Spring, 2022

ME 597 – Solid Mechanics II

Lecture 9
Hyperelastic solids
Stability and linearized equations

PURDUE UNIVERSITY | Mechanical Engineering

Instructor: Prof. Marcial Gonzalez

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Hyperelastic solids - Isotropic

Review: Incompressible hyperelastic materials

$$W\left(\lambda_1, \lambda_2, \lambda_3\right) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p} \left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3\right)$$ **Ogden model**

Limit of infinitesimal deformations: $\mu = \frac{1}{2}\sum_{p=1}^{N} \mu_p \alpha_p \quad \text{with} \quad \mu_p \alpha_p > 0$

shear modulus

$$W(I_1, I_2) = c_1(I_1 - 3) + c_2(I_2 - 3)$$ **Moony-Rivlin materials**

$$W(I_1, I_2) = c_1(I_1 - 3)$$ **neo-Hookean materials**

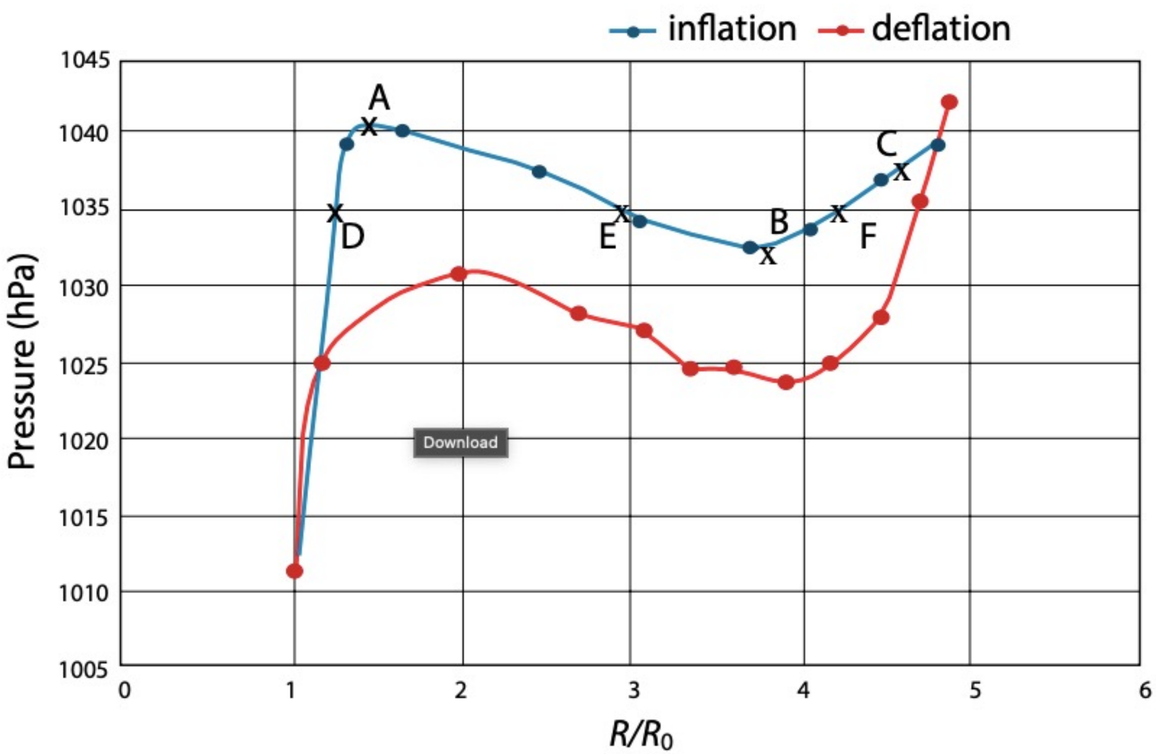
Limit of infinitesimal deformations: $\mu = 2(c_1 + c_2)$ (shear modulus) $\quad E = 6(c_1 + c_2)$ (Young's modulus)

Note: incompressibility is an internal constraint (or kinematic constraint) of the material $\quad \lambda_3 = 1/\lambda_1\lambda_2 \quad I_3 = 1$

Note: Pressure is introduced as a Lagrange multiplier and determined from boundary conditions.

Hyperelastic solids - Isotropic

Inflation of a spherical (incompressible rubber) balloon



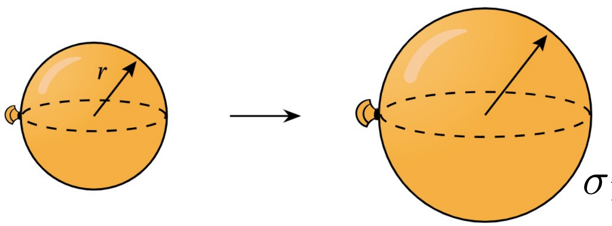
"It has been reposted [...] that smartphone barometers are good enough for real physics experiments"

Source: Julien Vandermarlière , "On the inflation of a rubber balloon" The Physics Teacher 54, 566-567 (2016)

Hyperelastic solids - Isotropic

Inflation of a spherical (incompressible rubber) balloon

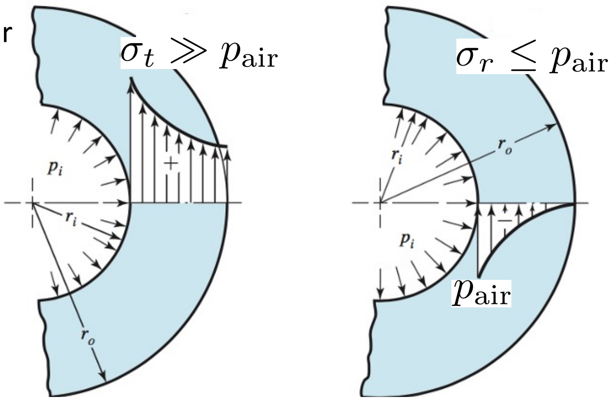
Spherical inflation mode $\lambda_1 = \lambda_2 = \frac{r}{R}$

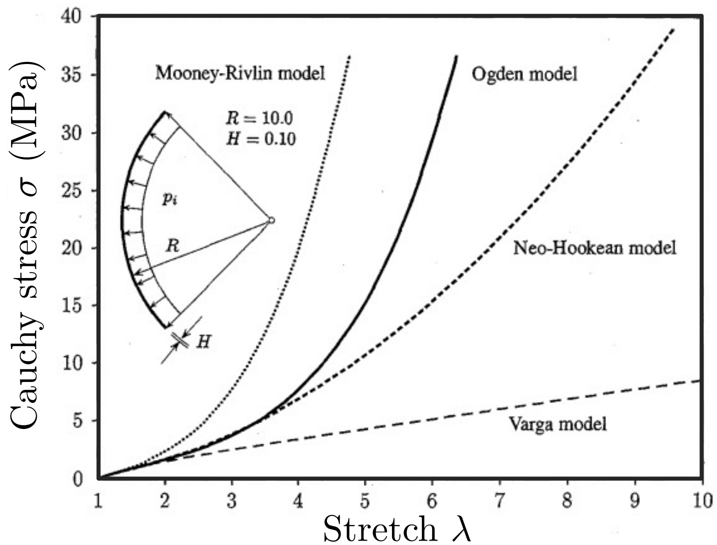


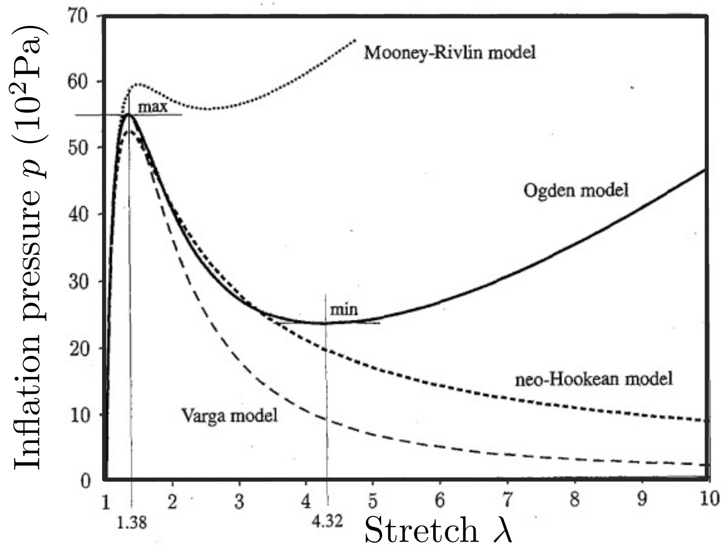
Uniform for thin walls

$t \ll r$

$$\sigma_{11} = \sigma_{22} = \frac{p_{\text{air}} r}{2t}$$







Source: Harold Alexander, "Tensile instability of initially spherical balloons", Int. J. Engng Sci. Vol. 9, 15I-162, 1971.

Hyperelastic solids - Isotropic

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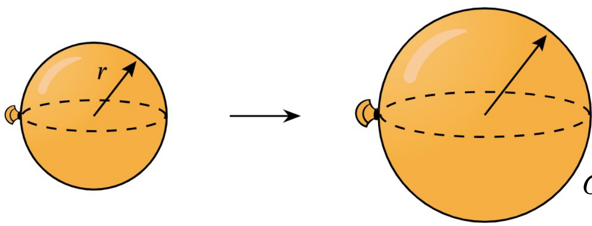
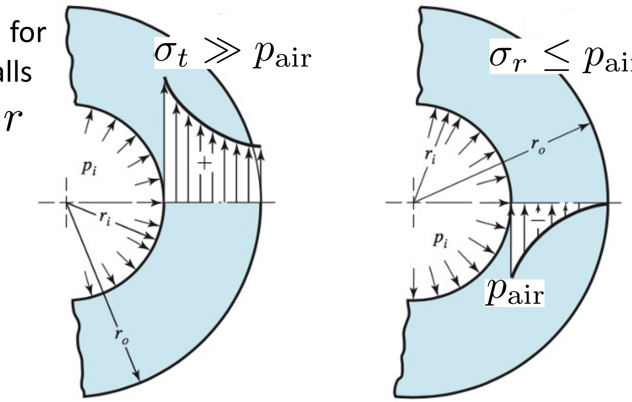


$$\sigma_{11} = \sigma_{22} = \frac{p_{\text{air}} r}{2t}$$

DIY

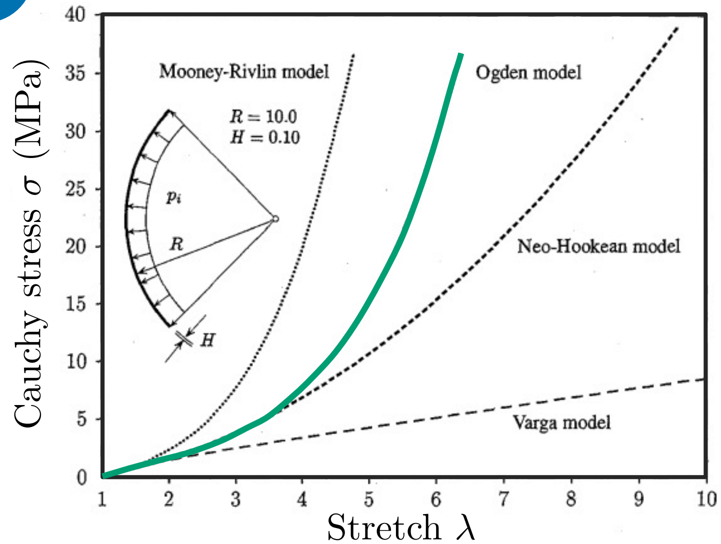
Uniform for thin walls $t \ll r$

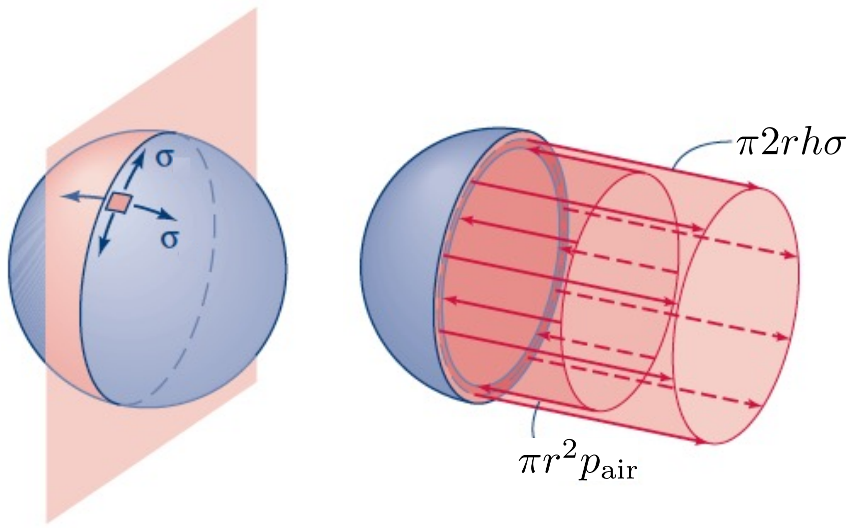


Hyperelastic solids - Isotropic

Inflation of a spherical (incompressible rubber) balloon

DIY





Source: Holzapfel G.A., “Nonlinear Solid Mechanics”, Wiley, 2000.

Hyperelastic solids - Isotropic

Inflation of a spherical (incompressible rubber) balloon

DIY

Questions …

How is the experiment carried out?

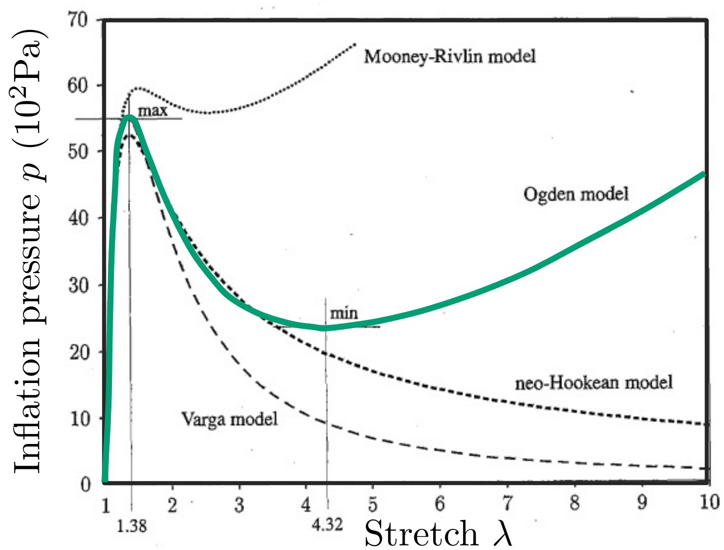
Is it a load-controlled or a displacement-controlled experiment?

What is a (stable) equilibrium configuration?

$\alpha_1 = 1.3$ $\quad \mu_1 = 6.3 \cdot 10^5 \mathrm{N/m^2}$

$\alpha_2 = 5.0$ $\quad \mu_2 = 0.012 \cdot 10^5 \mathrm{N/m^2}$

$\alpha_3 = -2.0$ $\quad \mu_3 = -0.1 \cdot 10^5 \mathrm{N/m^2}$



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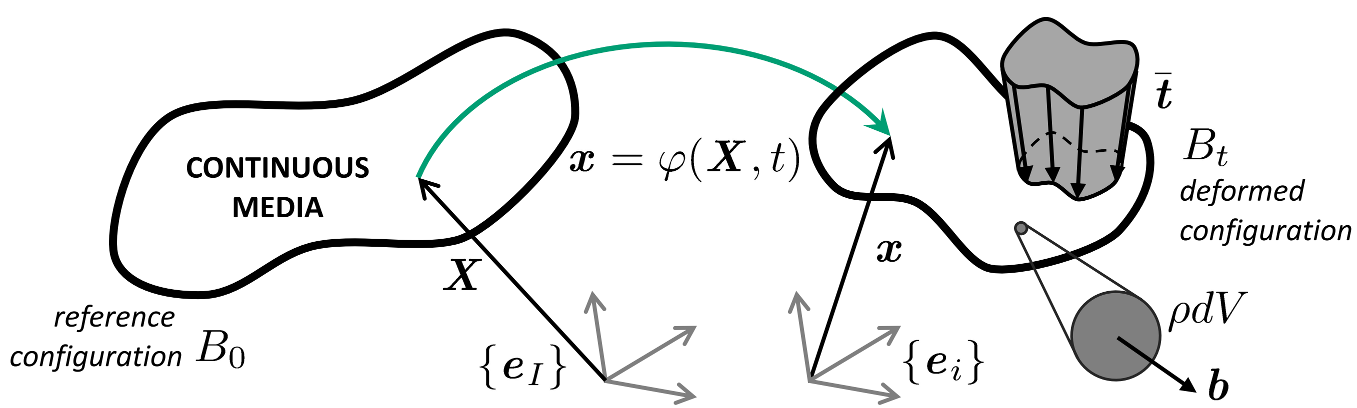
Stability of hyperelastic solids

Static equilibrium – Stability

- Principle of stationary potential energy:
  Given the set of admissible displacement fields for a conservative system, an equilibrium state will correspond to one for which the potential energy is stationary

$$\Pi = \underbrace{\int_{B_0} W(\boldsymbol{F})dV_0}_{\text{internal}} \underbrace{- \int_{B_0} \rho\boldsymbol{b}\cdot\boldsymbol{\varphi}dV_0 - \int_{\partial B_0} \boldsymbol{t}\cdot\boldsymbol{\varphi}dA_0}_{\text{external}}$$

- Stable solution?
  A stationary point could be a <u>minimum</u>, a maximum or a saddle point.



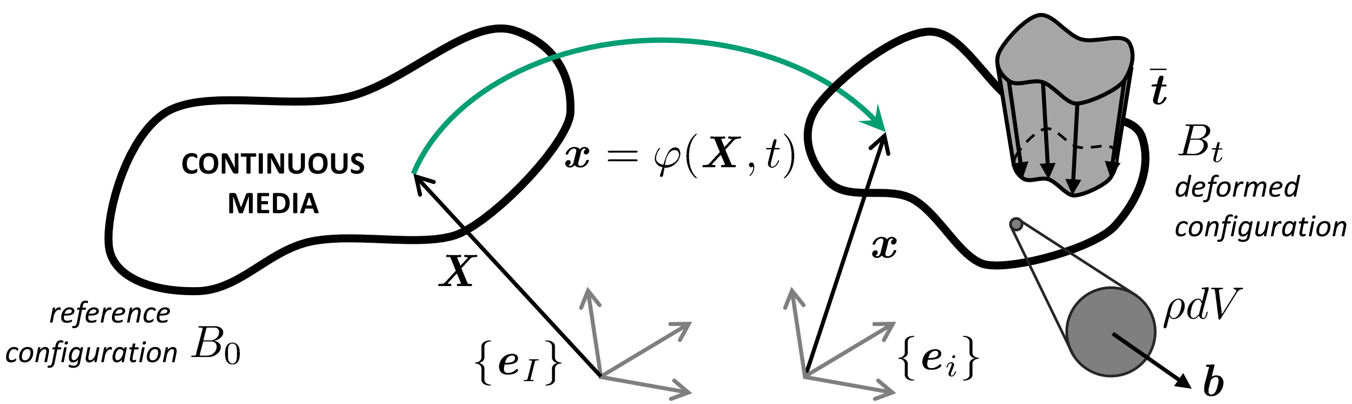
Stability of hyperelastic solids

Static equilibrium – Stability

- Stable solution?
  A stationary point could be a minimum, a maximum or a saddle point.

- Theory of stability of infinite-dimensional spaces is not trivial …

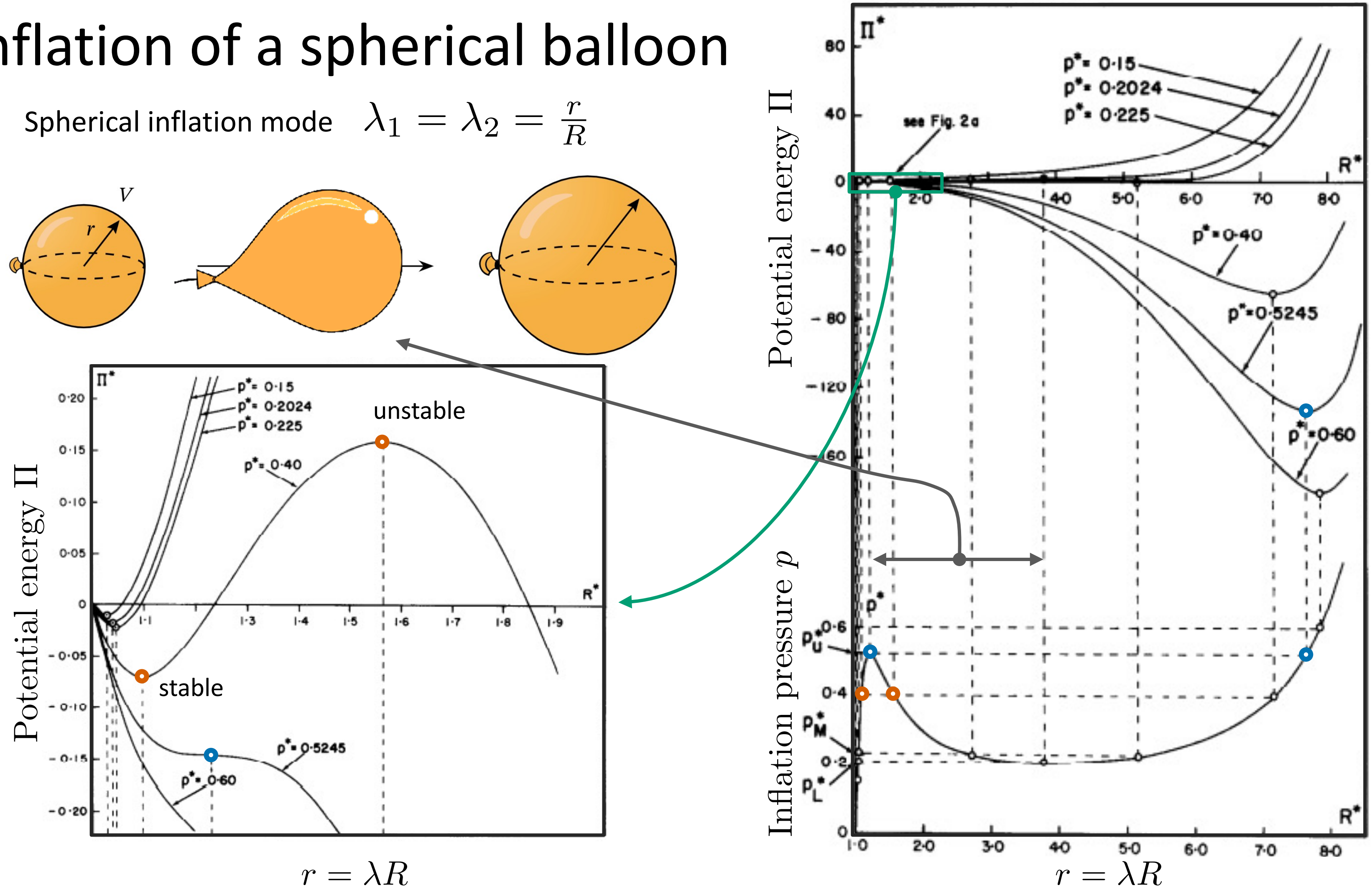
  …. but, ultimately, *necessary conditions* can be inferred from the strain energy density, and *sufficient conditions* come from nonlinear terms (e.g., in the deformation mapping) and from boundary conditions.



Stability of hyperelastic solids

Inflation of a spherical balloon

Spherical inflation mode $\lambda_1 = \lambda_2 = \frac{r}{R}$



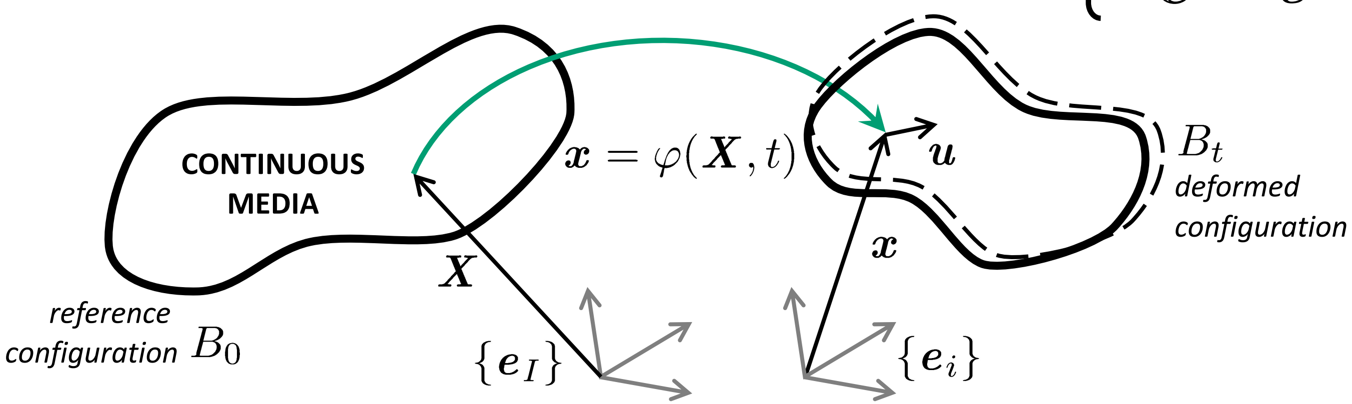
Source: Harold Alexander, “Tensile instability of initially spherical balloons”, Int. J. Engng Sci. Vol. 9, 15l-162, 1971.

Linearized constitutive equations

Review: Linearized kinematics

- Linearized or incremental expressions for the kinematic quantities are required when:
  (i) the deformation process is described as a series of small steps
  $$\boldsymbol{x} = \varphi(\boldsymbol{X}) \to \varphi(\boldsymbol{X}) + \boldsymbol{u}(\boldsymbol{X})$$
  (ii) displacements are indeed small
  $$\boldsymbol{X} \to \boldsymbol{X} + \boldsymbol{u}(\boldsymbol{X})$$
- If the linearization is evaluated in the 'stress free' undeformed configuration ($\boldsymbol{F}^* = \boldsymbol{I},\ J^* = 1,\ \boldsymbol{\sigma} = \boldsymbol{0}$), then

$$\boldsymbol{S} = \mathbb{C} : \boldsymbol{E} \quad \to \quad d\boldsymbol{S}^* = \mathbb{C}^* : d\boldsymbol{E}^* \qquad \begin{cases} d\boldsymbol{E}^* = d\boldsymbol{\epsilon} \to \boldsymbol{\epsilon} \\ d\boldsymbol{S}^* = d\boldsymbol{\sigma} \to \boldsymbol{\sigma} \\ \mathbb{C}^* = \boldsymbol{c} \end{cases}$$



Linearized constitutive equations

Generalized Hooke's law as a linearized hyperelasticity

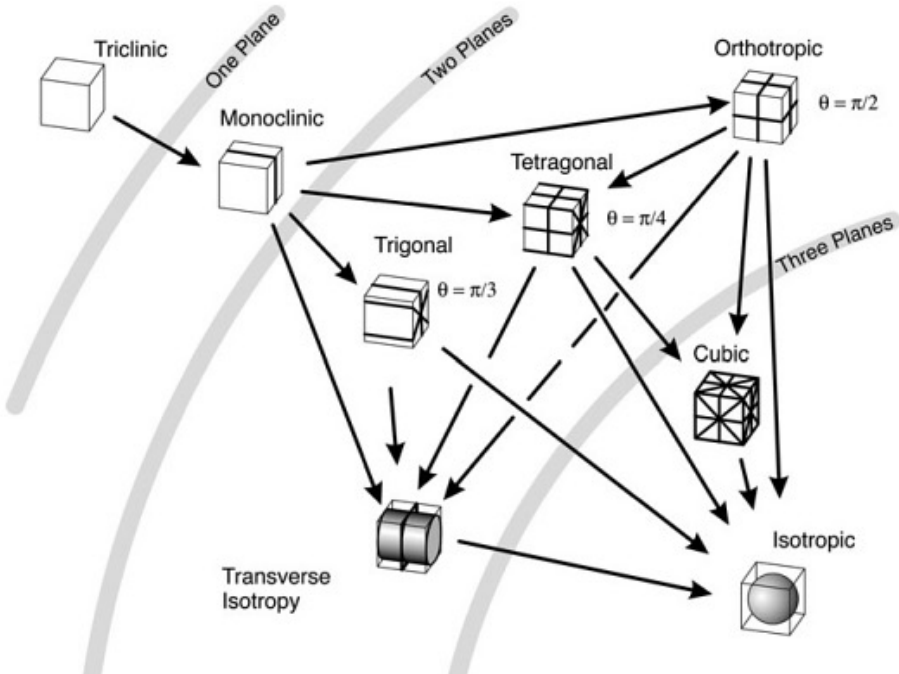
- Material symmetry

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} \iff \boldsymbol{\sigma} = \boldsymbol{c} : \boldsymbol{\epsilon}$$

(small strain) elasticity tensor

(*minor and major symmetries*)

$$c_{ijkl} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}c_{pqrs} \qquad \forall \boldsymbol{Q} = \boldsymbol{H} \in G \subset SO(3)$$



Symmetry Classes	Point Group	Symmetry Operation
Triclinic	1	One onefold
Monoclinic	2	One twofold
Orthotropic	222	Three twofold
Tetragonal	4	One fourfold
Trigonal	3	One threefold
Transv. Isotropy	6	One sixfold
Cubic	23	Four threefold
Isotropic	-	

"Continuum Mechanics of Anisotropic Materials" S.C. Cowin

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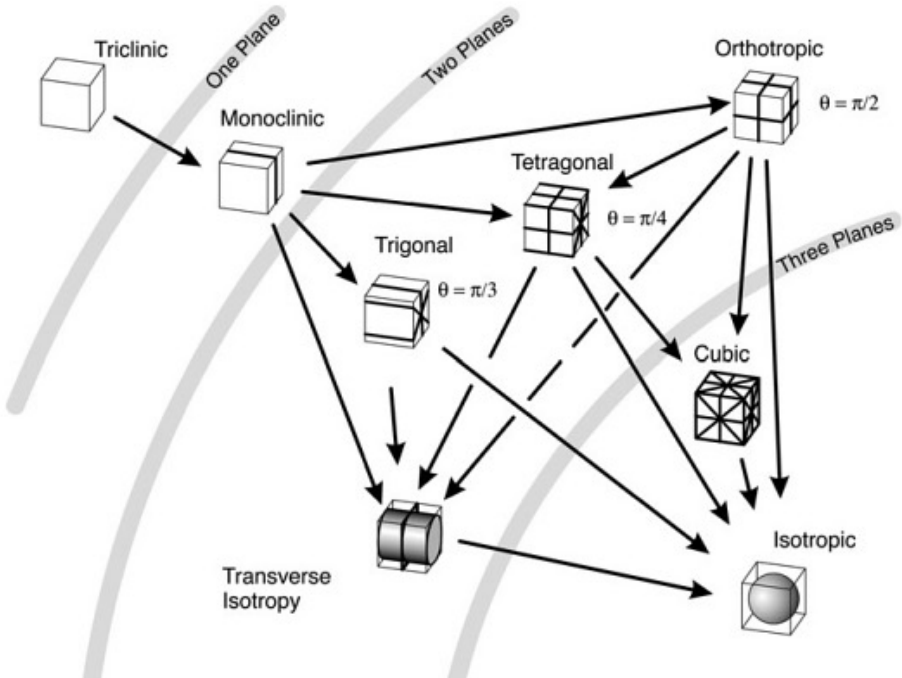

Note:
Symmetries are retained only under small deformations.

Although any symmetry can be cast into an incremental formulation, it might not result in an accurate representation.

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Linearized constitutive equations

Generalized Hooke's law as a linearized hyperelasticity

- Material symmetry: isotropic material

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(*minor and major symmetries*)

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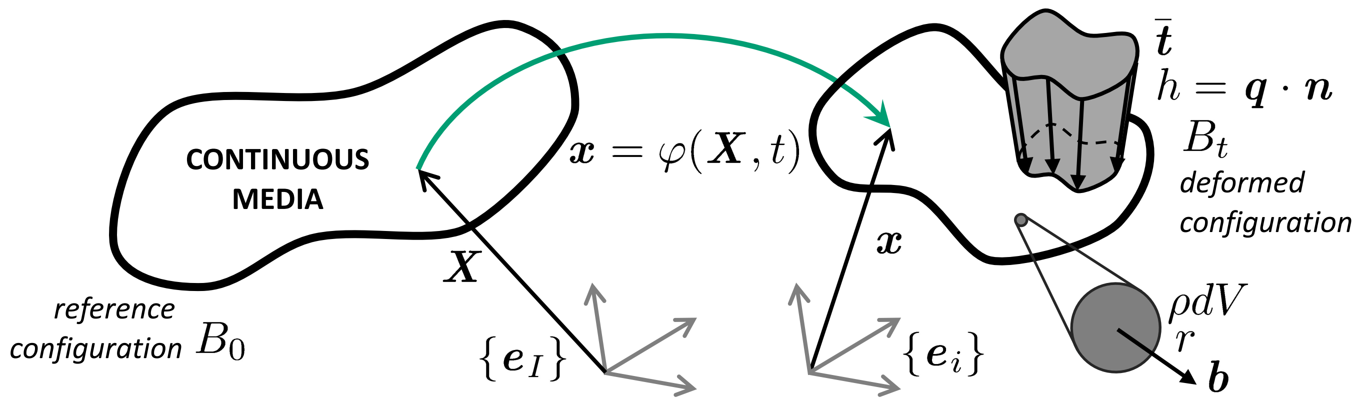
DIY

Isotropic (any proper rotation) $\boldsymbol{Q} = \boldsymbol{R}^T \in SO(3)$

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$\lambda = c_{1122} = c_{1133} = c_{2211} \qquad c_{1111} = c_{2222} = c_{3333}$$

$$\mu = c_{2323} = (c_{1111} - c_{1122})/2 = c_{1313} = c_{1212}$$



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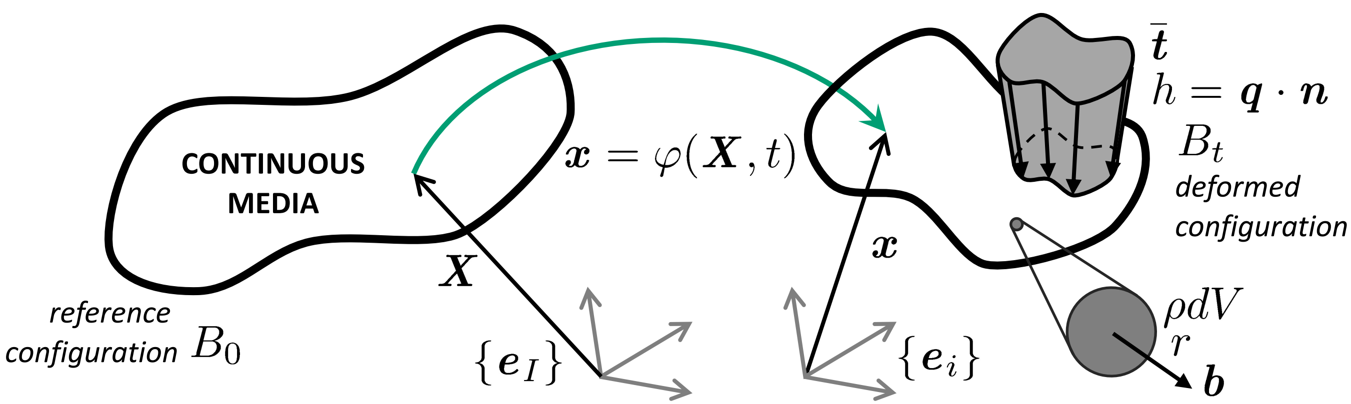
DIY

Isotropic (any proper rotation) $\boldsymbol{Q} = \boldsymbol{R}^T \in SO(3)$

$$\boldsymbol{\sigma} = \lambda(\mathrm{tr}\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}$$

$$\boldsymbol{\epsilon} = -\frac{\nu}{E}(\mathrm{tr}\boldsymbol{\sigma})\boldsymbol{I} + \frac{1+\nu}{E}\boldsymbol{\sigma}$$



Lecture 9 – Stability and linearized equations

Any questions?