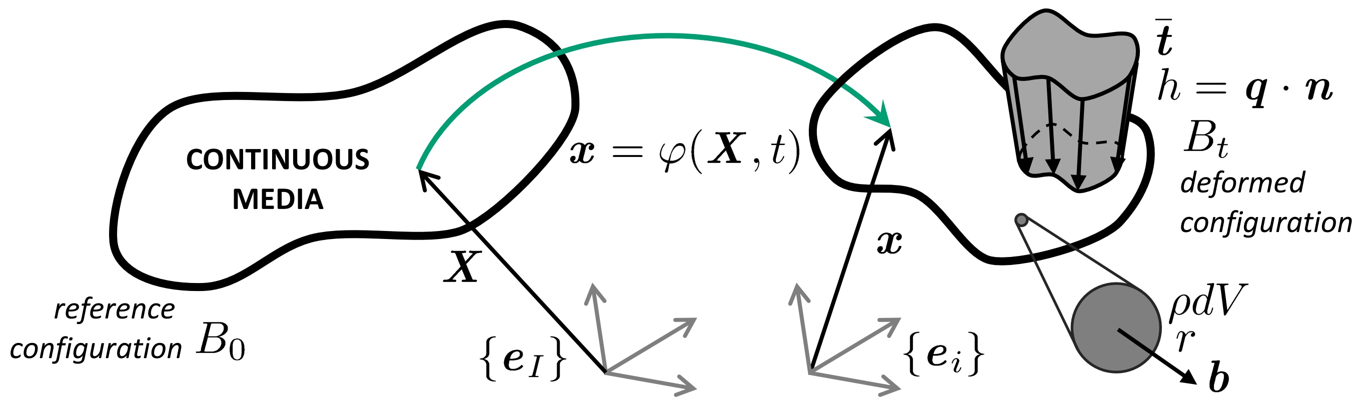
… based on the internal energy

$$\dot{s} = 0 \qquad \boldsymbol{q} = \boldsymbol{0} \qquad u = \frac{W}{\rho_0} + Ts = \overline{u}\left(\boldsymbol{F}, s\right)$$

$$T = \overline{T}\left(\boldsymbol{F}, s\right) \equiv \frac{\partial \overline{u}}{\partial s} \qquad \dot{T} \neq 0$$

$$\boldsymbol{\sigma}^{(e)} \equiv \frac{1}{J}\frac{\partial \rho_0 \overline{u}}{\partial \boldsymbol{F}}\boldsymbol{F}^T \qquad \bar{\boldsymbol{\sigma}}^{(v)}\left(\boldsymbol{F}, s, \boldsymbol{d}\right)$$

If $\boldsymbol{\sigma}^{(v)} = \boldsymbol{0}$ then the process is reversible, and the material is called **hyperelastic**



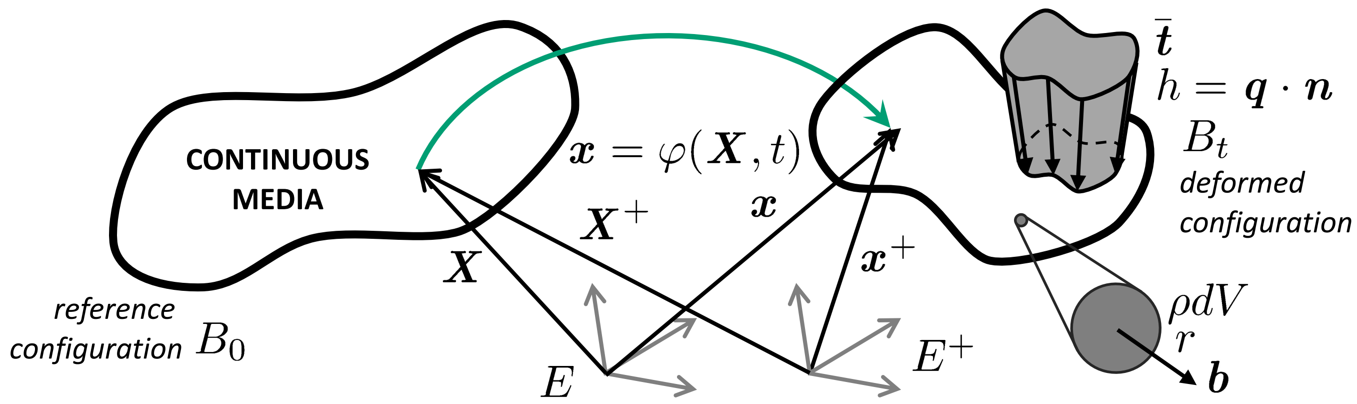
Constitutive relations

Constraints on constitutive relations

- Principle of material frame indifference (objectivity)

“All physical variables for which constitute relations are required must be objective tensors”

Definition: an objective tensor is a tensor which is physically the same in all frames of reference.
Recall: a frame of reference is an Euclidean point space, which represents points, and a clock, which represent time (relativistic phenomena is not considered).



Constitutive relations

Constraints on constitutive relations

- Principle of material frame indifference (objectivity)

Is the Cauchy stress tensor an objective tensor? **Yes.-**

$$\boldsymbol{t} = \boldsymbol{\sigma}\boldsymbol{n} \Longleftrightarrow \boldsymbol{t}^{+} = \boldsymbol{\sigma}^{+}\boldsymbol{n}^{+}$$

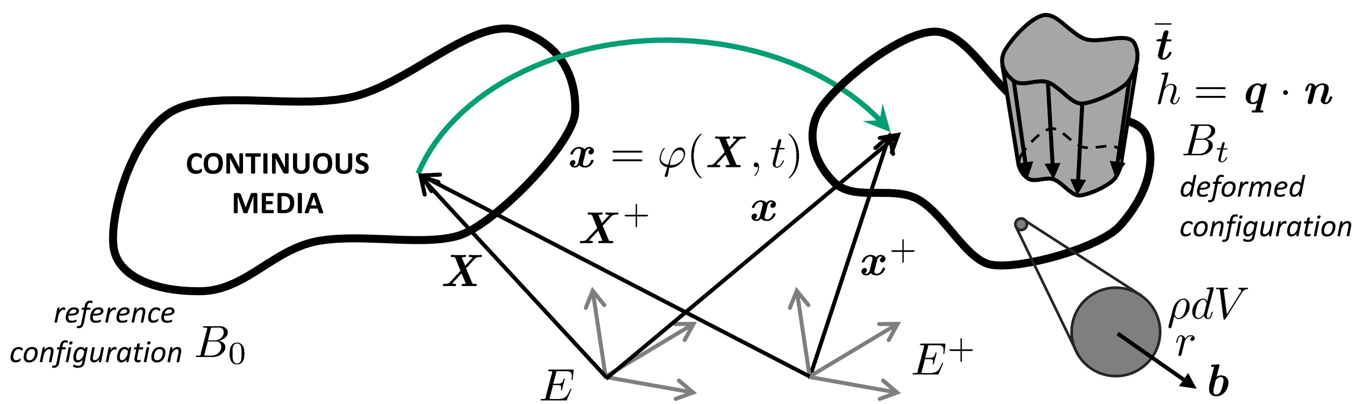
Constraint on constitutive relations

$$W = \overline{W}\left(\boldsymbol{F}, T\right) = \widehat{W}\left(\boldsymbol{U}, T\right) = \widetilde{W}\left(\boldsymbol{C}, T\right)$$

$$\boldsymbol{\sigma}^{(e)} = \frac{2}{J}\boldsymbol{F}\frac{\partial \widetilde{W}(\boldsymbol{C},T)}{\partial \boldsymbol{C}}\boldsymbol{F}^{T} \qquad \boldsymbol{S}^{(e)} = 2\frac{\partial \widetilde{W}(\boldsymbol{C},T)}{\partial \boldsymbol{C}}$$

Recall: $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$

A function of the right Cauchy-Green tensor $\boldsymbol{C} = \boldsymbol{U}^{2}$

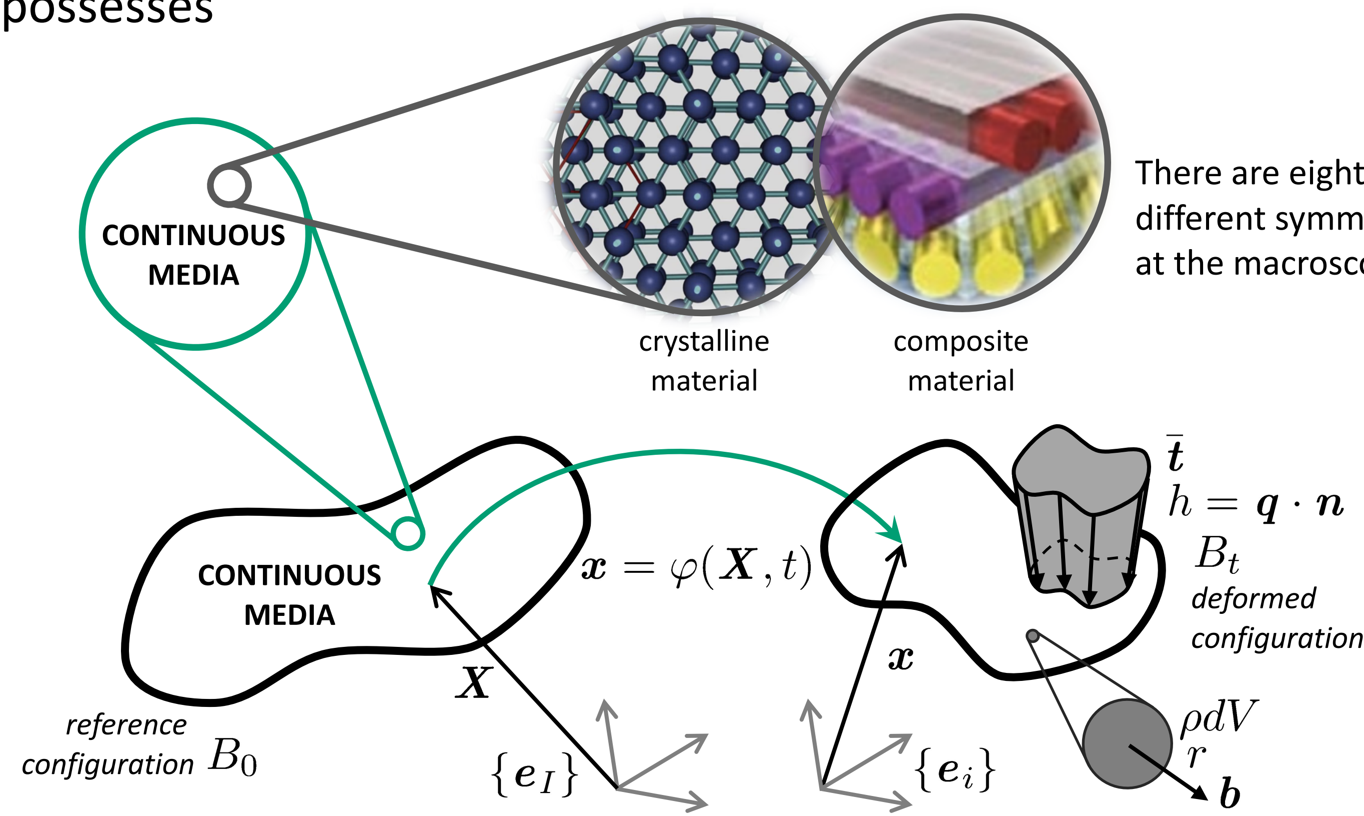


Constitutive relations

Constraints on constitutive relations

- Material symmetry

  "A constitutive relation must respect any symmetries that the material possesses"



There are eight possible different symmetries at the macroscopic level

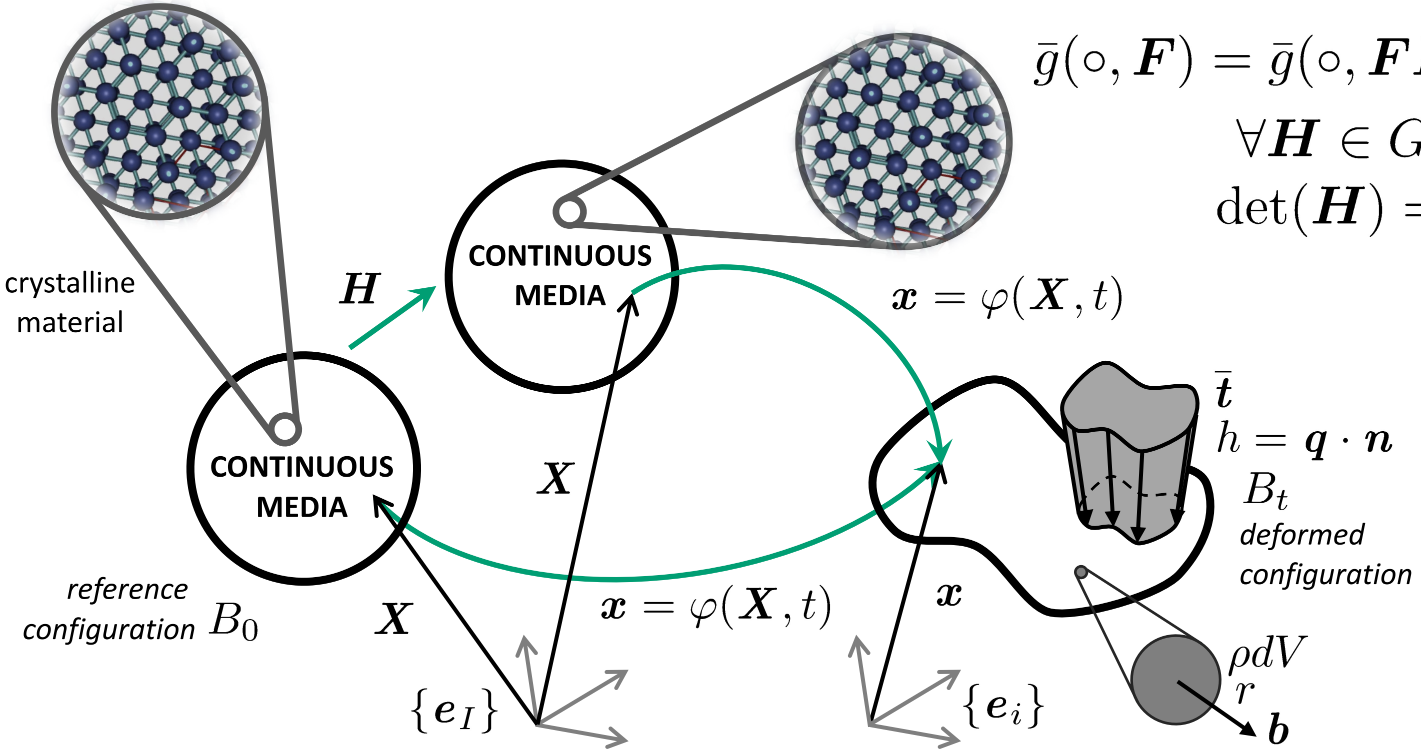
Constitutive relations

Constraints on constitutive relations

- Material symmetry

"A constitutive relation must respect any symmetries that the material possesses"

$$\bar{g}(\circ, \boldsymbol{F}) = \bar{g}(\circ, \boldsymbol{F}\boldsymbol{H})$$
$$\forall \boldsymbol{H} \in G_{\mathrm{ms}}$$
$$\det(\boldsymbol{H}) = 1$$



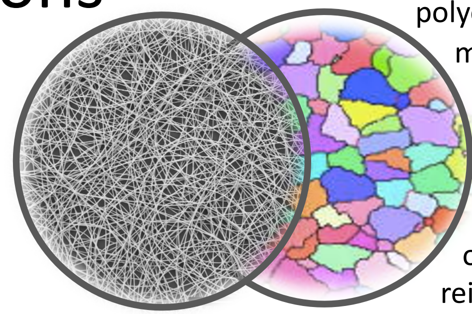
Constitutive relations

Constraints on constitutive relations

- Material symmetry – Examples

+ Isotropic solid

$$\boldsymbol{H} \in SO(3) \quad \text{equiv.} \quad \boldsymbol{H} = \boldsymbol{R}^T$$



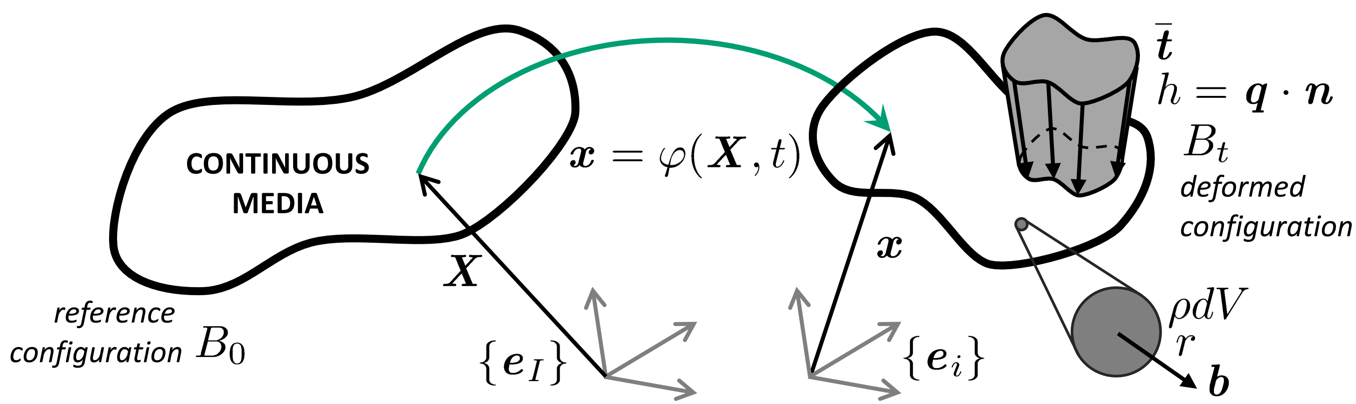
$$W = \overline{W}(\boldsymbol{F}, T) = \widehat{\widehat{W}}(\boldsymbol{V}, T) = \widetilde{\widetilde{W}}(\boldsymbol{B}, T)$$

Recall: $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$

A function of the left Cauchy-Green tensor

$$\boldsymbol{B} = \boldsymbol{V}^2$$

+ No symmetry at all $\quad \boldsymbol{H} = \boldsymbol{I}$



Constitutive relations

Isotropic hyperelastic solids (*no heat flux and viscous stress*)

- Frame indifference

$$W = \overline{W}(\boldsymbol{F}, T) = \widehat{W}(\boldsymbol{U}, T) = \widetilde{W}(\boldsymbol{C}, T)$$

- Material symmetry: isotropic

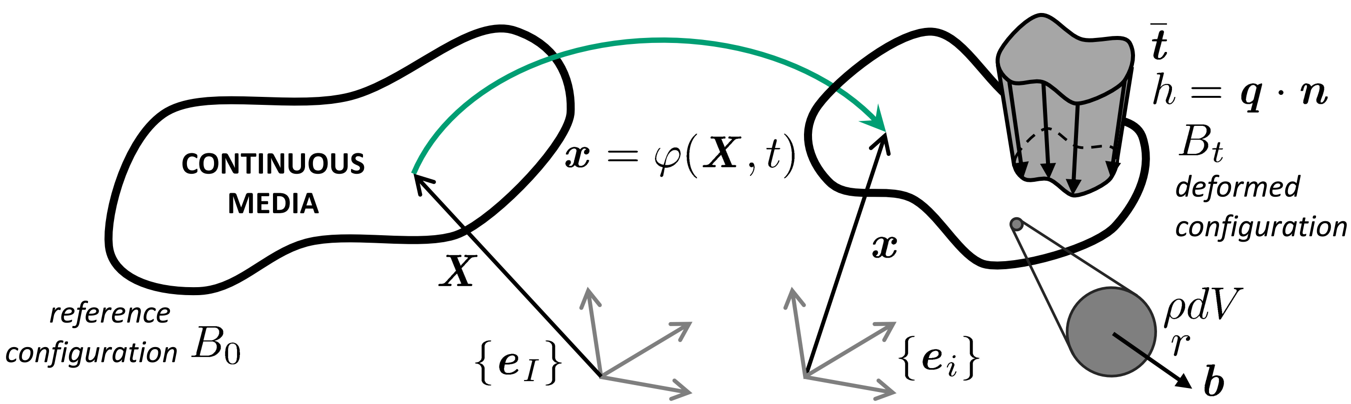
$$W = \overline{W}(\boldsymbol{F}, T) = \widehat{\widehat{W}}(\boldsymbol{V}, T) = \widetilde{\widetilde{W}}(\boldsymbol{B}, T)$$

A function of the right Cauchy-Green tensor **and** left Cauchy-Green tensor

- Strain energy density function: $W = W(I_1, I_2, I_3)$

DIY

The right and left Cauchy-Green tensors have the same principal invariants



Lecture 7 – Constitutive relations

Any questions?