Spring, 2022

ME 597 – Solid Mechanics II

Lecture 8
Hyperelastic solids

PURDUE UNIVERSITY | Mechanical Engineering

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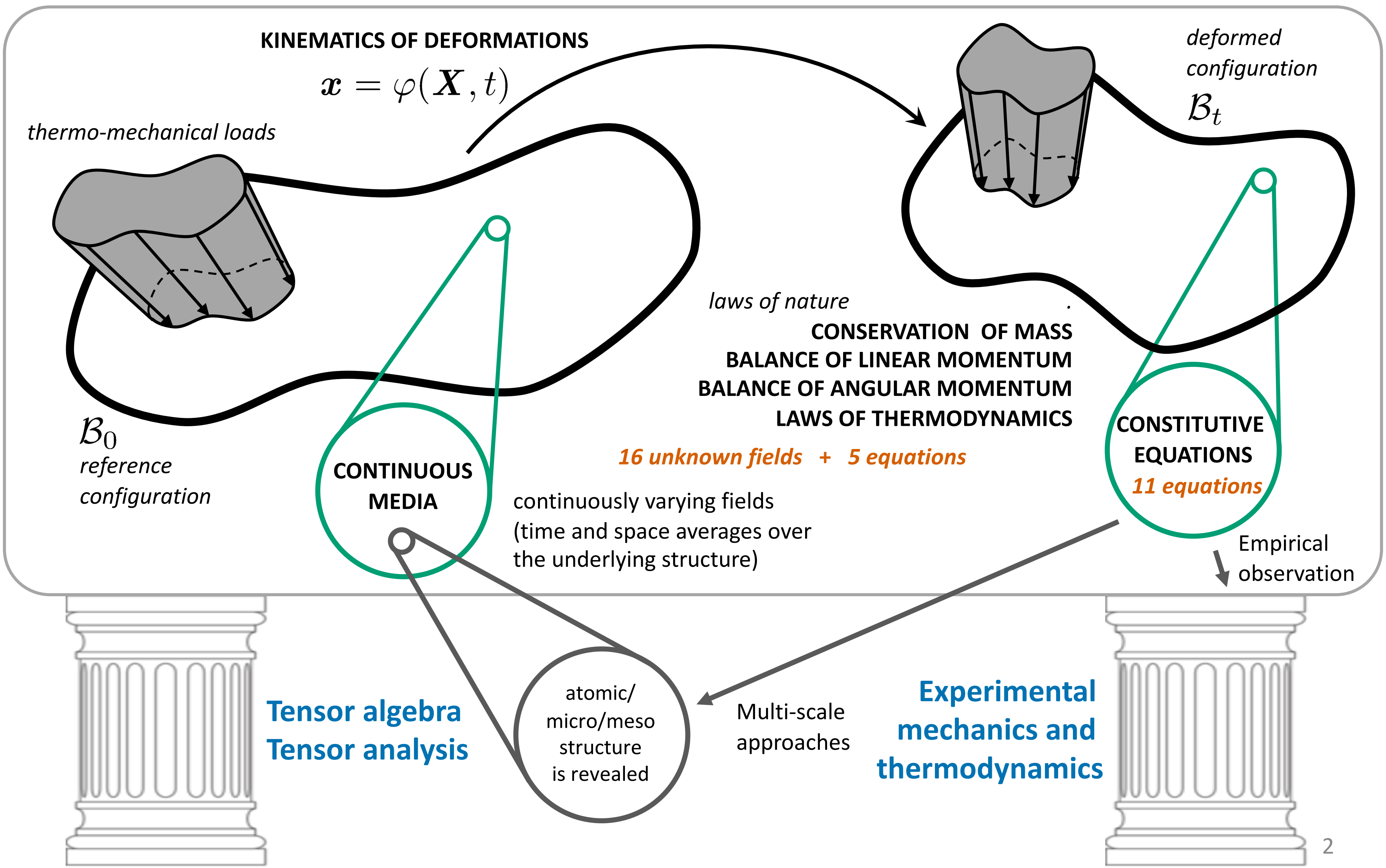
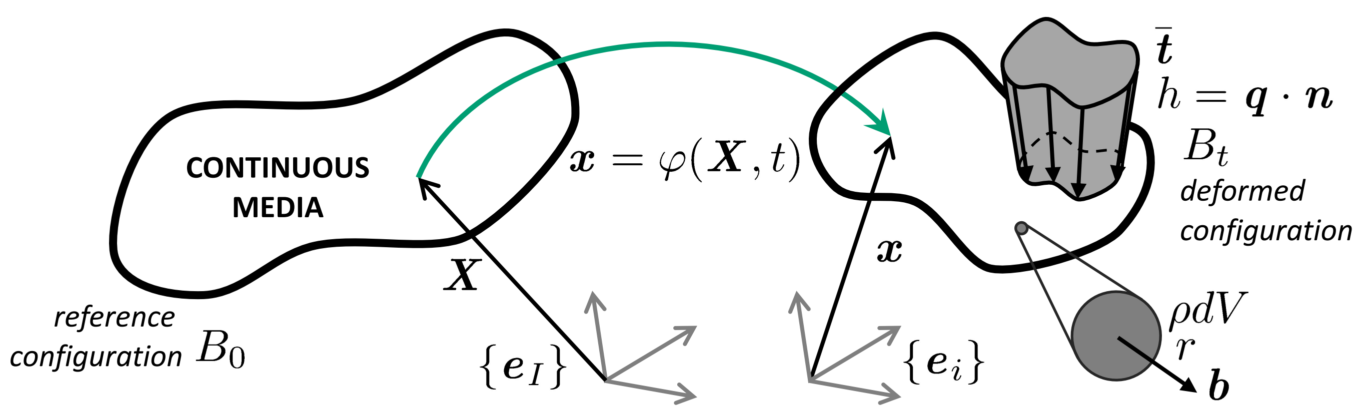
Lecture 8 – Constitutive relations



Laws of nature

Summary

$J\rho = \rho_0$	$\forall \boldsymbol{X} \in B_0$	**conservation of mass**	(1 equation)
$\mathrm{div}\boldsymbol{\sigma} + \rho\boldsymbol{b} = \rho\boldsymbol{a}$	$\forall \boldsymbol{x} \in B$	**balance of linear momentum**	(3 equations)
$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$	$\forall \boldsymbol{x} \in B$	**balance of angular momentum**	(constraint)
$\rho\dot{u} = \boldsymbol{\sigma} : \boldsymbol{d} + \rho r - \mathrm{div}\boldsymbol{q}$	$\forall \boldsymbol{x} \in B$	**conservation of energy**	(1 equation)
$\dot{s} \geq \frac{r}{T} - \frac{1}{\rho}\mathrm{div}\frac{\boldsymbol{q}}{T}$	$\forall \boldsymbol{x} \in B$	**Clausius-Duhem inequality**	(constraint)
		$\rho,\ \boldsymbol{x},\ \boldsymbol{\sigma},\ \boldsymbol{q},\ u,\ s,\ T$	(16 unknowns)



Constitutive relations

Constitutive relations

- Relations that describe the response of the material to mechanical and thermal loading, e.g., $\boldsymbol{\sigma},\ \boldsymbol{q},\ W,\ s$ (11 constitutive equations)
- Can these constitutive relations be selected arbitrarily? NO!

  They must follow fundamental principles:

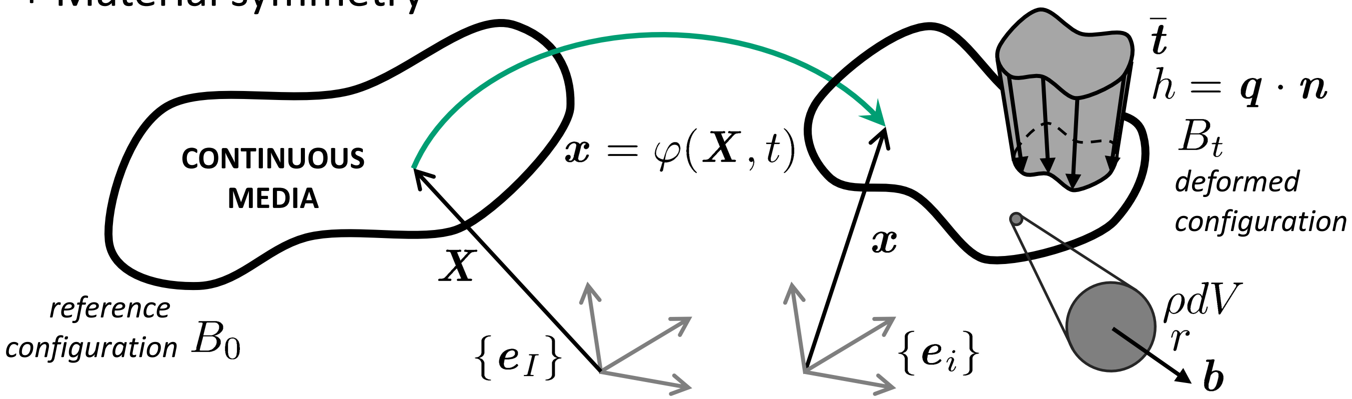
  \+ Principle of determinism

  \+ Principle of local action

  \+ Second law of thermodynamics restrictions (Clausius-Duhem inequality)

  \+ Principle of material frame indifference (objectivity)

  \+ Material symmetry



Hyperelastic solids - Isotropic

Coleman-Noll procedure + Frame indifference + Isotropy

$$W = W\left(I_1, I_2, I_3\right)$$

$$I_1 = \mathrm{tr}(\boldsymbol{C}) = \mathrm{tr}(\boldsymbol{B})$$

$$I_2 = \tfrac{1}{2}[\mathrm{tr}(\boldsymbol{C})^2 - \mathrm{tr}(\boldsymbol{C}^2)] = \tfrac{1}{2}[\mathrm{tr}(\boldsymbol{B})^2 - \mathrm{tr}(\boldsymbol{B}^2)]$$

$$I_3 = \det(\boldsymbol{C}) = \det(\boldsymbol{B}) = J^2$$

strain energy density function

(*a function of the principal invariants of the right/left Cauchy-Green tensor*)

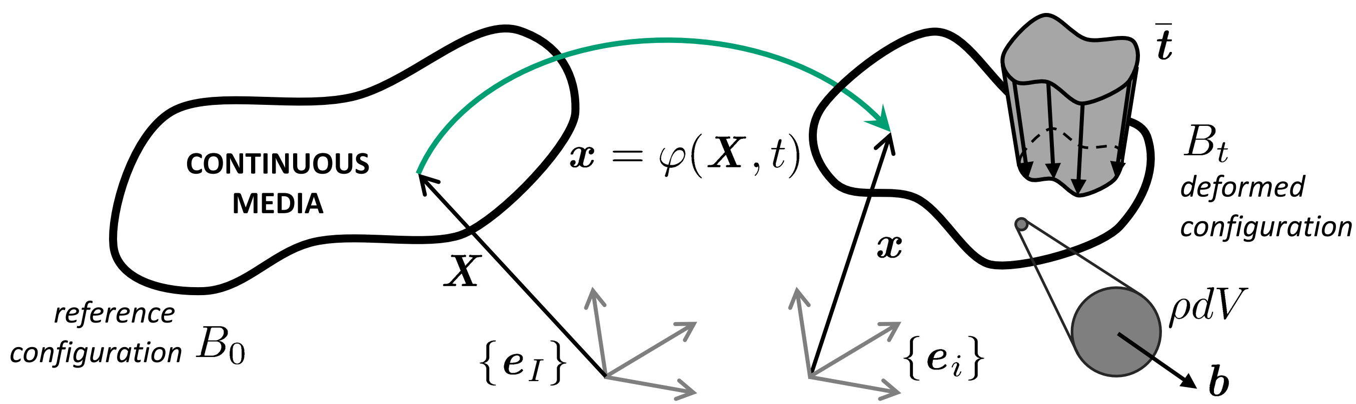
DIY

$$\boldsymbol{\sigma}^{(e)} = \frac{2}{J}\boldsymbol{F}\frac{\partial \widetilde{W}(\boldsymbol{C},T)}{\partial \boldsymbol{C}}\boldsymbol{F}^T$$

$$\boldsymbol{S}^{(e)} = 2\frac{\partial \widetilde{W}(\boldsymbol{C},T)}{\partial \boldsymbol{C}}$$

elastic part of the stress tensor

(*the viscous part is zero and thus, the process is reversible*)



Hyperelastic solids - Isotropic

Coleman-Noll procedure + Frame indifference + Isotropy

$$W = W(I_1, I_2, I_3) \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{(e)}$$

DIY

$$\boldsymbol{S} = 2\left[(W_{,I_1} + I_1 W_{,I_2})\boldsymbol{I} - W_{,I_2}\boldsymbol{C} + I_3 W_{,I_3}\boldsymbol{C}^{-1}\right]$$

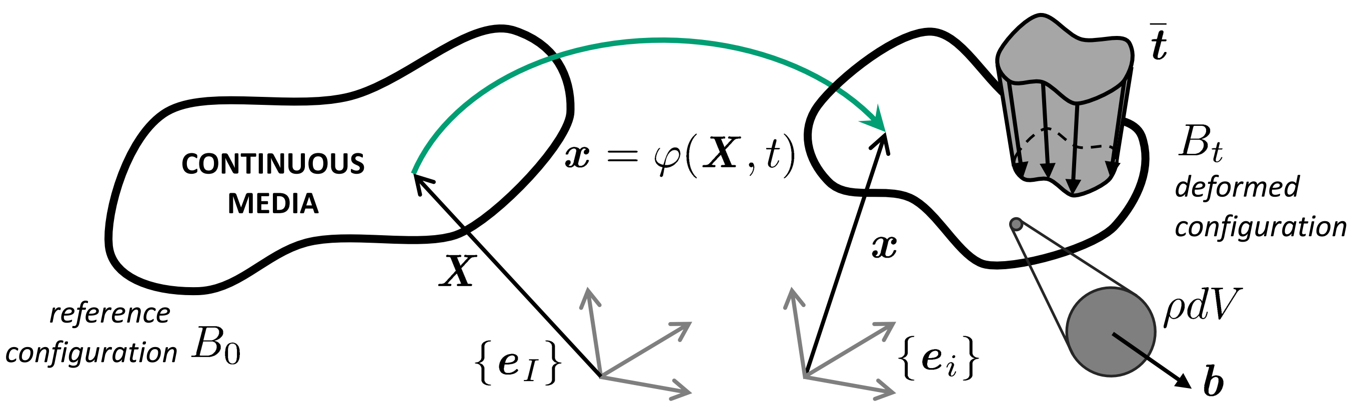
$$\boldsymbol{\sigma} = \frac{2}{I_3^{1/2}}\left[I_3 W_{,I_3}\boldsymbol{I} + (W_{,I_1} + I_1 W_{,I_2})\boldsymbol{B} - W_{,I_2}\boldsymbol{B}^2\right]$$

Hint: $\boldsymbol{S} = 2\sum_{i=1}^{3} \frac{\partial W(I_1, I_2, I_3)}{\partial I_i}\frac{\partial I_i}{\partial \boldsymbol{C}}$

$$\frac{\partial I_1}{\partial \boldsymbol{C}} = \boldsymbol{I}$$

$$\frac{\partial I_2}{\partial \boldsymbol{C}} = I_1\boldsymbol{I} - \boldsymbol{C}$$

$$\frac{\partial I_3}{\partial \boldsymbol{C}} = I_3\boldsymbol{C}^{-1}$$

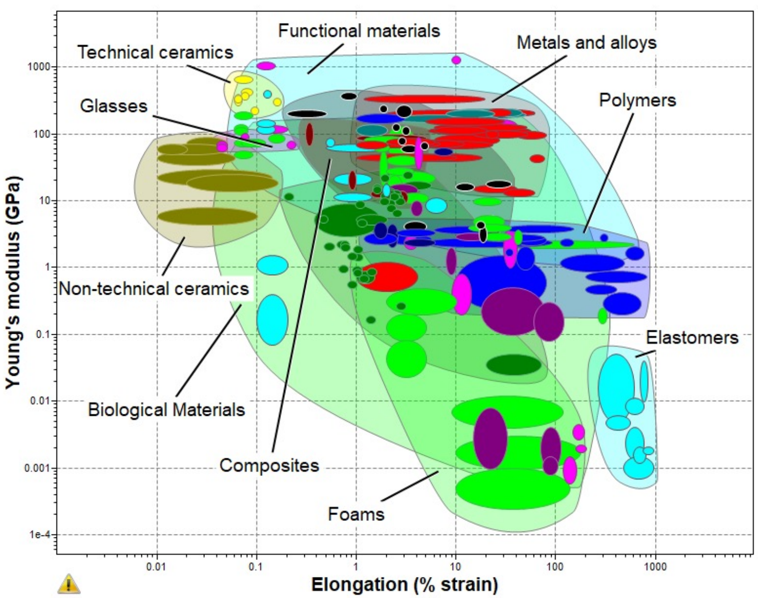


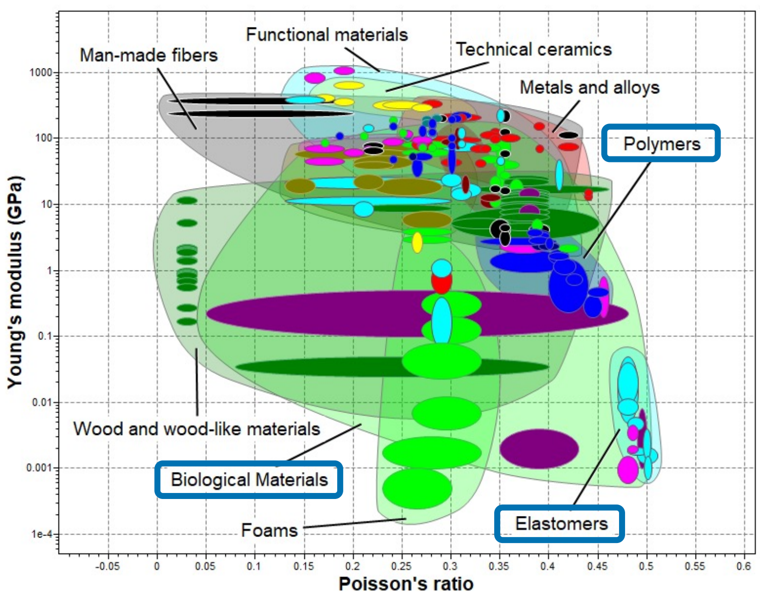
Hyperelastic solids - Isotropic

Incompressible hyperelastic materials

- Biomaterials such as biological soft tissues and solid polymers such as rubber-like materials undergo reversible finite strains.
- Vulcanized rubber undergoes very small volume changes at very high hydrostatic pressures. It is very much easier to change its shape than to change its volume. Rubber is often regarded as incompressible.

$$W = W(I_1, I_2) \text{ subject to } I_3 = J^2 = 1$$



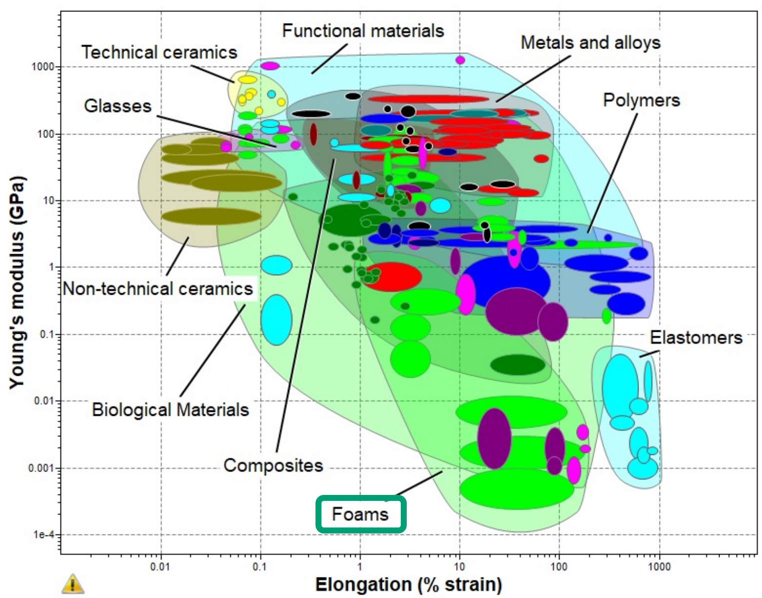


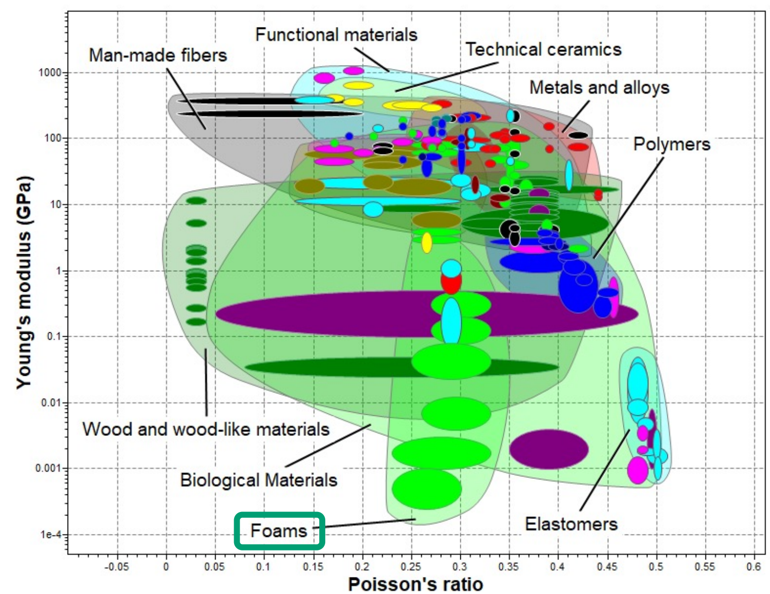
Hyperelastic solids - Isotropic

Compressible hyperelastic materials

- Foamed rubbers undergo reversible finite strain but cannot be regarded as being incompressible.

$$W = W(I_1, I_2, I_3)$$





Hyperelastic solids - Isotropic

Incompressible hyperelastic materials

$$W(I_1, I_2) = c_1(I_1 - 3) + c_2(I_2 - 3)$$

Moony-Rivlin materials

DIY

<u>Note</u>: incompressibility is an internal constraint (or kinematic constraint) of the material

$$I_3 = 1$$

<u>Question</u>: how is the pressure determined? The undetermined part of the pressure is introduced as a Lagrange multiplier and it is determined from boundary conditions

$$W(I_1, I_2) = c_1(I_1 - 3) + c_2(I_2 - 3) - c_0(I_3 - 1)$$

$$\rightarrow \boldsymbol{\sigma} = 2\left[(W_{,I_1} + I_1 W_{,I_2})\boldsymbol{B} - W_{,I_2}\boldsymbol{B}^2\right] - c_0 \boldsymbol{I}$$

<u>Limit of infinitesimal deformations:</u>

$$\mu = 2(c_1 + c_2) \qquad E = 6(c_1 + c_2)$$

shear modulus — Young's modulus

Hyperelastic solids - Isotropic

Incompressible hyperelastic materials

$$W(I_1, I_2) = c_1(I_1 - 3)$$

neo-Hookean materials

DIY

Note: incompressibility is an internal constraint (or kinematic constraint) of the material

$$I_3 = 1$$

Note: the neo-Hookean model is a special case of the Moony-Rivlin model

$$p = -\mathrm{tr}\boldsymbol{\sigma}/3 = c_0 - 2c_1 I_1/3$$

Hyperelastic solids - Isotropic

Almost incompressible hyperelastic materials

$$W(\boldsymbol{C}) = W_D(\overline{\boldsymbol{C}}) + W_H(I_3)$$

deviatoric hydrostatic

$$I_3 = J^2 \neq 1$$
$$\overline{\boldsymbol{C}} = J^{-2/3}\boldsymbol{C}$$
$$\overline{I_3} = 1$$

volume preserving or isochoric part

DIY

Example: Moony-Rivlin model extended to the compressible materials

$$W(\overline{I_1}, \overline{I_2}, I_3) = c_1(\overline{I_1} - 3) + c_2(\overline{I_2} - 3) + W_H(I_3)$$

with

$$W_H(I_3) = D_1(I_3^{1/2} - 1)^2$$

It is not a Lagrange multiplier but rather a penalization term that will generate 'almost' incompressible deformation mappings

Limit of infinitesimal deformations:

$\mu = 2(c_1 + c_2)$ shear modulus

$K = 2/D_1$ bulk modulus

Hyperelastic solids - Isotropic

Compressible hyperelastic materials

$$W = W(I_2, I_3) = c_1 \left[\frac{I_2}{I_3} + 2\sqrt{I_3} - 5 \right]$$

Blatz-Ko materials

DIY

<u>Note</u>: What is the dependency of the resulting pressure?
Does it depend only on the Jacobian of the deformation mapping? **NO!**

Hyperelastic solids - Isotropic

Isotropic hyperelastic materials *– Another take …*

- Frame indifference

$$W = \overline{W}(\boldsymbol{F}, T) = \widehat{W}(\boldsymbol{U}, T) = \widetilde{W}(\boldsymbol{C}, T)$$

- Material symmetry: isotropic

$$W = \overline{W}(\boldsymbol{F}, T) = \widehat{\widehat{W}}(\boldsymbol{V}, T) = \widetilde{\widetilde{W}}(\boldsymbol{B}, T)$$

A function of the right Cauchy-Green tensor **and** left Cauchy-Green tensor

- Strain energy density function: $W = W(\lambda_1, \lambda_2, \lambda_3)$

DIY

The right and left stretch tensors have the same principal stretches (eigenvalues).

Principal directions (eigenvectors).

$$\boldsymbol{\Lambda}_\alpha^{\boldsymbol{C}} = \boldsymbol{\Lambda}_\alpha^{\boldsymbol{U}} = \boldsymbol{N}_\alpha \quad \alpha = \{1, 2, 3\} \quad \|\boldsymbol{N}_\alpha\| = 1$$

$$\boldsymbol{U}\boldsymbol{N}_\alpha = \lambda_\alpha \boldsymbol{N}_\alpha \quad \alpha = \{1, 2, 3\}$$

$$\boldsymbol{C}\boldsymbol{N}_\alpha = \lambda_\alpha^2 \boldsymbol{N}_\alpha \quad \alpha = \{1, 2, 3\}$$

$$\frac{\partial \lambda_\alpha}{\partial \boldsymbol{C}} = \frac{1}{2\lambda_\alpha} \boldsymbol{N}_\alpha \otimes \boldsymbol{N}_\alpha \quad \alpha = \{1, 2, 3\}$$

Hyperelastic solids - Isotropic

Incompressible hyperelastic materials

$$W(\lambda_1, \lambda_2, \lambda_3) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p} \left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3\right)$$ **Ogden model**

DIY

Note: incompressibility is an internal constraint (or kinematic constraint) of the material $\lambda_3 = 1/\lambda_1\lambda_2$

Question: how is the pressure determined? The undetermined part of the pressure is introduced as a Lagrange multiplier and it is determined from boundary conditions

Limit of infinitesimal deformations: $\mu = \frac{1}{2}\sum_{p=1}^{N} \mu_p\alpha_p \quad \text{with} \quad \mu_p\alpha_p > 0$

shear modulus

Hyperelastic solids - Anisotropic

Hyperelastic materials – *One last take …*

- Frame indifference

$$W = \widetilde{W}(\boldsymbol{C}, T) = \check{W}(\boldsymbol{E}, T)$$

- Strain energy density function:

$$\check{W}(\boldsymbol{E}) = \frac{1}{2}(\mathbb{C} : \boldsymbol{E}) : \boldsymbol{E}$$

Saint Venant-Kirchhoff materials

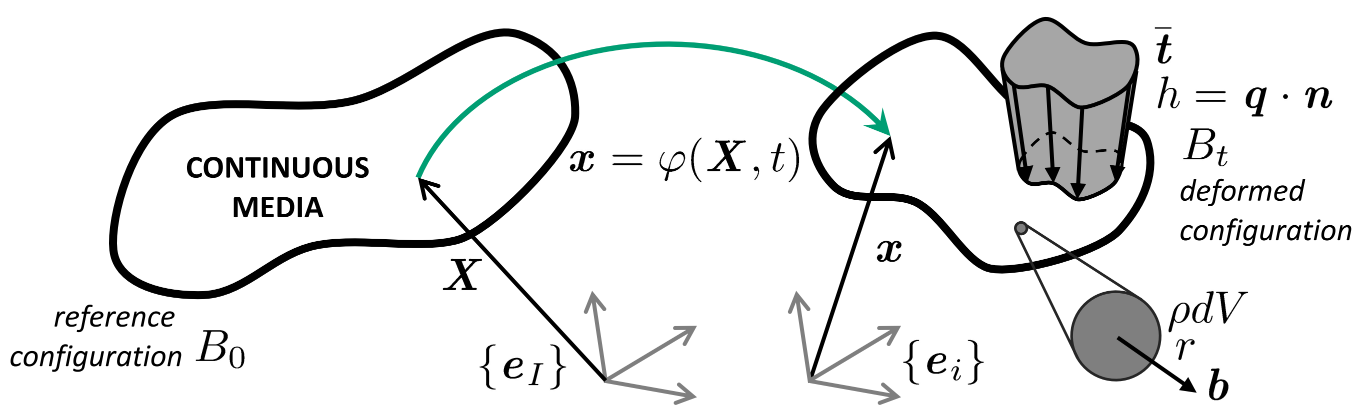
- Material symmetry: none

$$C_{IJKL} = C_{JIKL} = C_{IJLK} = C_{KLIJ}$$

material elastic tensor

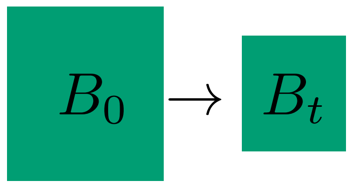
(*constant fourth-order tensor with minor and major symmetries*)

$$\boldsymbol{S} = \frac{\partial \check{W}(\boldsymbol{E})}{\partial \boldsymbol{E}} = \mathbb{C} : \boldsymbol{E}$$

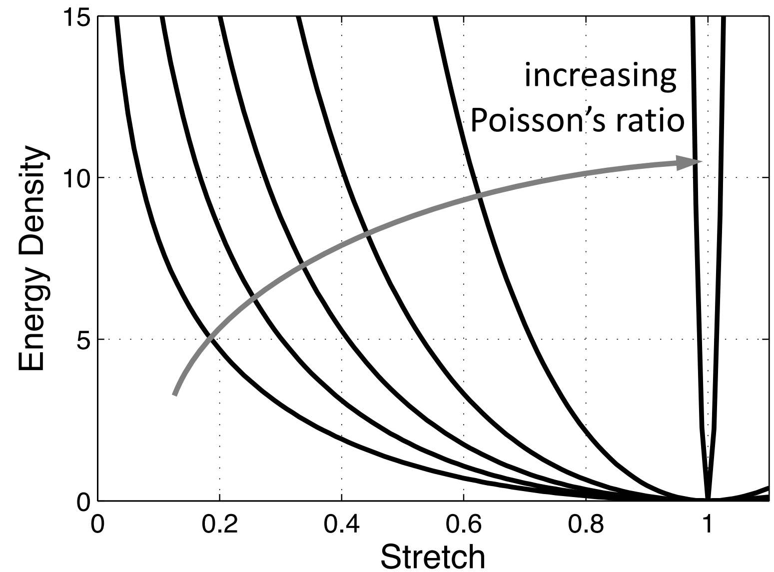


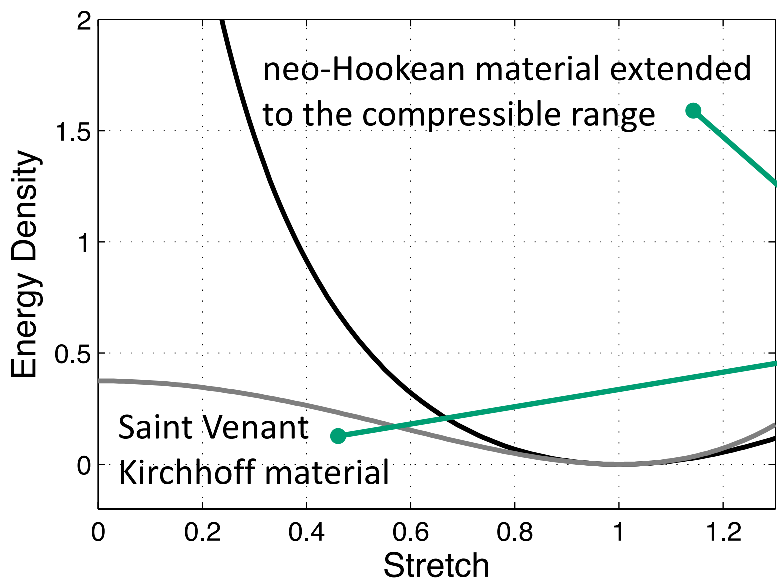
Hyperelastic solids – Isotropic – Food for thought

Isotropic elastic material under hydrostatic compression



$$[\boldsymbol{F}] = \begin{bmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix}$$



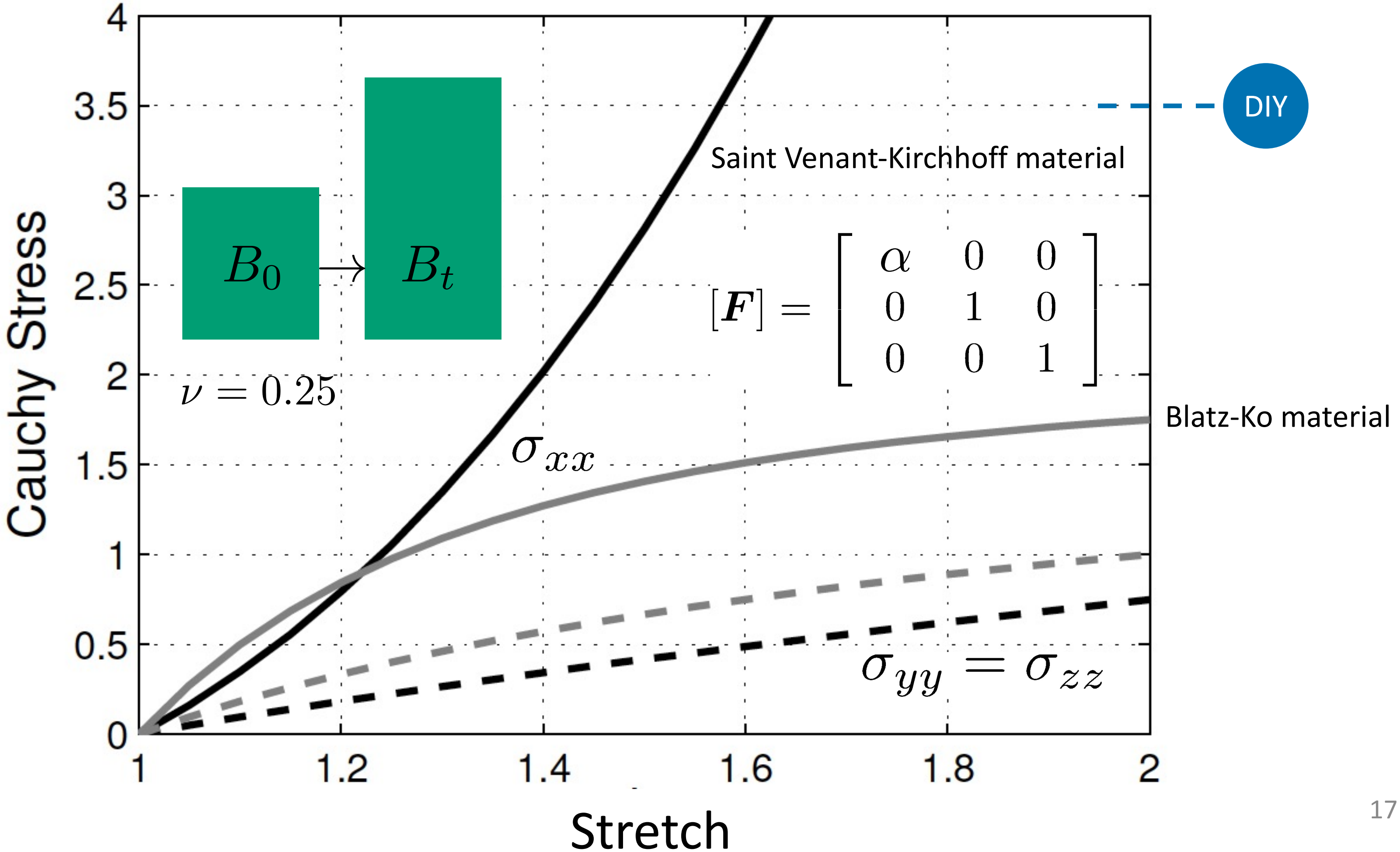


$$W(I_1, I_2, I_3) = \frac{\lambda}{2}\left[\ln\left(I_3^{1/2}\right)\right]^2 - \mu \ln\left(I_3^{1/2}\right) + \frac{\mu}{2}(I_1 - 3)$$

$$C_{IJKL} = \lambda\delta_{IJ}\delta_{KL} + \mu(\delta_{IK}\delta_{JL} + \delta_{IL}\delta_{JK})$$

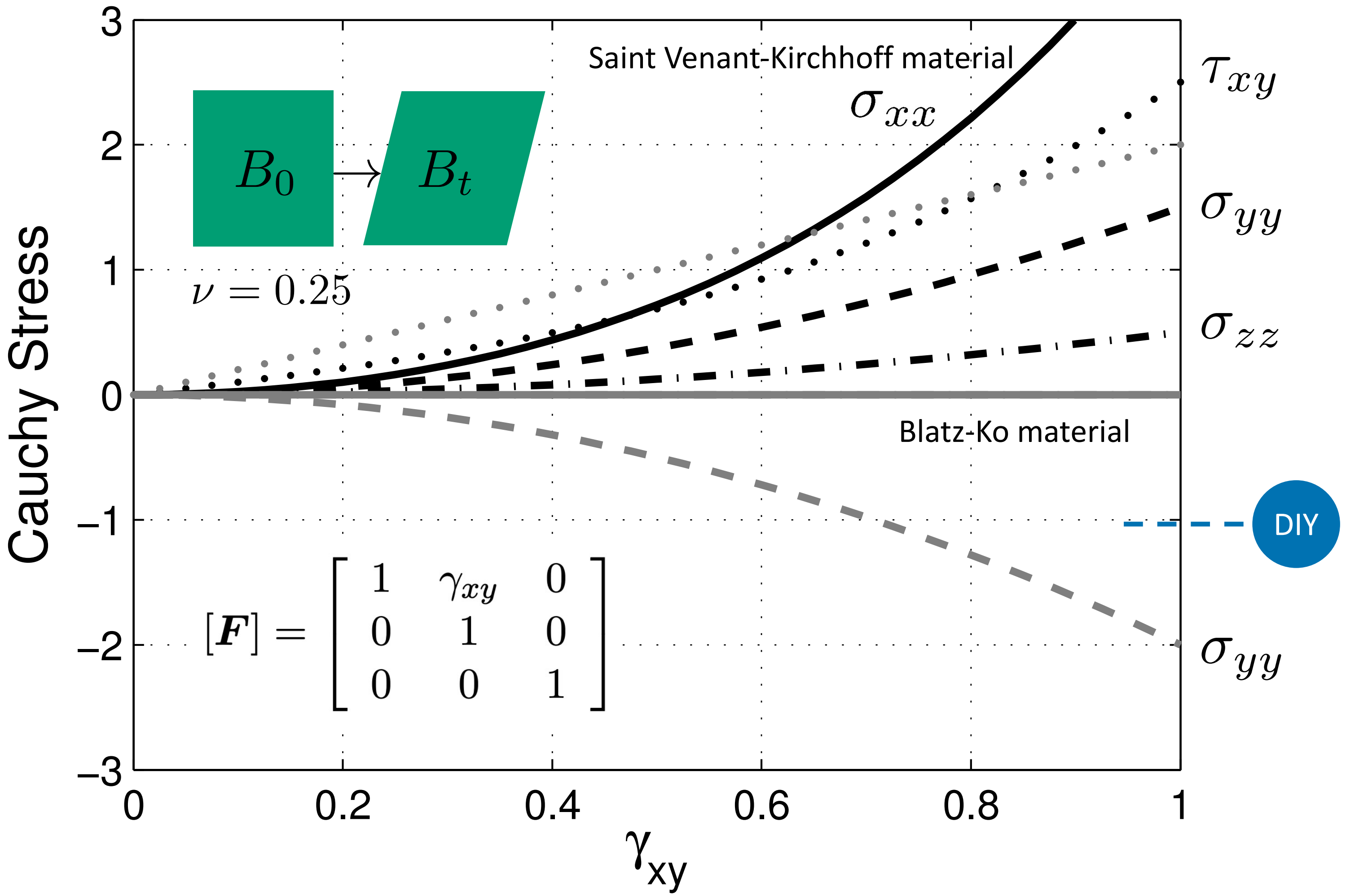
Hyperelastic solids – Isotropic – Food for thought

Isotropic elastic material under uniaxial stretch



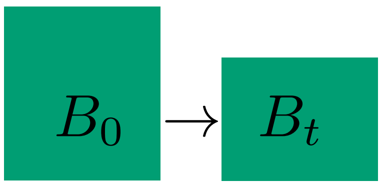
Hyperelastic solids – Isotropic – Food for thought

Isotropic elastic material under simple shear



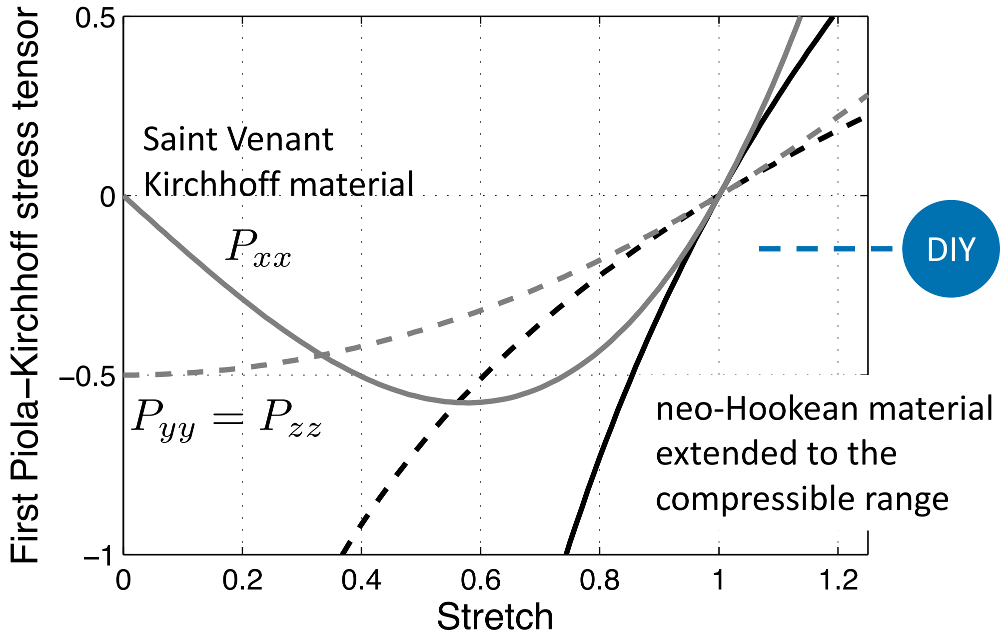
Hyperelastic solids – Isotropic – Food for thought

Isotropic elastic material under uniaxial compression



$\nu = 0.25$

$$[\boldsymbol{F}] = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



True traction: $t_i = \dfrac{df_i}{dA} = \sigma_{ij} n_j \iff \boldsymbol{t} = \boldsymbol{\sigma n}$

Cauchy stress tensor
(*symmetric, spatial tensor, a.k.a., true stress*)

Nominal traction: $T_i = \dfrac{df_i}{dA_0} = P_{iJ} N_J \iff \boldsymbol{T} = \boldsymbol{PN}$

first Piola-Kirchhoff stress tensor
(*non-symmetric, two-point tensor, a.k.a. engineering stress*)

Any questions?