Spring, 2022

ME 597 – Solid Mechanics II

Lecture 10
Viscoelastic solids

PURDUE UNIVERSITY | Mechanical Engineering

Instructor: Prof. Marcial Gonzalez

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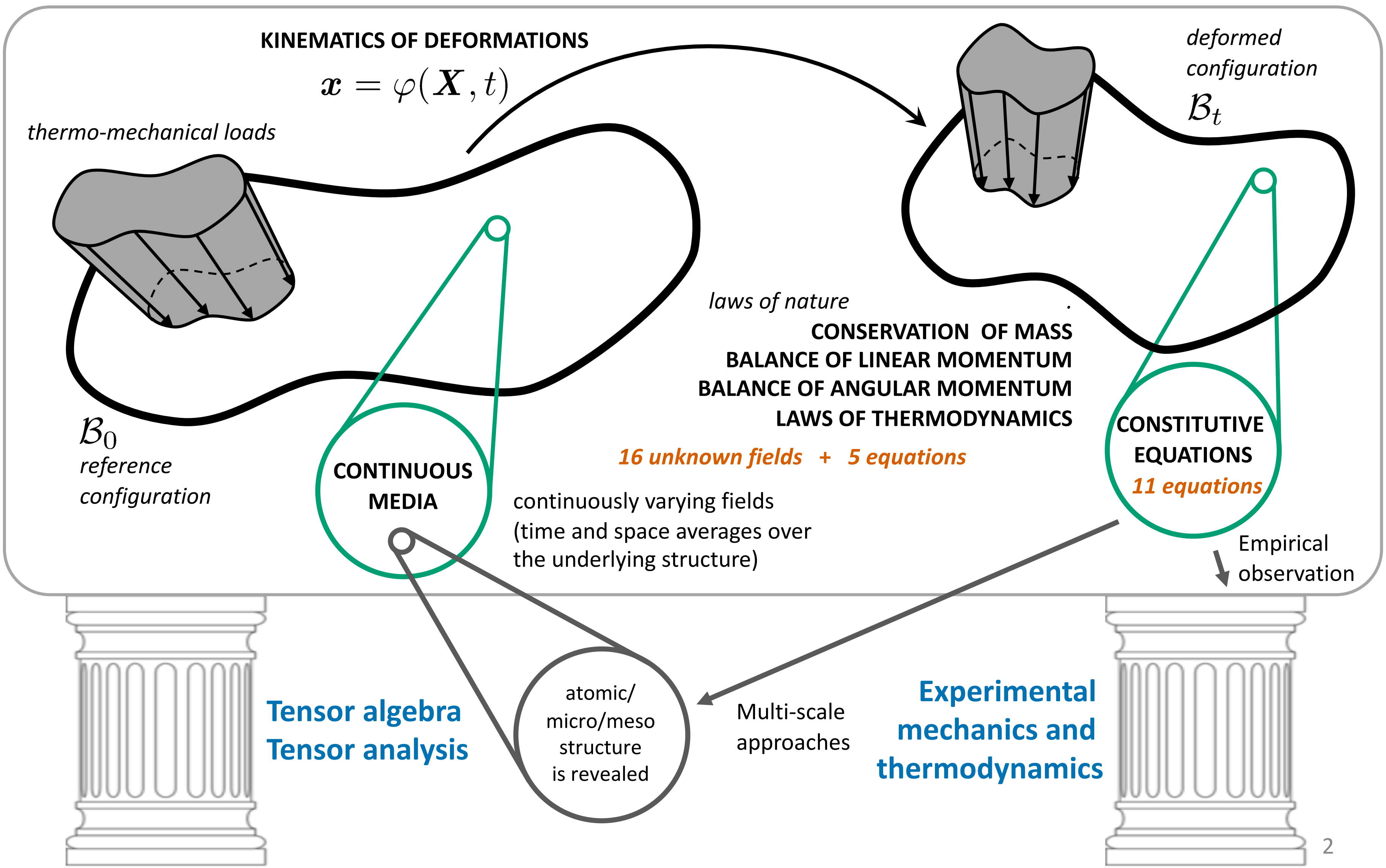
Lecture 10 – Constitutive relations



Constitutive relations

Constraints on constitutive relations

- Relations that describe the response of the material to mechanical and thermal loading, e.g., $\boldsymbol{\sigma},\ \boldsymbol{q},\ u,\ T$ (11 constitutive equations)

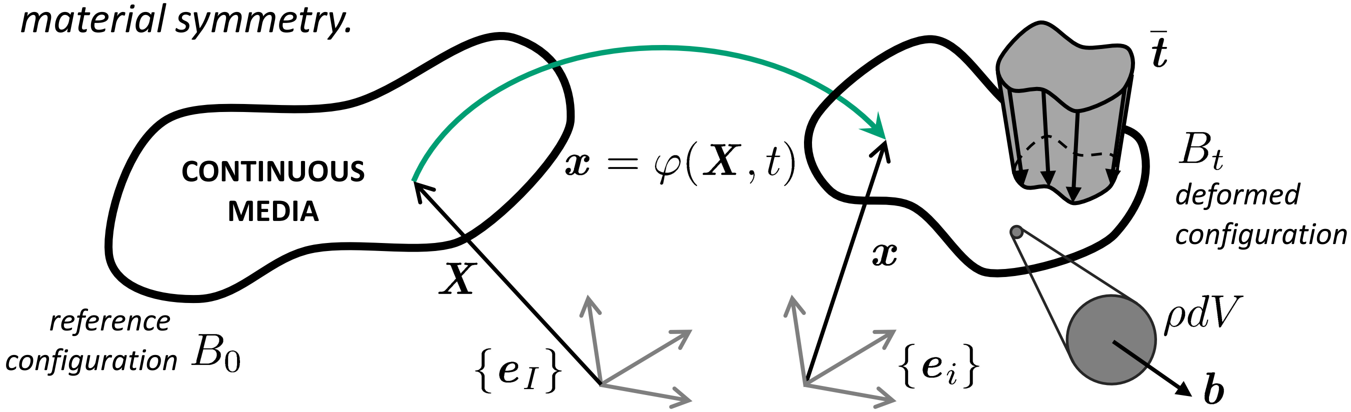
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(e)} + \boldsymbol{\sigma}^{(v)},\ \ \boldsymbol{q} = \mathbf{0}, \rho_0 u(\boldsymbol{F}, s),\ \ \dot{T} \neq 0 \ \ (\text{with } \dot{s} = 0)$$ isothermal processes

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(e)} + \boldsymbol{\sigma}^{(v)},\ \ \boldsymbol{q} = \mathbf{0}, W(\boldsymbol{F}, T),\ \ \dot{s} \neq 0 \ \ (\text{with } \dot{T} = 0)$$ isentropic processes

- Can these constitutive relations be selected arbitrarily? NO!

  They must follow the following fundamental principles:

  Principle of determinism, principle of local action, second law of thermodynamics restrictions (Clausius-Duhem inequality), principle of material frame indifference, material symmetry.



Constitutive relations

Constraints on constitutive relations

- Principle of determinism & Principle of local action

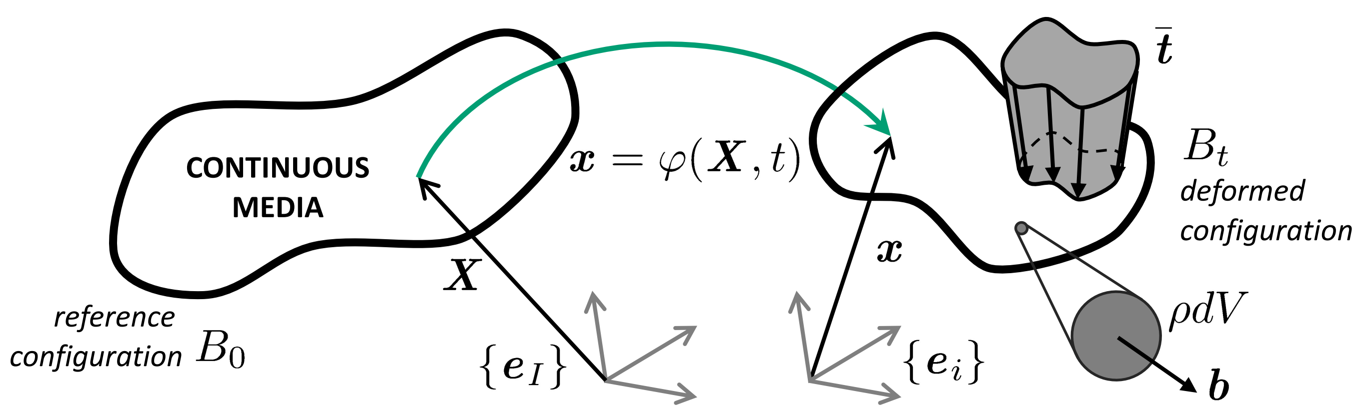
$$\boldsymbol{\sigma}(\boldsymbol{X},t) = \boldsymbol{f}(\varphi^t, \boldsymbol{F}^t, ..., T^t, \nabla_0 T^t, ..., \boldsymbol{X}, t)$$

time series $\square^t$ materials with memory

$\boldsymbol{f}(..., t)$ materials with aging

In general $\boldsymbol{F}(\boldsymbol{X},t)$, therefore $\boldsymbol{f}(..., \boldsymbol{F}, \dot{\boldsymbol{F}}, ...)$

Simple elastic material $W = \overline{W}(\boldsymbol{C},T)$ $\boldsymbol{S}^{(e)} = 2\frac{\partial \overline{W}(\boldsymbol{C},T)}{\partial \boldsymbol{C}}$ $\boldsymbol{\sigma}^{(v)}(\boldsymbol{C},T,\boldsymbol{d})$ — simple fluids —



Constitutive relations

Materials with memory or with hereditary effects

- Materials whose behavior exhibits dependence upon the history of kinetic variables.
  - Viscosity (Reiner–Rivlin fluid, Newtonian fluid)

$$\boldsymbol{\sigma}^{(v)}(\boldsymbol{d}) = \eta_0 \boldsymbol{I} + \eta_1 \boldsymbol{d} + \eta_2 \boldsymbol{d}^2$$

  - Internal processes (creep and relaxation) and hysteresis

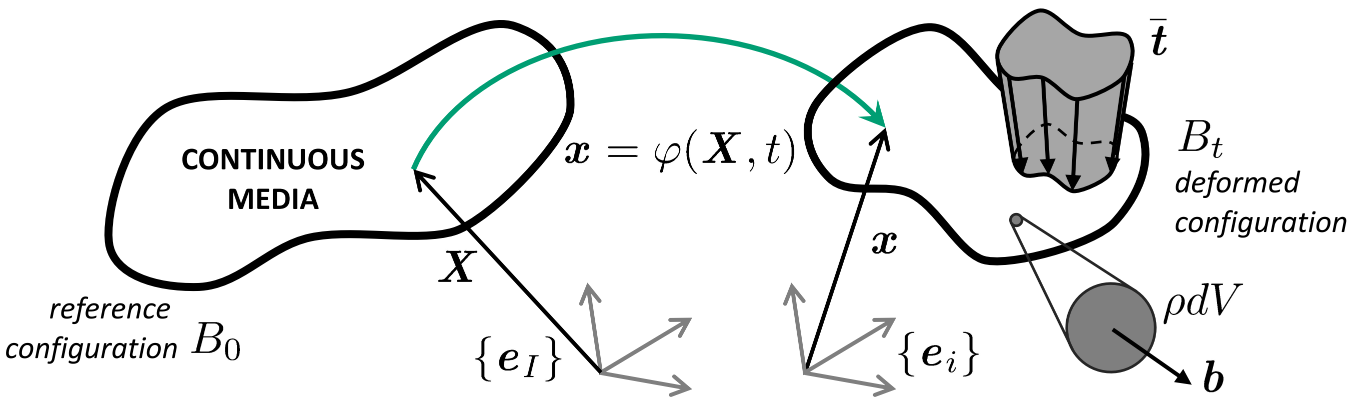
$$\boldsymbol{\sigma}(\boldsymbol{X}, t) = \boldsymbol{f}(\boldsymbol{C}, \boldsymbol{C}^t)$$

Limit of infinitesimal strains

$$\boldsymbol{d} \rightarrow \dot{\boldsymbol{\epsilon}}$$

$$\boldsymbol{\sigma}(\boldsymbol{X}, t) = \boldsymbol{f}(\boldsymbol{\epsilon}, \boldsymbol{\epsilon}^t)$$

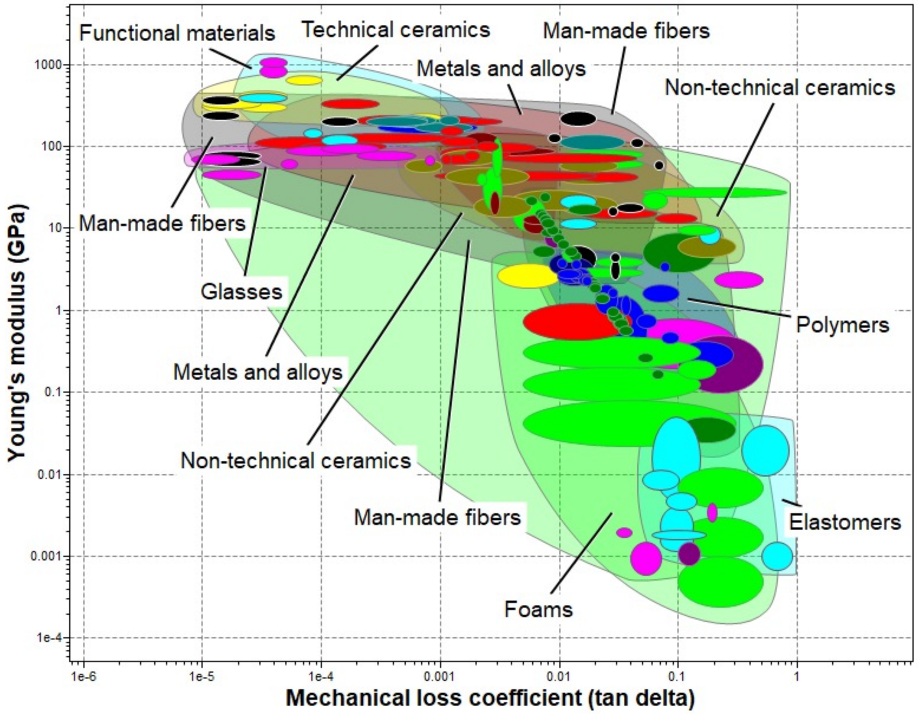
$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{(e)} + \boldsymbol{\epsilon}^{(i)}$$

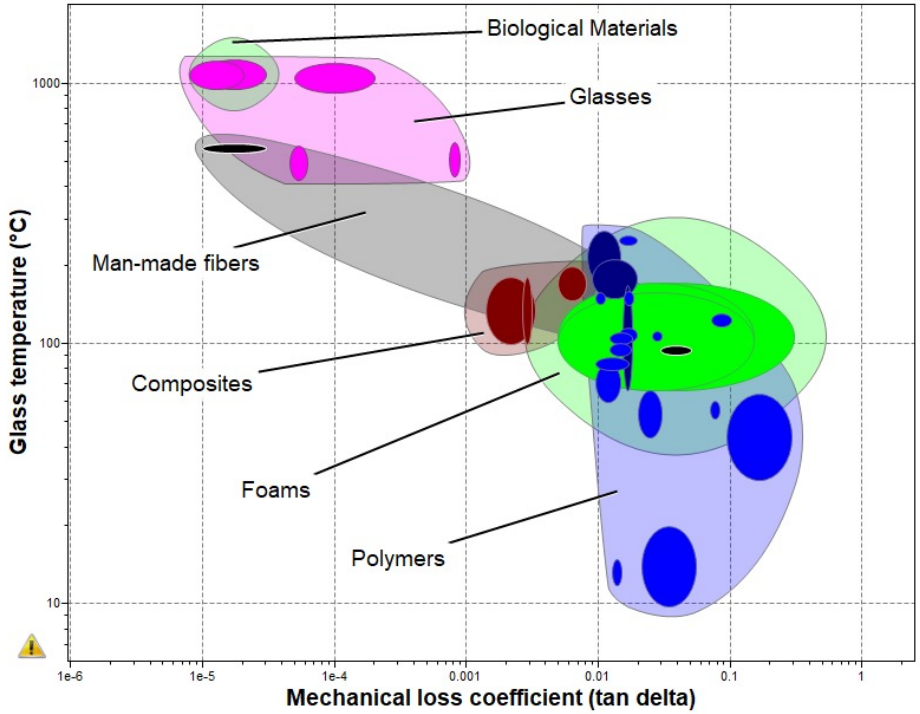


Viscoelasticity

Young's modulus – Mechanical loss – Glass temperature

- Reversible transition from hard/brittle 'glassy' state into viscous state.
- Materials with good damping behavior (i.e., high vibrational energy dissipation) are typical soft and have low glass-transition temperatures.

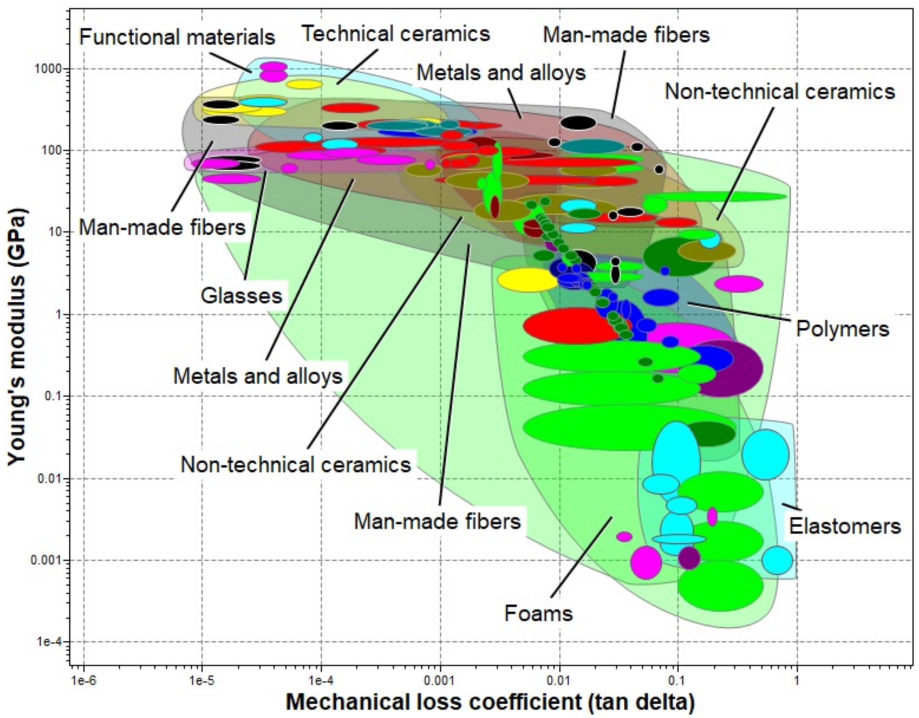


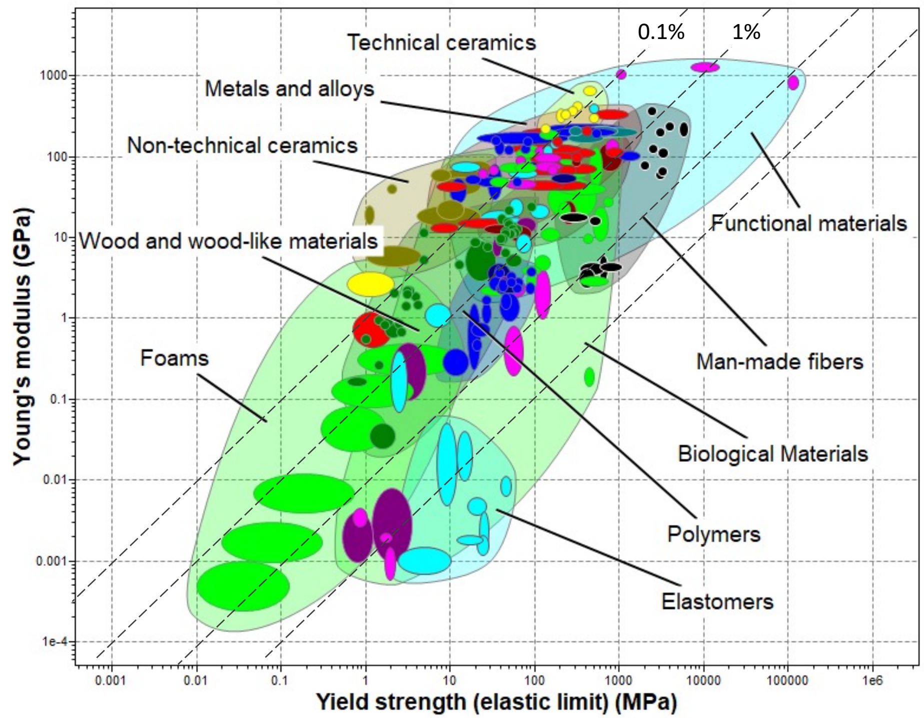


Viscoelasticity – Infinitesimal strain

Young's modulus – Yield stress– Density

- Irreversible transition from elastic to plastic behavior.
- Contours of constant yield strain are a family of parallel lines on the Young's modulus vs. yield strength plot.



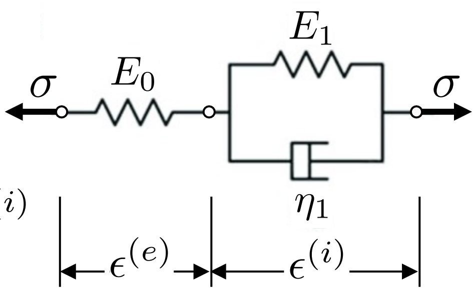


Viscoelasticity

Standard solid model

- Creep process (at constant stress)

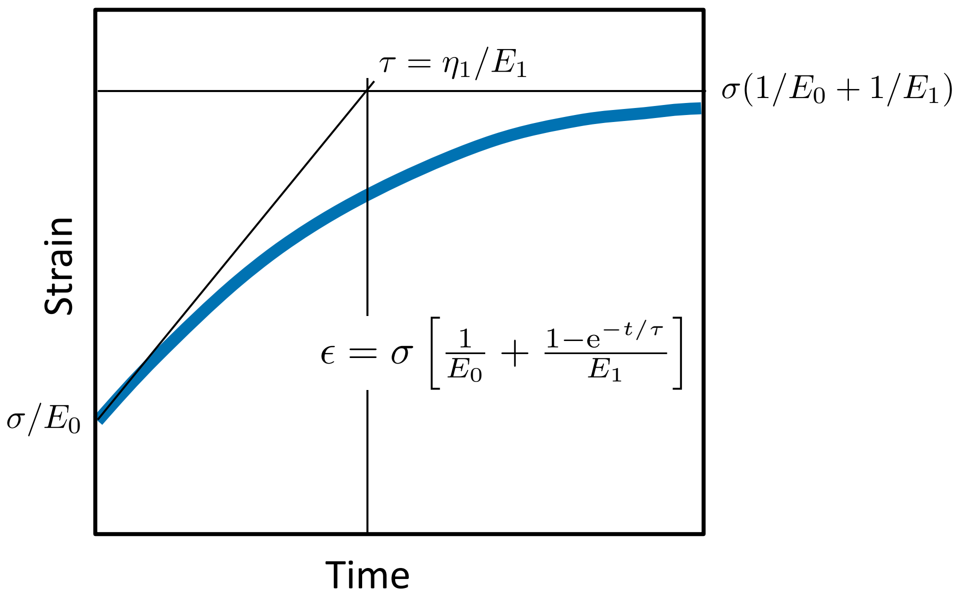
$$\epsilon = \epsilon^{(e)} + \epsilon^{(i)}$$
$$\sigma = E_0 \epsilon^{(e)}$$
$$\sigma = E_1 \epsilon^{(i)} + \eta_1 \dot{\epsilon}^{(i)}$$

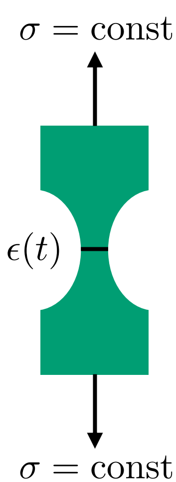


From equilibrium and compatibility:

$$\dot{\epsilon}^{(i)} = \frac{1}{\eta_1}\sigma - \frac{E_1}{\eta_1}\epsilon^{(i)}$$

Solution of ODE: $\epsilon^{(i)}(t) = \frac{1}{\eta_1}\int_{-\infty}^{t} \mathrm{e}^{-(t-t')/\tau}\sigma(t')dt'$





Viscoelasticity

Standard solid model

- Creep process (at constant stress)

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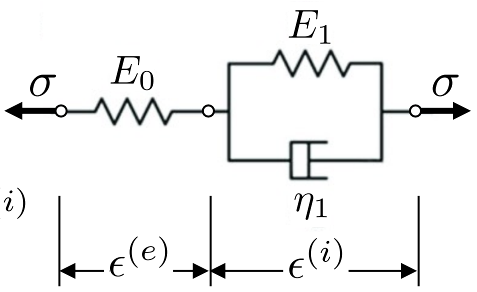

From equilibrium and compatibility:

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Solution of ODE: $\epsilon^{(i)}(t) = \frac{1}{\eta_1}\int_{-\infty}^{t} \mathrm{e}^{-(t-t')/\tau}\sigma(t')dt'$

DIY

Viscoelasticity

Standard solid model

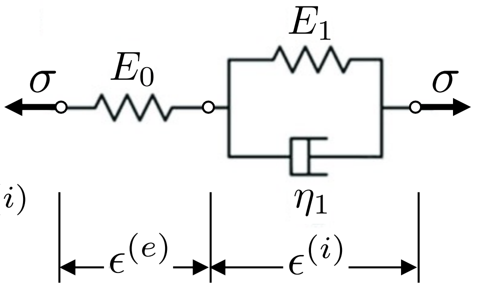
- Relaxation process (at constant strain)

$$\epsilon = \epsilon^{(e)} + \epsilon^{(i)}$$

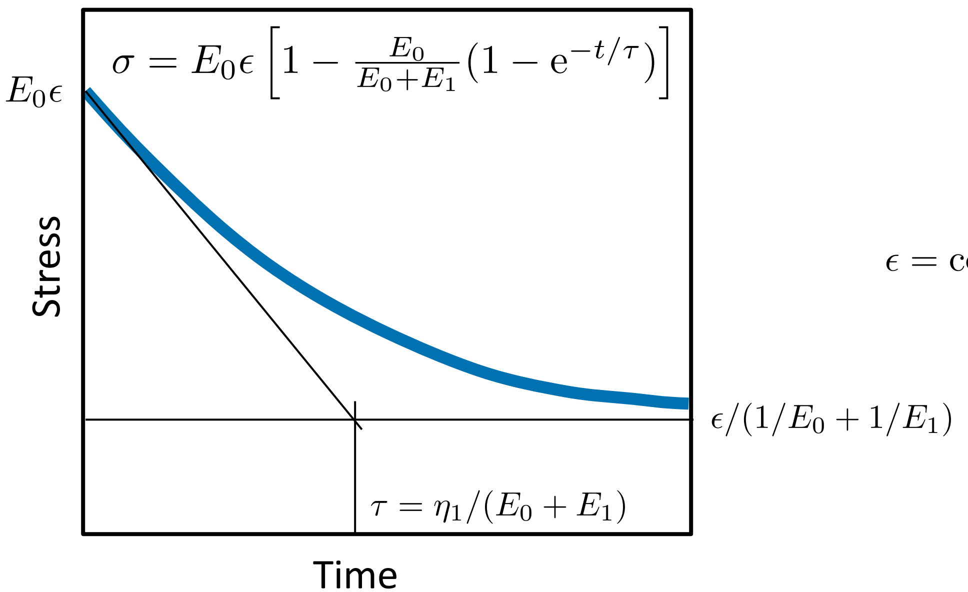
From equilibrium and compatibility:

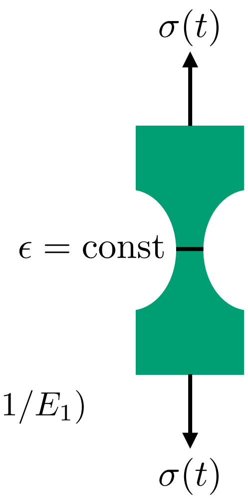
$$\sigma = E_0 \epsilon^{(e)}$$

$$\dot{\epsilon}^{(i)} = \frac{E_0}{\eta_1}\left(\epsilon - \epsilon^{(i)}\right) - \frac{E_1}{\eta_1}\epsilon^{(i)} \qquad \sigma = E_1 \epsilon^{(i)} + \eta_1 \dot{\epsilon}^{(i)}$$



Solution of ODE: $\epsilon^{(i)}(t) = \frac{E_0}{\eta_1}\int_{-\infty}^{t} \mathrm{e}^{-(t-t')/\tau}\epsilon(t')dt'$





Viscoelasticity

Standard solid model

- Relaxation process (at constant strain)

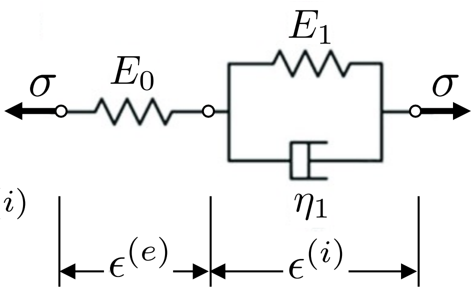
$$\epsilon = \epsilon^{(e)} + \epsilon^{(i)}$$

From equilibrium and compatibility:

$$\sigma = E_0 \epsilon^{(e)}$$

$$\dot{\epsilon}^{(i)} = \frac{E_0}{\eta_1}(\epsilon - \epsilon^{(i)}) - \frac{E_1}{\eta_1}\epsilon^{(i)}$$

$$\sigma = E_1 \epsilon^{(i)} + \eta_1 \dot{\epsilon}^{(i)}$$



Solution of ODE: $\epsilon^{(i)}(t) = \frac{E_0}{\eta_1}\int_{-\infty}^{t} \mathrm{e}^{-(t-t')/\tau}\epsilon(t')dt'$

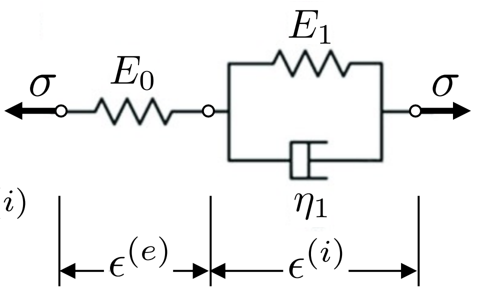
DIY

Viscoelasticity

Materials with memory

$$\sigma(t) = \int_{-\infty}^{t} R(t-t')\dot{\epsilon}(t')dt'$$

$$\epsilon = \epsilon^{(e)} + \epsilon^{(i)}$$
$$\sigma = E_0\epsilon^{(e)}$$
$$\sigma = E_1\epsilon^{(i)} + \eta_1\dot{\epsilon}^{(i)}$$



relaxation modulus or memory kernel

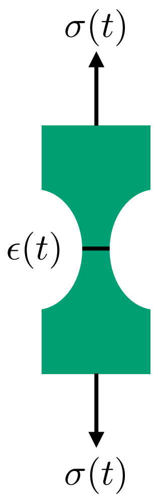
$$R(t) = R_\infty + (R_0 - R_\infty)\mathrm{e}^{-t/\tau}$$

- Example: standard solid model

$\tau = \eta_1/(E_0+E_1)$ relaxation time

$R_0 = E_0$ instantaneous modulus

$R_\infty = \left(\frac{1}{E_0}+\frac{1}{E_1}\right)^{-1}$ long-term modulus

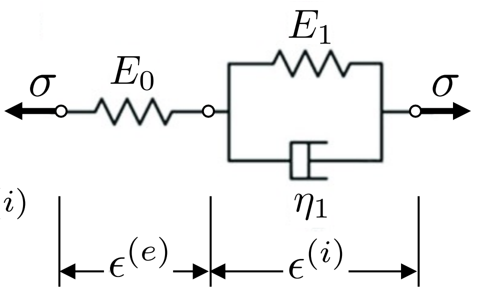


Viscoelasticity

Materials with memory - Hysteresis

$$\sigma(t) = \int_{-\infty}^{t} R(t-t')\dot{\epsilon}(t')dt'$$

$$\epsilon = \epsilon^{(e)} + \epsilon^{(i)}$$
$$\sigma = E_0 \epsilon^{(e)}$$
$$\sigma = E_1 \epsilon^{(i)} + \eta_1 \dot{\epsilon}^{(i)}$$



relaxation modulus
or memory kernel

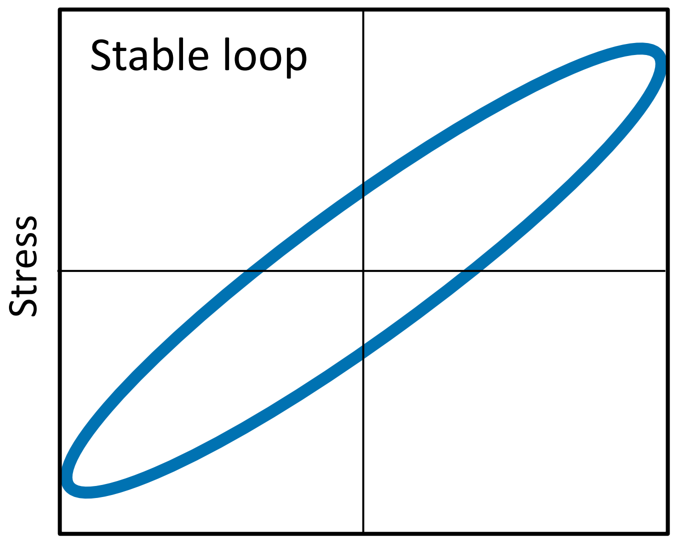
$$R(t) = R_\infty + (R_0 - R_\infty)\mathrm{e}^{-t/\tau}$$

- Example: standard solid model

$$\epsilon(t) = \epsilon_0 \sin(\omega t)$$
$$\sigma(t) = \sigma_0 \sin(\omega t + \delta)$$

Phase angle: phase difference between the dynamic stress and the dynamic strain

mechanical loss coefficient $\tan(\delta)$



Viscoelasticity

Materials with memory - Hysteresis

- Phase angle | Mechanical loss coefficient

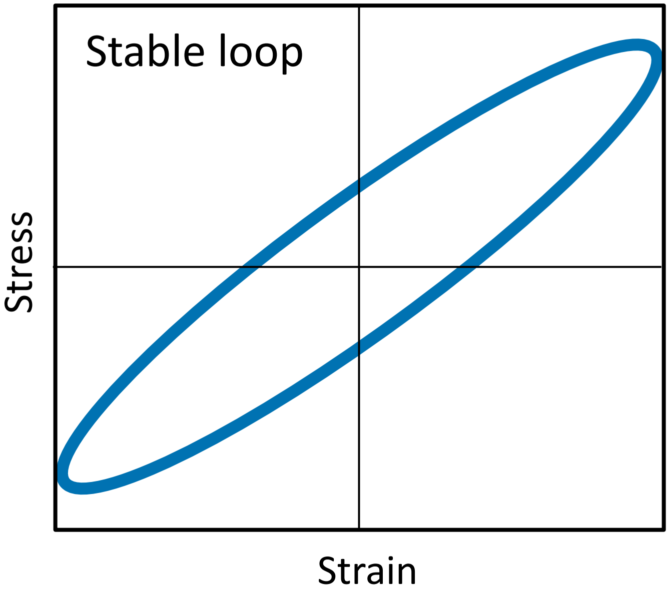
$$\epsilon(t) = \epsilon_0 \sin(\omega t)$$
$$\sigma(t) = \sigma_0 \sin(\omega t + \delta)$$

$$\tan(\delta) = E''/E'$$

storage and loss moduli of the material

DIY

$$E' = [\sigma_0 \cos(\delta)]/\epsilon_0$$
$$E'' = [\sigma_0 \sin(\delta)]/\epsilon_0$$



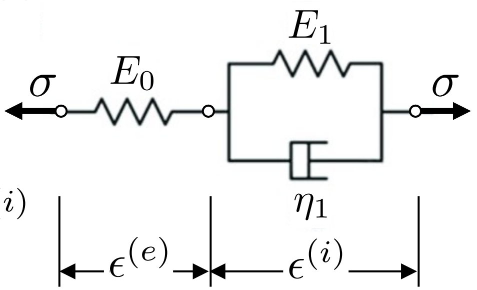
Viscoelasticity

Materials with memory – Time integration

kinetic equation

$$\dot{\epsilon}^{(i)} = \frac{E_0}{\eta_1}\left(\epsilon - \epsilon^{(i)}\right) - \frac{E_1}{\eta_1}\epsilon^{(i)}$$

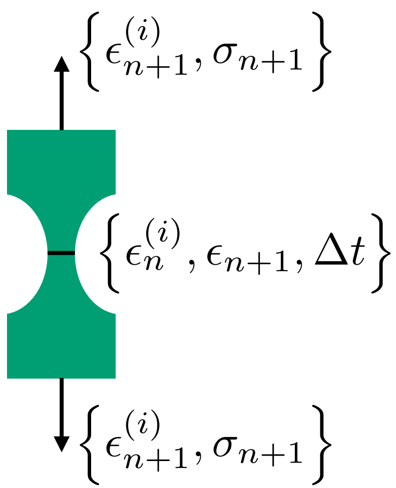
$$\epsilon = \epsilon^{(e)} + \epsilon^{(i)}$$
$$\sigma = E_0 \epsilon^{(e)}$$
$$\sigma = E_1 \epsilon^{(i)} + \eta_1 \dot{\epsilon}^{(i)}$$



- Time discretization $t_n \rightarrow t_{n+1} = t_n + \Delta t$

$$\epsilon^{(i)}_{n+1} = \frac{E_0 \epsilon_{n+1} + \eta_1/\Delta t \;\; \epsilon^{(i)}_n}{E_0 + E_1 + \eta_1/\Delta t}$$

$$\sigma_{n+1} = E_0\left(\epsilon_{n+1} - \epsilon^{(i)}_{n+1}\right)$$



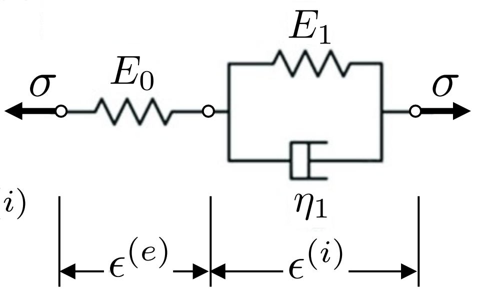
constitutive update

(*using a Backward-Euler scheme*)

Viscoelasticity

Materials with memory – Time integration

$$\epsilon = \epsilon^{(e)} + \epsilon^{(i)}$$
$$\sigma = E_0 \epsilon^{(e)}$$
$$\sigma = E_1 \epsilon^{(i)} + \eta_1 \dot{\epsilon}^{(i)}$$



kinetic equation

$$\dot{\epsilon}^{(i)} = \frac{E_0}{\eta_1}\left(\epsilon - \epsilon^{(i)}\right) - \frac{E_1}{\eta_1}\epsilon^{(i)}$$

- Time discretization $t_n \rightarrow t_{n+1} = t_n + \Delta t$

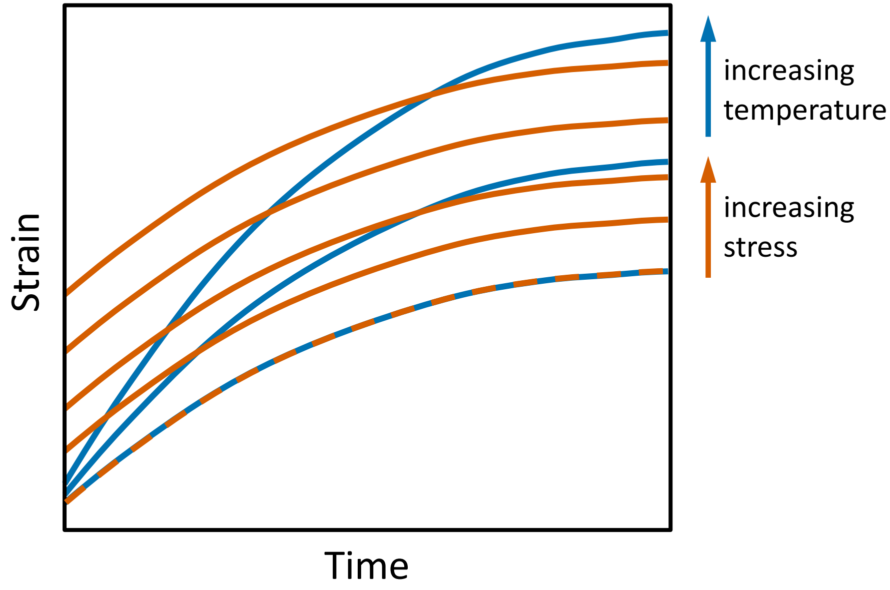
DIY

Viscoelasticity – Creep

Examples: polymers and metals

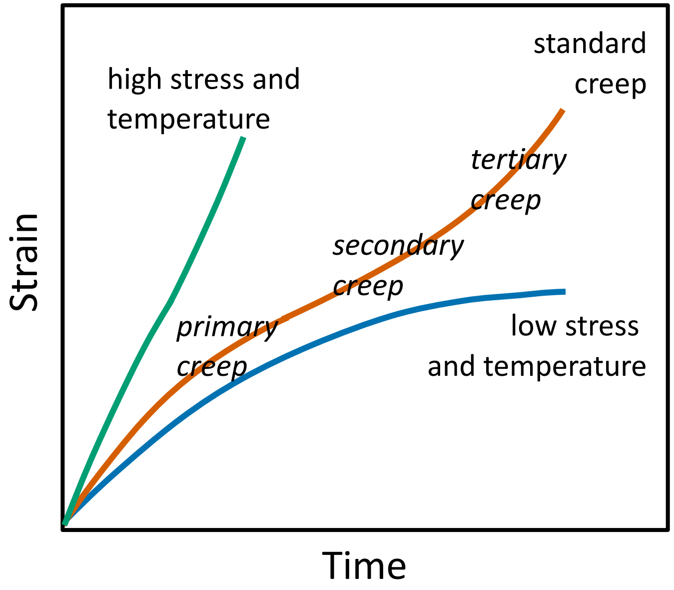
Polymers

Viscoeleastic models capture the behavior over a wide range of temperatures and stress levels



Metals

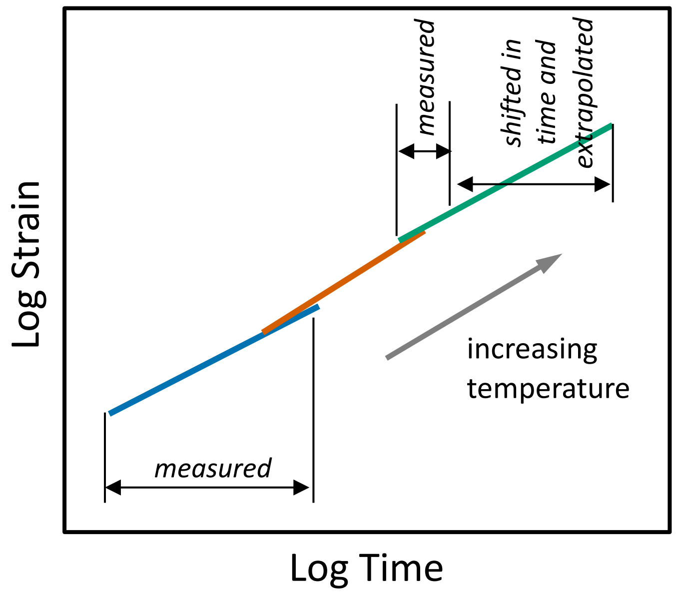
Ad-hoc creep models are needed under more extreme conditions



Viscoelasticity – Creep

Polymers: time-temperature superposition

Remark: It is usually not practical to perform a creep test for an extremely long period of time.



Time-temperature superposition (TTS):
Extrapolation technique to predict the long-term creep behavior using short-term testing. TTS assumes that the viscoelastic behavior of amorphous polymers at one temperature can be related to that at another temperature by a change in the time scale only.

How?!

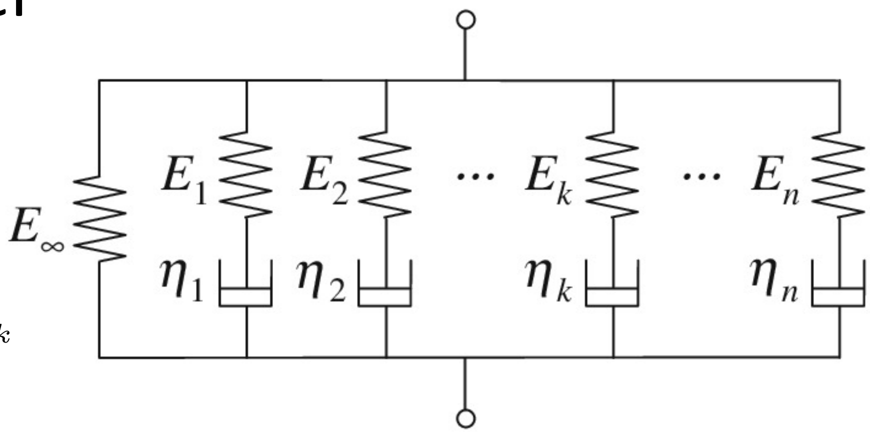
The curves from tests at different temperatures are **horizontally shifted** along a logarithmic time axis until the curves overlap to form one continuous master curve.

Viscoelasticity

Generalized Maxwell model

$$\sigma(t) = \int_{-\infty}^{t} R(t-t')\dot{\epsilon}(t')dt'$$

relaxation modulus or memory kernel



$$R(t) = E_\infty + \sum_{k=1}^{n} E_k \mathrm{e}^{-t/\tau_k}$$

$$\tau_k = \eta_k / E_k$$

Generalization to multiaxial behavior

$$R_{ijkl}(t) = R_0(t)\delta_{ij}\delta_{kl} + R_1(t)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

relaxation modulus
(*major and minor symmetries*)

Discussion

Relaxation functions can be generalized to nonlinear behavior, but most applications to real materials (i.e., nonlinear inelastic materials) require the use of **internal variables**.

Lecture 10 – Viscoelastic solids

Any questions?